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Noncommutative semialgebraic sets in nilpotent variables

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ABSTRACT. We solve the lifting problem in C^*-algebras for many sets of relations that include the relations $x_j^{N_j} = 0$ for all variables. The remaining relations must be of the form $\|p(x_1, \dots, x_n)\| \leq C$ for C a positive constant and p a noncommutative $*$-polynomial that is in some sense homogeneous. For example, we prove liftability for the set of relations

$$x^3 = 0, \quad y^4 = 0, \quad z^5 = 0, \quad xx^* + yy^* + zz^* \leq 1.$$

Thus we find more noncommutative semialgebraic sets that have the topology of noncommutative absolute retracts.

CONTENTS

1. Introduction

Lifting problems involving norms and star-polynomials are fundamental in C^*-algebras. They arise in basic lemmas in the subject, as we shall see in a moment. They also arise in descriptions of the boundary map in K-theory, in technical lemmas on inductive limits, and have of course been around in operator theory. Much of our understanding of the Calkin algebra comes from having found properties of its cosets that exist only when some operator in a coset has that property.

Let A denote a C^*-algebra and let I be an ideal in A. The quotient map will be denoted $\pi : A \to A/I$. Of course A/I is a C^*-algebra, but let us

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ponder how we know this. The standard proof uses an approximate unit u_λ and an approximate lifting property. The lemma used is that for any approximate unit u_λ, and any a in A,

$$\lim_\lambda \|a(1-u_\lambda)\| = \|\pi(a)\|$$

and trivially we obtain as a corollary

$$\lim_\lambda \|(1-u_\lambda)b(1-u_\lambda)\| = \|\pi(b)\|.$$

For a large λ, the lift $\bar{x} = a(1-u_\lambda)$ of $\pi(a)$ approximately achieves *two* norm conditions,

$$\|\bar{x}\| \approx \|\pi(\bar{x})\|, \quad \|\bar{x}^*\bar{x}\| \approx \|\pi(\bar{x})^*\pi(\bar{x})\|.$$

The equality $\|\bar{x}\|^2 = \|\bar{x}^*\bar{x}\|$ upstairs now passes downstairs, so A/I is a C^*-algebra.

We have an eye on potential applications in noncommutative real algebraic geometry [7, 8]. What essential differences are there between real algebraic geometry and noncommutative real algebraic geometry? Occam would cut between these fields with the equation

$$x^n = 0.$$

Could we just exclude this equation? Probably not. A search of the physics literature finds that polynomials in nilpotent variables are gaining popularity. Two examples to see are [3] in condensed matter physics, and [12] in quantum information.

Focusing back on lifting problems, we recall what is known about lifting nilpotents up from general C^*-algebra quotients. Akemann and Pedersen [1] showed the relation $x^2 = 0$ lifts, and Olsen and Pedersen [14] did the same for $x^n = 0$. Akemann and Pedersen [1] also showed that if $x^{n-1} \neq 0$ for some $x \in A/I$ then one can find a lift X of x with

$$\left\|X^j\right\| = \left\|x^j\right\|, \quad (j = 1, \ldots, n-1).$$

If $x^n = 0$ and $x^{n-1} \neq 0$ then we would like to combine these results, lifting both the nilpotent condition and the $n-1$ norm conditions. It was not until recently, in [16], that it was shown one could lift just the two relations

$$\|x\| \leq C, \quad x^n = 0$$

for $C > 0$.

Here we show how to lift a nilpotent and all these norm conditions, and so show the liftablity of the set of relations

$$\left\|x^j\right\| \leq C_j, \quad j = 1, \ldots, n,$$

even if $C_n = 0$. In the particular case where the quotient is the Calkin algebra and the lifting is to $\mathbb{B}(\mathbb{H})$, we proved this using different methods in [10], as a partial answer to Olsen's question [13].

More generally, we consider soft homogeneous relations (as defined below) together with relations $x_j^{N_j} = 0$. In one variable, another example of such a collection of liftable relations is

$$\|x\| \leq C_1, \quad \left\|x^*x - x^2\right\| \leq C_2, \quad x^3 = 0.$$

In two variables, we have such curiosities as

$$\|x\| \leq 1, \quad \|y\| \leq 1, \quad x^3 = 0, \quad y^3 = 0, \quad \|x - y\| \leq \epsilon$$

which we can now lift.

