New York Journal of Mathematics

New York J. Math. **18** (2012) 609–620.

Sequences of pseudo-Anosov mapping classes and their asymptotic behavior

Aaron D. Valdivia

Abstract. In this paper we provide a construction which produces sequences of pseudo-Anosov mapping classes on surfaces with decreasing Euler characteristic. The construction is based on Penner's examples used in the proof that the minimal dilatation, $\delta_{g,0}$, for a closed surface of genus g behaves asymptotically like $\frac{1}{g}$. We give a bound for the dilatation of the pseudo-Anosov elements of each sequence produced by the construction and use this bound to show that if $g_i = rn_i$ for some rational number $r > 0$ then δ_{g_i,n_i} behaves like $\frac{1}{|\chi(S_{g_i,n_i})|}$ where $\chi(S_{g_i,n_i})$ is the Euler characteristic of the genus g_i surface with n_i punctures.

Contents

1. Introduction

Consider a surface $S_{g,n}$ with genus g and n punctures with negative Euler characteristic. The group of orientation preserving self homeomorphisms of the surface up to isotopy is called the *mapping class group*, $\mathrm{Mod}^+(S_{g,n})$. By the Nielsen–Thurston classification [4] an element of the mapping class group is either periodic, reducible (i.e., there exists a nontrivial invariant family of disjoint curves), or pseudo-Anosov. A pseudo-Anosov element is one which fixes a pair of transverse measured singular foliations

$$\phi((\mathcal{F}^{\pm}, \mu^{\pm})) = (\mathcal{F}^{\pm}, \lambda^{\pm 1}\mu^{\pm})$$

up to scaling the measures $\mu^{\pm}$ by a constants $\lambda^{\pm 1}$ where $\lambda > 1$. The number λ is called the *dilatation* of ϕ. For fixed g and n the set of dilatations of pseudo-Anosov mapping classes in $\mathrm{Mod}^+(S_{g,n})$ is discrete and closed [1] [7]. Therefore for fixed g and n the minimum, $\delta_{g,n}$, is achieved by some mapping

Received February 15, 2012.
2010 *Mathematics Subject Classification.* 57M50.
Key words and phrases. Mapping class group, pseudo-Anosov, minimal dilatation.

ISSN 1076-9803/2012

class $\phi \in \mathrm{Mod}^+(S_{g,n})$. One open question about the spectrum of dilatations is the following one.

Question 1.1. Given a surface of negative Euler characteristic, $S_{g,n}$, what is the value of $\delta_{g,n}$?

This question is only answered for a handful of cases of small g and n, see [11], [5], [2]. More is known about the asymptotic behavior of these numbers. Penner proved a lower bound for minimal dilatations and explored the asymptotic behavior for closed surfaces.

Theorem 1.2 ([9])**.** *Given a surface $S_{g,n}$ with negative Euler characteristic the minimal dilatation, $\delta_{g,n}$, satisfies the following inequality:*

$$\log(\delta_{g,n}) \geq \frac{\log(2)}{12g-12+4n}. \tag{1}$$

Theorem 1.3 ([9])**.** *Consider the closed surfaces of negative Euler characteristic. Then the minimal dilatations, $\delta_{g,0}$, statisfy the following:*

$$\log(\delta_{g,0}) \asymp \frac{1}{g}. \tag{2}$$

By the notation $A \asymp B$ we mean there exists $C \in \mathbb{R}^+$ such that $\frac{B}{C} \leq A \leq BC$.

Tsai [12] (cf. [6]) continued this investigation for punctured surfaces and showed that for $g = 0$ or $g = 1$ and n even the behavior is

$$\log \delta_{g,n} \asymp \frac{1}{n}.$$

However for surfaces of fixed genus $g > 1$ the minimal dilatations behave like

$$\log(\delta_{g,n}) \asymp \frac{\log(n)}{n} \quad \text{where} \quad n \to \infty.$$

This leads to the following question.

Question 1.4 ([12])**.** What is the behavior for the minimal dilatations for different sequences of (g, n)?

In this paper we provide a partial answer to Tsai's question.

