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Combined additive and multiplicative properties near zero

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Abstract. It was proved that whenever $\mathbb{N}$ is partitioned into finitely many cells, one cell must contain arbitrary length geo-arithmetic progressions. It was also proved that arithmetic and geometric progressions can be nicely intertwined in one cell of partition, whenever $\mathbb{N}$ is partitioned into finitely many cells. In this article we prove that similar types of results also hold near zero for some suitable dense subsemigroups S of $\big((0,\infty),+\big)$ for which $S \cap (0,1)$ is a subsemigroup of $\big((0,1),\cdot\big)$.

Contents

1. Introduction

One of the famous Ramsey theoretic results is van der Waerden's Theorem [9], which says that whenever the set $\mathbb{N}$ of positive integers is divided into finitely many classes, one of these classes contains arbitrarily long arithmetic progressions. The analogous statement about geometric progressions is easily seen to be equivalent via the homomorphisms

$$b : (\mathbb{N},+) \to (\mathbb{N},\cdot) \quad \text{and} \quad l : (\mathbb{N}\setminus\{1\},\cdot) \to (\mathbb{N},+),$$

where $b(n) = 2^n$ and $l(n)$ is the length of the prime factorization of n. It has been shown in [1, Theorem 3.11] that any set which is multiplicatively large, that is a piecewise syndetic IP set in $(\mathbb{N},\cdot)$, must contain substantial combined additive and multiplicative structure; in particular it must contain arbitrarily large geo-arithmetic progressions, that is, sets of the form

$$\big\{r^j(a+id) : i,j \in \{0,1,\ldots,k\}\big\}.$$

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A well known extension of van der Waerden's Theorem allows one to get the additive increment of the arithmetic progression in the same cell as the arithmetic progression. Similarly for any finite partition of $\mathbb{N}$ there exist some cell A and $b, r \in \mathbb{N}$ such that $\{r, b, br, \ldots, br^k\} \subseteq A$. It is proved in [1, Theorem 1.5] that these two facts can be intertwined:

Theorem 1.1. *Let $r, k \in \mathbb{N}$ and $\mathbb{N} = \bigcup_{i=1}^{r} A_i$. Then there exist $s \in \{1, 2, \ldots, r\}$ and $a, b, d \in A_s$, such that*

$$\begin{aligned}\big\{b(a+id)^j : i, j \in \{0, 1, \ldots, k\}\big\} \cup \big\{bd^j : j \in \{0, 1, \ldots, k\}\big\} \\ \cup \big\{a + id : i \in \{0, 1, \ldots, k\}\big\} \subseteq A_s.\end{aligned}$$

In the present article our aim is to establish the above two facts for some suitable dense subsemigroups $\big((0, \infty), +\big)$. In fact we will prove that given suitable dense subsemigroup S of $\big((0, \infty), +\big)$ and any finite coloring of S we get monochromatic geo-arithmetic progressions near zero, as well as that arithmetic progressions and geometric progressions can be nicely intertwined in one cell of partition near zero. We will use algebraic structure of the Stone–Čech compactification βS for proving our results.

Given a discrete semigroup $(S, \cdot)$, we take the points of βS to be the ultrafilters on S, identifying the principal ultrafilters with the points of S and thus pretending that $S \subseteq \beta S$. Given $A \subseteq S$,

$$c\ell\, A = \bar{A} = \{p \in \beta S : A \in p\}$$

which form a basis for the topology of βS. The operation $\cdot$ can be extended to the Stone–Čech compactification βS of S so that $(\beta S, \cdot)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q \cdot p$ is continuous) with S contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x \cdot q$ is continuous). A nonempty subset I of a semigroup T is called a *left ideal of* S if $TI \subset I$, a *right ideal* if $IT \subset I$, and a *two sided ideal* (or simply an *ideal*) if it is both a left and right ideal. A *minimal left ideal* is a left ideal that does not contain any proper left ideal. Similarly, we can define *minimal right ideal* and *smallest ideal*.