Given a $*$-polynomial in $x_1, \ldots, x_n$ we have the usual relation

$$p(x_1, \ldots, x_n) = 0,$$

where now the x_j are in a C^*-algebra. In part due to the shortage of semiprojective C^*-algebras, Blackadar [2] suggested that we would do well to study the relation $\|p(x_1, \ldots, x_n)\| \leq C$ for some $C > 0$. Following Exel's lead [6], we call this a *soft polynomial relation.* Softened relations come up naturally when trying to classifying C^*-algebras that are inductive limits, as in [5], when exact relations in the limit lead only to inexact relations in a building block in the inductive system.

The homogeneity we need is only that there be a subset, say $x_1, \ldots, x_r$, of the variables and an integer $d \geq 1$ so that every monomial in p contains exactly d factors from $x_1, x_1^* \ldots, x_r, x_r^*$.

The relation $x^N = 0$ is "more liftable" than most liftable relations in that it can be added to many liftable sets while maintaining liftability. Other relations that behave this way are $x^* = x$ and $x \geq 0$. We explored semialgebraic sets (as NC topological spaces) in positive and hermitian variables in [11].

There are still other relations that are "more liftable" in this sense. We consider in this note $xyx^* = 0$ and $xy = 0$. This is not the end of the story. We might have a rare case of too little theory and too many examples.

We use many technical results from our previous work [11]. We also have use for the Kasparov Technical Theorem. Indeed we use only a simplified version, but the fully technical version can probably be used to find even more lifting theorems in this realm. For a reference, a choice could be made from [4, 9, 14].

We will use the notation $a \ll b$ to mean b acts like unit on a, i.e.,

$$ab = a = ba.$$

A trick we use repeatedly is to replace a single element c so that $0 \leq c \leq 1$ and

$$x_j c = x_j, \quad c y_k = 0$$

for some sequences x_j and y_k with two elements a and b with

$$0 \leq a \ll b \leq 1 \tag{1.1}$$

and

$$x_j a = x_j, \quad b y_k = 0. \tag{1.2}$$

These are found with basic functional calculus. The simplified version of Kasparov's technical theorem we need can be stated as follows: for $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ in a corona algebra $C(A) = M(A)/A$ (for A σ-unital) with $x_j y_k = 0$ for all j and k, there are elements a and b in $C(A)$ satisfying (1.1) and (1.2).

2. Lifting nilpotents while preserving various norms

Lemma 2.1. *Suppose A is σ-unital C^*-algebra, n is at least 2, and consider the quotient map $\pi : M(A) \to M(A)/A$.*

(1) *If x is an element of $M(A)$ so that $\pi(x^n) = 0$ then there are elements $p_1, \ldots, p_{n-1}$ and $q_1, \ldots, q_{n-1}$ of $M(A)$ with*

$$j > k \implies p_j q_k = 0$$

and

$$\pi\left(\sum_{j=1}^{n-1} q_j x p_j\right) = \pi(x).$$

(2) *If $\pi(\tilde{x}) = \pi(x)$ and we set*

$$\bar{x} = \sum_{j=1}^{n-1} q_j \tilde{x} p_j,$$

then $\pi(\bar{x}) = \pi(x)$ and $\bar{x}^n = 0$.

Proof. This is the essential framework that assists the lifting of nilpotents, going back to [14]. Other than a change of notation, this is an amalgam of Lemmas 1.1, 8.1.3, 12.1.3 and 12.1.4 of [9]. □

Theorem 2.2. *If x is an element of a C^*-algebra A, and I is an ideal and $\pi : A \to A/I$ is the quotient map, then for any natural number N, there is an element $\bar{x}$ in A so that $\pi(\bar{x}) = \pi(x)$ and*

$$\|\bar{x}^n\| = \|\pi(x^n)\|, \quad (n = 1, \ldots, N).$$

Proof. If $\pi(x^N) \neq 0$, then this is the first statement in Theorem 3.8 of [1].