Theorem 1.5. *Given any rational number r the asymptotic behavior of $\delta_{g,n}$ along the ray defined by $g = rn$ is*

$$\log(\delta_{g,n}) \asymp \frac{1}{|\chi(S_{g,n})|},$$

where $\chi(S_{g,n})$ is the Euler characteristic of the surface $S_{g,n}$.

The proof follows Penner's proof of Theorem 1.3. Penner proves a general lower bound in Theorem 1.2 and defines a sequence of pseudo-Anosov mapping classes $\phi_g : S_{g,0} \to S_{g,0}$ such that $\lambda((\phi_g)^g)$ is bounded by some constant. We use Penner's lower bound and generalize his examples. This generalization allows us to construct sequences with the logarithm of the dilatation bounded by some constant multiple of $\frac{1}{|\chi(S_{g,n})|}$. Making certain choices we find examples giving the upper bound for Theorem 1.5.

In Section 2 we recall some known results about pseudo-Anosov mapping classes and train tracks and some techniques of Penner's used in providing the upper bound for closed surfaces. In Section 3 we define generalized Penner sequences and begin to prove that such a sequence ϕ_m has $\log(\lambda(\phi_m)) \asymp \frac{1}{|\chi(S_{g,n})|}$. In Section 4 we apply our construction and its behavior to the proof of Theorem 1.5.

Acknowledgments. I would like to thank Eriko Hironaka for suggesting the problem solved in this paper and for many helpful conversations. I would like to thank Beson Farb for reading an earlier version and for his helpful comments. I would also like to thank the referee for a number of suggestions that greatly improved the readability of the paper.

2. Background

In this section we recall some facts about pseudo-Anosov mapping classes and train tracks that will be used later in this paper.

Dehn showed in [3] that the mapping class group is generated by finitely many Dehn twists d_x where x is a simple closed curve in the surface. The following theorem of Penner's gives a partial answer to the question of which compositions of Dehn twists define pseudo-Ansov mapping classes.

Theorem 2.1 (Penner's Semigroup Criterion [8])**.** *Suppose C and D are each collections of disjointly embedded simple closed curves in an oriented surface S. Suppose C interesects D minimally and $C \cup D$ fills S (i.e., the connected components of $S \backslash C \cup D$ have nonnegative Euler characteristic). Let $R(C^+, D^-)$ be the semigroup generated by $\{d_c \mid c \in C\} \cup \{d_d^{-1} \mid d \in D\}$. If $\omega \in R(C^+, D^-)$, then ω is pseudo-Anosov if each d_c and d_d^{-1} appears in ω.*

A *train track* is graph embedded in a surface with a definable tangent direction at the vertices. An example can be found in Figure 3. Every pseudo-Anosov mapping class $\phi : S_{g,n} \to S_{g,n}$ has an *invariant train track* $\tau \subset S_{g,n}$. Invariant here means we pick a representative of ϕ which we will also call ϕ. Then $\phi(\tau)$ is *carried* by τ, or there exists a homotopy $H_t(x) : S_{g,n} \to S_{g,n}$ such that $H_0(x) = \text{id}$, $H_1(\phi(\tau)) \subset \tau$, $H_1(x)|_{\phi(\tau)}$ is C^1, and the restriction of the differential $dH_1(p)$ is nonzero for every point $p \in \phi(\tau)$. This homotopy defines a transition matrix $T[a_{i,j}]$ on the edges of τ. The entry $a_{i,j}$ is the incidence of the edge i with the edge j after applying ϕ followed by the homotopy. To be more precise let p be a point on edge j of τ we have $a_{i,j} = |H_1^{-1}(p) \cap \phi(e_i)|$.

Definition 2.2. An *admissible measure* is a set of real non negative numbers, or *weights*, assigned to each edge of a train track τ with the sum of weights for the incoming edges equal to the sum of weights for the outgoing edges at each vertex. Incoming and outgoing are defined by an arbitrary choice of tangent direction at each vertex.

The matrix $T[a_{i,j}]$ defines a linear action on the vector space of admissable measures of τ. Each admissable measure of τ defines a measured singular foliation of $S_{g,n}$. The real positive eigenvector of T is the invariant foliation and the eigenvalue is the dilatation. For a further discussion of train tracks see [10] [8].