Any compact Hausdorff right topological semigroup T has a smallest two sided ideal

$$\begin{aligned}K(T) &= \bigcup\{L : L \text{ is a minimal left ideal of } T\} \\ &= \bigcup\{R : L \text{ is a minimal right ideal of } T\}.\end{aligned}$$

Given a minimal left ideal L and a minimal right ideal R, $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is a *minimal* idempotent. If p and q are idempotents in T we write $p \leq q$ if and only if $pq = qp = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal.

Given $p, q \in \beta S$ and $A \subseteq S$, we have

$$A \in p \cdot q \quad \text{if and only if} \quad \{x \in S : x^{-1}A \in q\} \in p,$$

where $x^{-1}A = \{y \in S : x \cdot y \in A\}$. See [8] for an elementary introduction to the algebra of βS and for any unfamiliar details.

Definition 1.2. A subset $C \subseteq S$ is called *central* if and only if there is an idempotent $p \in K(\beta S)$ such that $C \in p$.

Central sets are ideal objects for Ramsey theoretic study and rich in combinatorial properties. The Central Sets Theorem was first discovered by H. Furstenberg [5, Proposition 8.21] for the semigroup $\mathbb{N}$ and considering sequences in $\mathbb{Z}$. However the most general version of Central Sets Theorem is available in [3, Theorem 2.2].

If S is a dense subsemigroup of $\big((0,\infty),+\big)$ then we have the following definition.

Definition 1.3. $0^+(S) = \{p \in \beta S_d : (\forall \epsilon > 0)\big((0,\epsilon) \cap S \in p\big)\}$.

It is proved in [6, Lemma 2.5], that $0^+(S)$ is a compact right topological semigroup of $(\beta S_d, +)$ which is disjoint from $K(\beta S_d)$ and hence gives some new information which is not available from $K(\beta S_d)$. Being a compact right topological semigroup $0^+(S)$ contains minimal idempotents of $0^+(S)$. We recall the following from [6, Definition 4.1].

Definition 1.4. Let S be a dense subsemigroup of $\big((0,\infty),+\big)$. A set $C \subseteq S$ is central near 0 if and only if there is some idempotent $p \in K\big(0^+(S)\big)$ with $C \in p$.

Now if we turn our attention to a dense subsemigroup S of $\big((0,\infty),+\big)$ we get another version of Central Sets Theorem.

Theorem 1.5. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$. Let $\mathcal{T}$ be the set of sequences $\langle y_n\rangle_{n=1}^{\infty}$ in $S \cup \{0\}$ such that $\lim\limits_{n\to\infty} y_n = 0$. Let C be a subset of S which is central near zero and let $F \in \mathcal{P}_f(\mathcal{T})$. There exist a sequence $\langle a_n\rangle_{n=1}^{\infty}$ in C such that $\sum_{n=1}^{\infty} a_n$ converges and such that $FS(\langle a_n\rangle_{n=1}^{\infty}) \subseteq C$ and a sequence $\langle H_n\rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ such that for each $n \in \mathbb{N}$, $\max H_n < \min H_{n+1}$ and for each $L \in \mathcal{P}_f(\mathbb{N})$ and each $f \in F$, $\sum_{n\in L}\big(a_n + \sum_{t\in H_n} f(t)\big) \in C$.*

Proof. [2, Corollary 4.7] or [6, Theorem 4.11]. □

An immediate simple application of the above theorem yields the following version of van der Waerden's Theorem that has been proved in [6, Corollary 5.1].