Assume then that $\pi(x^N) = 0$. Standard reductions (Theorem 10.1.9 of [9]) allow us to assume $A = M(E)$ and $I = E$ for some separable C^*-algebra E. The first part of Lemma 2.1 provides elements $p_1, \ldots, p_{N-1}$ and $q_1, \ldots, q_{N-1}$ in $M(E)$ with

$$j > k \implies p_j q_k = 0$$

and

$$\pi\left(\sum_{j=1}^{N-1} q_j x p_j\right) = \pi(x).$$

Let $C_n = \|\pi(x^n)\|$. Each norm condition

$$\left\|\left(\sum_{j=1}^{N-1} q_j \tilde{x} p_j\right)^n\right\| \leq C_n \quad (n = 1, \ldots, N-1)$$

is a norm-restriction of a NC polynomial that is homogeneous in $\tilde{x}$. We can apply Theorem 3.2 of [11] to find $\hat{x}$ in $M(E)$ with $\pi(\hat{x}) = \pi(\tilde{x})$ and

$$\left\|\left(\sum_{j=1}^{N-1} q_j \hat{x} p_j\right)^n\right\| \leq C_n \quad (n = 1, \ldots, N-1).$$

Since $\pi(\hat{x}) = \pi(x)$ we may apply the second part of Lemma 2.1 to conclude that

$$\bar{x} = \sum_{j=1}^{N-1} q_j \hat{x} p_j$$

is a lift of $\pi(x)$, is nilpotent of order N, and

$$\|\bar{x}^n\| \leq C_n = \|\pi(x^n)\|$$

for $n = 1, \ldots, N-1$. $\square$

There was nothing special about the homogeneous $*$-polynomials x^n, and we can deal with more than one nilpotent variable x at a time. We say a $*$-polynomial is *homogeneous of degree r for some subset S* of the variables when the total number of times either x or x^* for $x \in S$ appears in each monomial is r. Staying consistent with the notation in [11], we use

$$p(\mathbf{x}, \mathbf{y}) = p(x_1, \ldots, x_r, y_1, y_2, \ldots)$$

as so keep to the left the variables in subset where there is homogeneity.

Theorem 2.3. *Suppose $p_1, \ldots, p_J$ are NC $*$-polynomials in infinitely many variables that are homogeneous in the set of the first r variables, each with degree of homogeneity d_j at least one. Suppose $C_j > 0$ are real constants and $N_k \geq 2$ are integer constants, $k = 1, \ldots, r$. For every C^*-algebra A and $I \lhd A$ an ideal, given $x_1, \ldots, x_r$ and $y_1, y_2, \ldots$ in A with*

$$(\pi(x_k))^{N_k} = 0$$

and

$$\|p_j(\pi(\mathbf{x}, \mathbf{y}))\| \leq C_j,$$

there are $z_1, \ldots, z_r$ in A with $\pi(\mathbf{z}) = \pi(\mathbf{x})$ and

$$z_k^{N_k} = 0$$

and

$$\|p_j(\mathbf{z}, \mathbf{y})\| \leq C_j.$$

Proof. Again we use standard reductions to assume $A = M(E)$ and $I = E$ for some separable C^*-algebra E. Now we apply Lemma 2.1 to each x_k and find $p_{k,1}, \dots, p_{k,N_k-1}$ and $q_{k,1}, \dots, q_{k,N_k-1}$ in $M(E)$ with

$$b > c \implies p_{k,b}q_{k,c} = 0$$

and

$$\pi\left(\sum_{b=1}^{N_k-1} q_{k,b}x_k p_{k,b}\right) = \pi\left(x_k\right).$$

We know that any $\tilde{\mathbf{x}}$ we take with $\pi\left(\tilde{\mathbf{x}}\right) = \pi\left(\mathbf{x}\right)$ will give us

$$\pi\left(\sum_{b=1}^{N_k-1} q_{k,b}\tilde{x}_k p_{k,b}\right) = \pi\left(x_k\right)$$

and

$$\left(\sum_{b=1}^{N_k-1} q_{k,b}\tilde{x}_k p_{k,b}\right)^{N_k} = 0,$$

so we need only fix the relations

$$\left\| p_j\left(\sum_{b=1}^{N_1-1} q_{1,b}\tilde{x}_1 p_{1,b}, \dots, \sum_{b=1}^{N_r-1} q_{r,b}\tilde{x}_r p_{r,b}, \mathbf{y}\right)\right\| \le C_j.$$

These are homogeneous in $\{\tilde{x}_1, \dots, \tilde{x}_r\}$ so we are done, by Theorem 3.2 of [11]. □

We could add various relations on the variables $y_1, y_2, \dots,$ and include in the p_j some $*$-polynomials that ensure that there is an associated universal C^*-algebra which is then projective. For example, we could zero out the extra variables (so just omit them) and impose a soft relation known to imply all the x_j are contractions. Let us give one specific class of examples.