Recall that a nonnegative matrix M such that M^n is positive for some $n > 0$ is said to be Perron–Frobenius. Such a matrix has a unique positive real eigenvector with real eigenvalue which is the spectral radius of the matrix. We will use the following lemma to bound the spectral radii of these matrices. The result is well known, and we include the proof for the convenience of the reader.

Lemma 2.3. *The spectral radius of a Perron–Frobenius matrix is bounded by the largest column sum.*

Proof. Let M be a Perron–Frobenius matrix with spectral radius λ and corresponding left eigenvector v with norm 1. This eigenvector is real and positive.

$$|vM| = \lambda = \frac{\sum_{i=1}^{n} v_i m_{ij}}{v_j} \text{ for all } j = 1, \ldots, n.$$

Let v_j be the largest component of v. Then we have:

$$|vM| = \lambda = \sum_{i=1}^{n} \frac{v_i}{v_j} m_{ij}.$$

But each term $\frac{v_i}{v_j} \leq 1$ and each term $m_{ij} \geq 0$ and so we have the inequality:

$$\lambda \leq \sum_{i=1}^{n} m_{ij}.$$

Thus λ is bounded by the largest column sum. □

We will construct pseudo-Anosov maps and corresponding Perron–Frobenius matrices.

As we have already mentioned a pseudo-Anosov mapping class defines a pair of transverse measured singular foliations $(\mathcal{F}^{\pm}, \mu^{\pm})$. The following lemma tells us when a pseudo-Anosov mapping class extends under the forgetful map to another with the same dilatation, see [6].

Lemma 2.4. *If ϕ is a pseudo-Anosov mapping class on the surface $S_{g,n}$ with some subset I of the punctures fixed setwise, if none of the points in I are 1-pronged then the punctures in I may be filled in and the induced mapping class $\tilde{\phi} : S_{g,n-|I|} \to S_{g,n-|I|}$ is pseudo-Anosov with $\lambda(\phi) = \lambda(\tilde{\phi})$.*

An m-*gon* is a possibly punctured disc in $S_{g,n}\backslash\tau$ with m cusps on its boundary. We will be able to apply Lemma 2.4 with the use of the following Lemma, see [10].

Lemma 2.5. *Let ϕ be a pseudo-Anosov mapping class and τ an invariant train track. Then we have the following:*

(1) *The singularities and punctures of S defined by the stable foliation of ϕ are in one-to-one correspondence with the the m-gons of τ.*
(2) *A singularity or puncture is m-pronged if and only if it is contained in an m-gon of τ.*

We will use this information in order to construct examples, find transition matrices that bound their dilatation and lastly extend these examples to ones suitable for the gn-rays we are interested in.

3. Penner sequences

In this section we will define Penner sequences and give the stepping stones to prove the following theorem.

Theorem 3.1. *Given a Penner sequence of mapping classes $\phi_m : F_m \to F_m$,*

$$\log(\lambda(\phi_m)) \asymp \frac{1}{|\chi(F_m)|}$$

where $\chi(F_m)$ is the Euler characteristic of the surface F_m.

First we contstruct the surfaces the mapping classes are defined on. Consider an oriented surface $S_{g,n,b}$ with genus g, n punctures, and b boundary components, and with two sets of disjoint arcs on the boundary components a^- and a^+ such that

$$a^- \cap a^+ = \emptyset$$

and an orientation reversing homeomorphism

$$\iota : a^+ \to a^-.$$

Let Σ_i be homeomorphic copies of $S_{g,n,b}$ and let $h_i : S_{g,n,b} \to \Sigma_i$ be identity homeomorphism for each $i \in \mathbb{Z}$. Set

$$F_\infty = \bigcup_{i\in\mathbb{Z}} \Sigma_i/\sim$$

where $y_i \sim y_j$ if, for some $x \in a^+$ and $k \in \mathbb{Z}$,

$$(y_i, y_j) = (h_k(x), h_{k+1}(\iota(x))).$$

Given the map

$$\rho = h_{i+1}h_i^{-1}$$

the group $\langle\rho\rangle$ acts properly discontinuously on F_∞. Then we define

$$F_m = F_\infty/\rho^m.$$

The map $\pi_m : F_\infty \to F_m$ denotes the projection defined above. Further we define the map $\rho_m : F_m \to F_m$ to be the map induced by ρ such that $\rho_m \pi_m = \pi_m \rho$.