Theorem 1.6. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ and let $\langle x_n\rangle_{n=1}^{\infty}$ be a sequence in S such that $\lim_{n\to\infty} x_n = 0$. Assume $r, k \in \mathbb{N}$, $\delta > 0$ and let $S = \bigcup_{i=1}^{r} B_i$. Then there exists $i \in \{1, 2, \ldots, r\}$, $a \in S$ and $d \in FS(\langle x_n\rangle_{n=1}^{\infty})$ such that $\{a, a+d, \ldots, a+kd\} \subseteq B_i \cap (0,\delta)$.*

Now if S is a dense subsemigroup of $\big((0,\infty),+\big)$ for which $S\cap(0,1)$ is a subsemigroup of $\big((0,1),\cdot\big)$, then any central set in $(S\cap(0,1),\cdot)$ contains a geometric progression $\{b, br, \ldots, br^k\}$ of arbitrary length $k\in\mathbb{N}$. In Section 2 of this article we first prove that geo-arithmetic progressions as well as geometric progressions can be found in one cell of a partition. Further we shall prove that a geometric progression and an arithmetic progression can be nicely intertwined in one cell of a partition in the direction of Theorem 1.6.
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2. Combined additive and multiplicative properties near zero

In this section, first we prove that for dense subsemigroups S of $\big((0,\infty),+\big)$ for which $S\cap(0,1)$ is a subsemigroup of $\big((0,1),\cdot\big)$ with some extra property we can find geo-arithmetic progressions of arbitrary length in one cell of partition of $S\cap(0,1)$.

As an application of Theorem 1.5 it was proved in Theorem 1.6 that given any sequence $\langle x_n\rangle_{n=1}^\infty$ with $\lim_{n\to\infty} x_n = 0$, any central set near zero contains arbitrary long arithmetic progressions with increment in $FS(\langle x_n\rangle_{n=1}^\infty)$. Further it can be proved that central sets near zero also contain arbitrary long arithmetic progressions as well as their increments. First we need the following simple consequence of Theorem 1.5, which holds for piecewise syndetic sets near zero as well. The notion of piecewise syndetic sets near zero was first introduced in [6] in the course of central sets near zero. We recall the definition first.

Definition 2.1. Let S be a dense subsemigroup of $\big((0,\infty),+\big)$. A set $A\subseteq S$ is *piecewise syndetic near* 0 if and only if there is some $p\in K\big(0^+(S)\big)$ with $A\in p$.

Theorem 2.2. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$, C be a piecewise syndetic near zero set in $(S,+)$ and for each $i\in\{1,2,\ldots,k\}$, let $\langle y_{i,n}\rangle_{n=1}^\infty$ be a sequence in S such that $\sum_{n=1}^\infty y_{i,n}$ converges. Then there exist $b\in C$, a sequence $\langle a_n\rangle_{n=1}^\infty$ in S, and a sequence $\langle H_n\rangle_{n=1}^\infty$ in $\mathcal{P}_f(\mathbb{N})$ with* $\max H_n < \min H_{n+1}$ *for each $n\in\mathbb{N}$ such that*

$$b+FS\big(\langle a_n\rangle_{n=1}^\infty\big)\subseteq C \quad \textit{and} \quad b+FS\left(\left\langle a_n+\sum_{t\in H_n} y_{i,t}\right\rangle_{n=1}^\infty\right)\subseteq C$$

for all $i\in\{1,2,\ldots,k\}$. In particular there exist $F\in\mathcal{P}_f(\mathbb{N})$ and $b\in S$ such that

$$\{b\}\cup\left\{b+\sum_{t\in F} y_{i,t} : i\in\{1,2,\ldots,k\}\right\}\subseteq C.$$

Proof. Pick $p \in K\big(0^+(S)\big)$ such that $C \in p$. We choose a minimal left ideal L of $0^+(S)$ containing p and let e be an idempotent in L. Then $p = p + e$ so $\{x \in S : -x + C \in e\} \in p$. Pick $b \in C$ such that $-b + C \in e$. Then $-b + C$ is central near zero. Therefore by Theorem 1.5 we can find a sequence $\langle a_n \rangle_{n=1}^{\infty}$ in S such that $\sum_{n=1}^{\infty} a_n$ converges and a sequence $\langle H_n \rangle_{n=1}^{\infty}$ in $\mathcal{P}_f(\mathbb{N})$ with $\max H_n < \min H_{n+1}$ for each $n \in \mathbb{N}$ such that $FS(\langle a_n \rangle_{n=1}^{\infty}) \subseteq -b + C$ and $FS(\langle a_n + \sum_{t \in H_n} y_{i,t} \rangle_{n=1}^{\infty}) \subseteq -b + C$ for each $i \in \{1, 2, \ldots, k\}$. □