Example 2.4. Let A be the universal C^*-algebra on $x_1, \dots, x_n$ subject to the relations

$$x^{N_k} = 0, \quad \left\|\sum x_k x_k^*\right\| \le 1, \quad \left\|p_j\left(x_1, \dots, x_n\right)\right\| \le C_j$$

for $C_j > 0$ and where the p_j are all NC $*$-polynomials that are homogeneous in $x_1, \dots, x_n$. Then A is projective.

3. The relation $xyx^* = 0$

We now explore setting xyx^* to zero. This word is unshrinkable, in the sense of [17]. We show that many sets of relations involving $xyx^* = 0$ are liftable. One example, chosen essentially at random, is the set consisting of the relations

$$\|x\| \le 1, \quad \|y\| \le 1, \quad \|xy + yx\| \le 1, \quad xyx^* = 0.$$

Lemma 3.1. *Suppose A is σ-unital and $C(A) = M(A)/A$. If x and y are elements of $M(A)$ so that $xyx^* = 0$, then there are elements*

$$0 \leq e \ll f \ll g \leq 1$$

so that

$$x(1-g) = x$$

and

$$ey + (1-e)yf = y.$$

Proof. We apply Kasparov's technical theorem to the product $x\,(yx^*) = 0$ to find

$$0 \leq d \leq 1$$

in $C(A)$ with

$$xd = x, \tag{3.1}$$

$$dyx^* = 0. \tag{3.2}$$

We rewrite (3.1) as

$$(1-d)x^* = 0 \tag{3.3}$$

and apply Kasparov's technical theorem to (3.2) and (3.3) to find

$$0 \leq f \ll g \leq 1$$

in $C(E)$ with

$$\begin{gathered}(1-d)f = (1-d)\\ dyf = dy\\ gx^* = 0.\end{gathered} \tag{3.4}$$

Thus we have $xg = 0$ and

$$0 \leq 1-d \ll f \ll g \leq 1.$$

We are done, with $e = 1-d$, since (3.4) gives us

$$ey + (1-e)yf = (1-d)y + dyf = y.$$

∎

Lemma 3.2. *Suppose A is σ-unital and consider the quotient map*

$$\pi : M(A) \to M(A)/A.$$

(1) *If x and y are elements of $M(A)$ so that $\pi\,(xyx^*) = 0$, then there are elements e, f and g in $M(A)$ with*

$$\begin{gathered}0 \leq e \ll f \ll g \leq 1,\\ \pi\,(x(1-g)) = \pi(x)\end{gathered} \tag{3.5}$$

and

$$\pi\,(ey + (1-e)yf) = \pi(y).$$

(2) *If* $\pi(\tilde{x}) = \pi(x)$ *and* $\pi(\tilde{y}) = \pi(y)$ *then, if we set*

$$\bar{x} = \tilde{x}(1-g),$$
$$\bar{y} = e\tilde{y} + (1-e)\tilde{y}f,$$

we have $\pi(\bar{x}) = \pi(x)$, $\pi(\bar{y}) = \pi(y)$ *and* $\bar{x}\bar{y}\bar{x}^* = 0$.

Proof. In $C(A)$, the product $\pi(x)\pi(y)\pi(x)^*$ is zero, so Lemma 3.1 produces e_0, f_0 and g_0 in $C(A)$ with

$$0 \le e_0 \ll f_0 \ll g \le 1,$$

$$\pi(x)(1-g_0) = \pi(x)$$

and

$$e_0\pi(y) + (1-e_0)\pi(y)f_0 = \pi(y).$$

Lemma 1.1.1 of [9] tells us there are lifts e, f and g in $M(A)$ of e_0, f_0 and g_0 satisfying (3.5). Then

$$\pi\left(x(1-g)\right) = \pi\left(x\right)\left(1-g_0\right) = \pi\left(x\right)$$