We then pick two sets of multicurves C and D on Σ_1 satisfying Theorem 2.1. A *connecting curve* is a curve, γ, on F_∞ such that $\gamma \subset \Sigma_1 \cup \Sigma_2$, $C \cup \rho(C) \cup \gamma$ is a multicurve, γ intersects $D \cup \rho(D)$ minimally, and the set of curves

$$J = \{\rho^i(C \cup D \cup \gamma)\}_{-\infty}^{\infty}$$

fills the surface F_∞.

Definition 3.2. A sequence of mapping classes $\phi_m : F_m \to F_m$ is called a Penner sequence if for some (C, D) as in Theorem 2.1, some curve γ as above, and some fixed word $\omega \in R(\pi_m(C)^+, \pi_m(D)^-)$ which is pseudo-Anosov on $\pi_m(\Sigma_1)$ we have

$$\phi_m = \rho_m d_{\pi_m(\gamma)} \omega.$$

Next we want to show that these mapping classes are pseudo-Anosov. We start with the following lemma about train tracks. Here we allow our train tracks to have bigons.

Lemma 3.3. *Given a Penner sequence ϕ_m there exists an invariant train track on each surface F_m such that the curves in $\pi_m(C)$ and $\pi_m(D)$, and the curve $\pi_m(\gamma)$ are carried on the train track.*

Proof. We construct a train track on the surface F_∞ and then project it to F_m. Consider the set of curves J in the definition of F_∞. Then assign positive orientation to all curves

$$\{\rho^i(C \cup \gamma)\}_{-\infty}^{\infty}$$

and negative orientation to all curves

$$\{\rho^i(D)\}_{-\infty}^{\infty}.$$

We then smooth the intersections according to Figure 1 where + denotes a positively oriented curve and − denoted a negatively oriented curve.

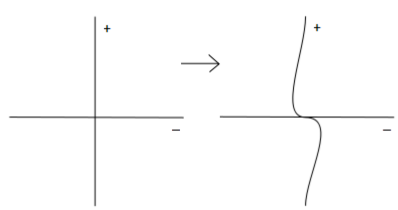

FIGURE 1. Smoothing

This provides a train track τ which may be projected to a track τ_m on F_m. It is easy to see that all the curves in J are carried by τ and thus the projections of these curves are carried by τ_m. One can check that the image of the train track under any single positive Dehn twist about an element of $\pi_m(C \cup \gamma)$ or a negative Dehn twist about any element of $\pi_m(D)$ leaves τ_m invariant. Lastly the map ρ_m permutes the edges of τ_m and so τ_m is invariant with respect to ϕ_m. $\square$

The next lemma allows us to compute the entries of a transition matrix on this train track. First we establish some notation. Let the set of curves $\pi_m(C) = \{\Gamma_1, \Gamma_2 \dots, \Gamma_r\}$ and $\pi_m(D) = \{\Delta_1, \Delta_2 \dots, \Delta_s\}$ now set

$$N = r + s + 1 = |\pi_m(C) \cup \pi_m(\gamma)| + |\pi_m(D)|.$$

Each curve of $\pi_m(C) \cup \pi_m(D) \cup \pi_m(\gamma)$ is carried by the train track and so induces an admissable measure on the train track. The weight for any given edge is the cardnality of the carrying map's preimage over a point on the edge. We think of admissable measures as vectors where the entries are weights on the edges. Let the vectors $\mu_{nN+1}, \dots, \mu_{(n+1)N}$ correspond to the admissable measures defined by the curves $\rho_m^n(\Gamma_1), \dots, \rho_m^n(\Gamma_r)$, $\rho_m^n(\Delta_1), \dots, \rho_m^n(\Delta_s), \rho_m^n(\gamma)$ respectively where $n = 0, \dots, m-1$.