Theorem 2.3. *Let S be a dense subsemigroup of $\big((0, \infty), +\big)$, Let $r, k \in \mathbb{N}$, let $\delta > 0$ and let $S = \bigcup_{i=1}^{r} B_i$. Then there exists $i \in \{1, 2, \ldots, r\}$ and $b, d \in S$ such that $\{d, b, b + d, \ldots, b + kd\} \subseteq B_i \cap (0, \delta)$.*

Proof. There exists $i \in \{1, 2, \ldots, r\}$ such that $B_i \cap (0, \delta)$ is central near zero. Let us choose $\langle x_n \rangle_{n=1}^{\infty}$ in S such that $FS(\langle x_n \rangle_{n=1}^{\infty}) \subseteq B_i \cap (0, \delta)$. For each $i \in \{1, 2, \ldots, k\}$ and $n \in \mathbb{N}$, let $y_{i,n} = i \cdot x_n$. Pick b and F as guaranteed by Theorem 2.2 and take $d = \sum_{t \in F} x_t$, which implies that

$$\{d\} \cup \{b\} \cup \left\{ b + \sum_{t \in F} (i \cdot x_n) : i \in \{1, 2, \ldots, k\} \right\} \subseteq B_i \cap (0, \delta). \qquad \square$$

The following follows from [1, Corollary 2.7].

Theorem 2.4. *Let S be a subsemigroup of $\big((0, 1), \cdot\big)$, A be a central set in $(S, \cdot)$ and $k \in \mathbb{N}$. Then there exist $a \in S$ and $r \in A$ such that*

$$\{r, a, ar, ar^2, \ldots, ar^k\} \subseteq A.$$

We will apply the above Theorem to prove existence of geo-arithmetic progression in one cell of partition.

Definition 2.5. A family $\mathcal{A}$ of subsets of a set X is partition regular provided that whenever X is partitioned into finitely many classes, one of these classes contains a member of $\mathcal{A}$.

We recall the following from [1, Theorem 1.1(b)].

Theorem 2.6. *Let $(S, \cdot)$ be a semigroup and $\mathcal{F}$ and $\mathcal{G}$ be partition regular families of of subsets of S with all members of $\mathcal{F}$ finite. let $r \in \mathbb{N}$ and let $S = \bigcup_{i=1}^{r} A_i$. Then there exists $i \in \{1, 2, \ldots, r\}$, $B \in \mathcal{F}$ and $C \in \mathcal{G}$ such that $B \cdot C \subseteq A_i$.*

The following theorem is our promised geo-arithmetic progression in one cell of the partition.

Theorem 2.7. *Let S be a dense subsemigroup of $\big((0, \infty), +\big)$ for which $S \cap (0, 1)$ is a subsemigroup of $\big((0, 1), \cdot\big)$ and assume that for each $y \in S \cap (0, 1)$ and for each $x \in S$, $x/y \in S$ and $yx \in S$. Let $s, k \in \mathbb{N}$ and let $S \cap (0, 1) = \bigcup_{i=1}^{s} B_i$. Then there exists $i \in \{1, 2, \ldots, s\}$, $a, d \in S \cap (0, 1)$, and $r \in B_i$ such that*

$$\big\{r^s(a + td) : t, s \in \{0, 1, \ldots, k\}\big\} \cup \big\{dr^s : s \in \{0, 1, \ldots, k\}\big\} \subseteq B_i.$$