and

$$\pi\left(ey + (1-e)yf\right) = e_0\pi(y) + (1-e_0)\pi(y)f_0 = \pi(y).$$

As for the second statement,

$$\pi(\bar{x}) = \pi\left(\tilde{x}(1-g)\right) = \pi(x)(1-g_0) = \pi(x),$$

$$\pi(\bar{y}) = \pi(e\tilde{y} + (1-e)\tilde{y}f) = e_0\pi(y) + (1-e_0)\pi(y)f_0 = \pi(y)$$

and

$$\bar{x}\bar{y}\bar{x}^* = \tilde{x}(1-g)e\tilde{y}(1-g)\tilde{x}^* + \tilde{x}(1-g)(1-e)\tilde{y}f(1-g)\tilde{x}^* = 0$$

since $(1-g)e = 0$ and $(1-g)f = 0$. $\square$

Theorem 3.3. *Suppose* $p_1, \ldots, p_J$ *are NC* $*$*-polynomials in infinitely many variables that are homogeneous in the set of the first* $2r$ *variables, each with degree of homogeneity* d_j *at least one. Suppose* $C_j > 0$ *are real constants and* $N_j \ge 2$ *are integer constants. For every* C^**-algebra* A *and* $I \lhd A$ *an ideal, given* $x_1, \ldots, x_r$ *and* $y_1, \ldots, y_r$ *and* $z_1, z_2, \ldots$ *in* A *with*

$$\pi\left(x_k\right)\pi\left(y_k\right)\pi\left(x_k\right)^* = 0, \quad (k = 1, \ldots, r)$$

and

$$\left\|p_j\left(\pi\left(\mathbf{x}, \mathbf{y}, \mathbf{z}\right)\right)\right\| \le C_j, \quad (j = 1, \ldots, J)$$

there are $\bar{x}_1, \ldots, \bar{x}_r$ *and* $\bar{y}_1, \ldots, \bar{y}_r$ *in* A *with* $\pi\left(\bar{\mathbf{x}}\right) = \pi(\mathbf{x})$ *and* $\pi\left(\bar{\mathbf{y}}\right) = \pi(\mathbf{y})$ *and*

$$\bar{x}_k\bar{y}_k\bar{x}_k^* = 0, \quad (k = 1, \ldots, r)$$

and

$$\left\|p_j\left(\bar{\mathbf{x}}, \bar{\mathbf{y}}, \bar{\mathbf{z}}\right)\right\| \le C_j, \quad (j = 1, \ldots, J).$$

Proof. Without loss of generality, assume $A = M(E)$ and $I = E$ for some separable C^*-algebra E. Now we apply Lemma 3.2 to each pair x_j and y_j and find e_j, f_j and g_j in $M(E)$ so that, given any lifts $\tilde{x}_j$ and $\tilde{y}_j$ of $\pi(x_j)$ and $\pi(y_j)$, setting

$$\bar{x}_j = \tilde{x}_j(1 - g_j)$$

and

$$\bar{y}_j = e_j\tilde{y}_j + (1 - e_j)\tilde{y}_j f_j$$

produces again lifts of the $\pi(x_j)$ and $\pi(y_j)$ with

$$\bar{x}_j\bar{y}_j\bar{x}_j^* = 0.$$

The needed norm conditions

$$\begin{aligned}\big\| p_j\big(&\tilde{x}_1(1-g_1), \ldots, \tilde{x}_r(1-g_r),\\ &e_1\tilde{y}_1 + (1-e_1)\tilde{y}_1 f_1, \ldots, e_r\tilde{y}_r + (1-e_r)\tilde{y}_r f_r, \bar{\mathbf{z}}\big)\big\| \le C_j\end{aligned}$$

involve NC $*$-polynomials that are homogeneous in $\{x_1, \ldots, x_r, y_1, \ldots, y_r\}$, so Theorem 3.2 of [11] again finishes the job. □

Example 3.4. For any r, the C^*-algebra

$$C^*\left\langle x_1, \ldots, x_r, y_1, \ldots, y_r \;\middle|\; \begin{array}{c} x_j y_j x_j^* = 0, \\ \left\|\sum x_j x_j^* + y_j y_j^*\right\| \le 1 \end{array} \right\rangle$$

is projective. In particular, since projective implies residually finite dimensional, if one could show that the $*$-algebra

$$\mathbb{C}\left\langle x_1, \ldots, x_r, y_1, \ldots, y_r \;\middle|\; \begin{array}{c} x_j y_j x_j^* = 0, \\ \left\|\sum x_j x_j^* + y_j y_j^*\right\| \le 1 \end{array} \right\rangle$$

is C^*-representable (as in [15]), then it would have a separating family of finite dimensional representations.