Proof. By [6, Theorem 5.6], there exists $i \in \{1, 2, \ldots, s\}$ such that B_i is central near 0 in the additive semigroup $(S, +)$ as well as B_i is central in $(S \cap (0,1), \cdot)$. Let

$$\mathcal{F} = \big\{br^s : s \in \{0, 1, \ldots, k\} : b \in S \cap (0,1) \text{ and } r \in B_i\big\},$$
$$\mathcal{G} = \big\{\{d\} \cup \big\{(a + td) : t \in \{0, 1, \ldots, k\}\big\} : a, d \in S \cap (0,1)\big\}.$$

Then $\mathcal{F}$ is a partition regular family of $S \cap (0,1)$. In fact let $S \cap (0,1) = \bigcup_{j=1}^{r} C_j$ and put $B_{i,j} = B_i \cap C_j$ for $j \in \{1, 2, \ldots, r\}$. Then $B_i = \bigcup_{j=1}^{r} B_{i,j}$. But since B_i is central in $(S \cap (0,1), \cdot)$ there exists a minimal idempotent $p \in K\big(\beta\big(S \cap (0,1)\big)_d, \cdot\big)$ such that $B_i \in p$. Now as $B_i = \bigcup_{j=1}^{r} B_{i,j}$ we have some $B_{i,j} \in p$, so that $B_i \cap C_j$ is central in $(S \cap (0,1), \cdot)$. Therefore by Theorem 2.4 there exist $b \in S \cap (0,1)$ and $r \in B_i$ such that

$$F = \big\{br^s : s \in \{0, 1, \ldots, k\}\big\} \subset B_i \cap C_j;$$

in particular there exists $F \in \mathcal{F}$ such that $F \subset C_j$. Again by Theorem 2.3 we have $\mathcal{G}$ is a partition regular family of $(S \cap (0,1), \cdot)$. Now Theorem 2.6 implies that

$$\mathcal{H} = \{B \cdot C : B \in \mathcal{F} \text{ and } C \in \mathcal{G}\}$$

is also partition regular. Now observe that for any $t \in S \cap (0,1)$ and $H \in \mathcal{H}$, $tH \in \mathcal{H}$. Again since B_i is central in $(S \cap (0,1), \cdot)$, by [1, Lemma 2.3] we can pick some $B \in \mathcal{F}$ and $C \in \mathcal{G}$ such that $B \cdot C \subseteq B_i$. Now pick $x \in S \cap (0,1)$ and $r \in B_i$ such that

$$B = \big\{xr^s : s \in \{0, 1, \ldots, k\}\big\}$$

and pick $u, v \in S \cap (0,1)$ such that

$$C = \{v\} \cup \big\{(u + tv) : t \in \{0, 1, \ldots, k\}\big\}.$$

Let us put $a = ux$ and $d = vx$. $\square$

We end this article by proving that arithmetic and geometric progressions can be nicely intertwined near zero in one cell of a finite partition for some suitable dense subsemigroups S of $\big((0, \infty), +\big)$. The following theorem is the basis for our final theorem.

Theorem 2.8. *Let S be a dense subsemigroup of $\big((0,1), \cdot\big)$. Let C be a central subset of $(S, \cdot)$ and $\mathcal{F}$ be a partition regular family of finite subsets of S and $k \in \mathbb{N}$. Then there exists $b, r \in S \cap (0,1)$ and $F \in \mathcal{F}$ such that*

$$rF \cup \big\{b(rx)^j : x \in F \text{ and } j \in \{0, 1, 2, \cdots, k\}\big\} \subseteq C.$$

Proof. See [1, Theorem 4.2]. $\square$

As mentioned in [6], we recall the fact that an ultrafilter on $S \cap (0,1)$ is not quite the same thing as an ultrafilter on S with $(0,1)$ as a member and so pretend that $0^+(S) \subseteq \beta\big(S \cap (0,1)\big)_d$.

Theorem 2.9. *Let S be a dense subsemigroup of $\big((0,\infty),+\big)$ for which $S\cap(0,1)$ is a subsemigroup of $\big((0,1),\cdot\big)$ and assume that for each $y\in S\cap(0,1)$ and for each $x\in S$, $x/y\in S$ and $yx\in S$. Let $r,k\in\mathbb{N}$, let $\delta>0$, and let $S\cap(0,1)=\bigcup_{i=1}^r B_i$. Then there exist $i\in\{1,2,\ldots,r\}$ and $a,b,d\in S\cap(0,1)$ such that*

$$\big\{b(a+id)^j : i,j\in\{0,1,\ldots,k\}\big\}\cup\{bd^j : j\in\{0,1,\ldots,k\}\}$$
$$\cup\big\{a+id : i\in\{0,1,\ldots,k\}\big\}\cup\{d\}\subseteq B_i\cap(0,\delta).$$

Proof. By [6, Theorem 5.6], there exists $i\in\{1,2,\ldots,r\}$ such that B_i is central near 0 in the additive semigroup $(S,+)$ as well as B_i is central in $(S\cap(0,1),\cdot)$. Hence $B_i\cap(0,\delta)$ is also central in $(S\cap(0,1),\cdot)$. Now let

$$\mathcal{F}=\big\{\{d,a,a+d,\ldots,a+kd\} : a,d\in S\cap(0,1)\big\}.$$

Then by Theorem 2.3 we have $\mathcal{F}$ is a partition regular family of $(S\cap(0,1),\cdot)$. We choose by Theorem 2.8, $b,u\in S\cap(0,1)$ and $F\in\mathcal{F}$ such that

$$uF\cup\big\{b(ux)^j : x\in F \text{ and } j\in\{0,1,2,\cdots,k\}\big\}\subseteq B_i\cap(0,\delta).$$

Pick $c,s\in S\cap(0,1)$ such that $F=\{c,s,s+c,\ldots,s+kc\}$. Choose $d=uc$ and $a=us$. □

References

[1] Beiglböck, Mathias; Bergelson, Vitaly; Hindman, Neil; Strauss, Dona. Some new results in multiplicative and additive Ramsey theory. *Trans. Amer. Math. Soc.* **360** (2008), no. 2, 819–847. MR2346473 (2008g:05214), Zbl 1136.05074.

[2] De, Dibyendu; Hindman, Neil. Image partition regularity near zero. *Discrete Math.* **309** (2009), no. 10, 3219–3232. MR2526740 (2010h:05325), Zbl 1202.05146.

[3] De, Dibyendu; Hindman, Neil; Strauss, Dona. A new and stronger central sets theorem. *Fund. Math.* **199** (2008), no. 2, 155–175. MR2410923 (2009c:05248), Zbl 1148.05052.

[4] De, Dibyendu; Paul, Ram Krishna. Image partition regularity of matrices near 0 with real entries. *New York J. Math.* **17** (2011), 149–161. MR2526740 (2010h:05325), Zbl 1148.05052.

[5] Furstenberg, H. Recurrence in ergodic theory and combinatorial number theory. M. B. Porter Lectures, *Princeton University Press, Princeton, N. J.* 1981, xi+203 pp. ISBN: 0459.28023. MRR0603625 (82j:28010), Zbl 0459.28023.

[6] Hindman, Neil; Leader, Imre. The semigroup of ultrafilters near 0. *Semigroup Forum* **59** (1999), no. 1, 33–55. MR1847941 (2002h:22004), Zbl 0942.22003.

[7] Hindman, Neil; Leader, Imre; Strauss, Dona. Image partition regular matrices — bounded solutions and preservation of largeness. *Discrete Math.* **242** (2002), no. 1-3, 115–144. MR1874760 (2002j:05146), Zbl 1007.05095.

[8] Hindman, Neil; Strauss, Dona. Algebra in the Stone-Čech compactification. Theory and applications. de Gruyter Expositions in Mathematics, 27. *Walter de Gruyter & Co., Berlin*, 1998. xiv+485 pp. ISBN: 3-11-015420-X. MR1642231 (99j:54001), Zbl 0918.22001.

[9] van der Waerden, Bartel Leendert. Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wiskunde* **15** (1927), 212–216.

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