4. The relations $x_j x_k = 0$

We can work with variables that are "half-orthogonal" in that any product $x_j x_k$ is zero. The $*$-monoid here contains only monomials of the forms

$$x_{j_1} x_{j_2}^* \cdots x_{j_{2N-1}} x_{j_{2N}}^*, \; x_{j_1} x_{j_2}^* \cdots x_{j_{2N}}^* x_{j_{2N+1}}$$

and their adjoints.

Lemma 4.1. *Suppose A is σ-unital and $C(A) = M(A)/A$. If $x_1 \ldots, x_r$ are elements of $M(A)$ so that $x_j x_k = 0$ for all j and k then there are elements $0 \le f, g \le 1$ so that*

$$fg = 0$$

and

$$f x_j g = x_j$$

for all j.

Proof. We apply Kasparov's technical theorem to find a and b with

$$0 \leq a \ll b \leq 1$$

and

$$x_j a = a, \quad b x_j = 0.$$

Let $f = 1 - b$ and $g = a$. □

Lemma 4.2. *Suppose A is σ-unital and consider the quotient map*

$$\pi : M(A) \to M(A)/A.$$

(1) *If $x_1, \ldots, x_r$ are elements of $M(A)$ so that $\pi(x_j x_k) = 0$ for all j and k, then there are elements f and g in $M(A)$ with*

$$0 \leq f, g \leq 1, \tag{4.1}$$

$$fg = 0 \tag{4.2}$$

and

$$\pi(f x_j g) = \pi(x_j).$$

(2) *If $\pi(\tilde{x}_j) = \pi(x_j)$ then, if we set*

$$\bar{x}_j = f \tilde{x}_j g,$$

we have $\pi(\bar{x}_j) = \pi(x_j)$ and

$$\bar{x}_j \bar{x}_k = 0$$

for all f and g.

Proof. The products $\pi(x_j)\pi(x_k)$ are zero, so Lemma 4.1 gives us elements $0 \leq f_0, g_0 \leq 1$ in $C(A)$ with $f_0 g_0 = 0$ and

$$f_0 \pi(x_j) g_0 = \pi(x_j).$$

Orthogonal positive contractions lift to orthogonal positive contractions, so there are f and g in $M(A)$ satisfying (4.1) and (4.2) that are lifts of f_0 and g_0, which means

$$\pi(f x_j g) = f_0 \pi(x_j) g_0 = \pi(x_j).$$

With $\bar{x}_j$ as indicated,

$$\pi(\bar{x}_j) = \pi(f \tilde{x}_j g) = f_0 \pi(x_j) g_0 = \pi(x_j)$$

and

$$\bar{x}_j \bar{x}_k = f \tilde{x}_j g f \tilde{x}_k g = 0.$$

□

Theorem 4.3. *Suppose $p_1, \ldots, p_J$ are NC $*$-polynomials in infinitely many variables that are homogeneous in the set of the first r variables, each with degree of homogeneity d_j at least one. Suppose $C_j > 0$ are real constants. For every C^*-algebra A and $I \lhd A$ an ideal, given $x_1, \ldots, x_r$ and $y_1, y_2, \ldots$ in A with*

$$\pi(x_k) \pi(x_l) = 0, \quad (k, l = 1, \ldots, r)$$

and

$$\|p_j(\pi(\mathbf{x}, \mathbf{y}))\| \leq C_j, \quad (j = 1, \ldots, J)$$

there are $\bar{x}_1, \ldots, \bar{x}_r$ *in* A *with* $\pi(\bar{\mathbf{x}}) = \pi(\mathbf{x})$ *and*

$$\bar{x}_k \bar{x}_l = 0, \quad (k, l = 1, \ldots, r)$$

and

$$\|p_j(\bar{\mathbf{x}}, \bar{\mathbf{y}})\| \leq C_j, \quad (j = 1, \ldots, J).$$

Proof. The proof is essentially the same as that of Theorem 3.3. □

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