Lemma 3.4. *Consider admissable measures μ_x and μ_y as above and $d_x^\star$ is the map on admissable measures induced by d_x then*

$$(d_x^\star)^{s(x)}(\mu_y) = \mu_y + i(x,y)\mu_x.$$

Here $d_x(y)$ is understood to be the Dehn twist corresponding to the curve defining μ_x twisting the curve defining μ_y. Also

$$s(x) = \begin{cases} 1 & \textit{if } x \in \rho_m^n(\pi_m(C \cup \gamma)) \textit{ for some } n, \\ -1 & \textit{otherwise.} \end{cases}$$

Proof. We assume that the Dehn twists are performed in a neighbohood about τ_m. Then if the intersection number of these curves is k, $d_x(y)^{s(x)}$ will have one strand parallel to the curve defining μ_y and k strands parallel to the curve defining μ_x away from the vertices of τ_m. The sign $s(x)$ insures that near the vertices of τ_m the curve $d_x(y)^{s(x)}$ moves according to the smoothing in Lemma 3.3. We then see the curve is carried by the train track τ_m and the measure induced gives the desired result. $\square$

There is a vector space corresponding to all real weights on the edges of the train track with standard basis $\{e_i\}$ for each edge e_i of τ_m. The admissable measures we have defined are each a sum of these standard basis vectors. By construction they partition the standard basis and so are a linearly independent set of vectors. They form an ordered basis $\{\mu_1, \dots, \mu_{mN}\}$ for a subspace of the weight space and every nonnegative vector of this subspace is an admissable measure. We will continue to use this basis throughout the rest of the paper.

The above lemma also shows us that the transition matrix for an appropriately signed Dehn twist on this subspace can be defined by the intersection numbers of the curves in $J_m = \{\rho_m^i(\pi_m(C \cup D \cup \gamma))\}_{i=1}^m$. Explicitly a Dehn twist about the curve defining the kth basis element has transition matrix $T[a_{i,j}]$ with $a_{i,j} = 1$ if $i = j$, $a_{k,j} = i(k,j)$ and 0 otherwise.

Lemma 3.5. *The matrix T_{ϕ_m} is Perron–Frobenius for each $m \geq 2$.*

Proof. We consider the mapping class

$$\phi_m^m = (\rho_m d_{\pi_m(\gamma)}\omega)(\rho_m d_{\pi_m(\gamma)}\omega)\dots(\rho_m d_{\pi_m(\gamma)}\omega).$$

With the observation that ρ_m^m is the identity map on F_m one can see that this map is a composition of Dehn twists with appropriate sign about all the curves in J_m. Since J_m is connected and is a union of mN curves using Lemma 3.4 we see that if v is a nonnegative vector and not the zero vector then the nonzero entries of $T_{\phi_m}^m(v)$ cannot decrease in value. Furthermore if v has components with zeros, the number of zero components of $T_{\phi_m}^m(v)$ must be less than the number of zero components of v. Since v has length mN, $(T_{\phi_m}^m)^{mN}(v)$ is strictly positive. In particular $T_{\phi_m}^{m^2N}(\mu_x)$ is strictly positive for each standard basis vector μ_x, $x = 1,\dots,mN$. Therefore $T_{\phi_m}^{m^2N}$ is strictly positive. □

Remark. An admissable measure on a bigon track defines a measured foliation up to an equivalence of the admissable measures [10]. We only consider a subset of the admissable measures when computing transition matrices. Since we have shown that the transition matrix on these measures is Perron–Frobenius the Perron–Frobenius eigenvector is positive and defines an invariant foliation for the mapping class. We will prove that these mapping classes are pseudo-Anosov in the next section, therefore the invariant expanding foliation is unique and we need not worry about the equivalent measures or measures outside the considered subset.

4. Asymptotic behavior

Proof of Theorem 3.1. Recalling the observation that ϕ_m^m is a composition of Dehn twists about all the curves in J_m, with appropriate sign, we see that the mapping classes $(\phi_m)^m$ are pseudo-Anosov by Penner's semigroup criteria. If ϕ_m^m is psuedo-Anosov then the mapping class ϕ_m is as well. Lemma 3.3 gives an invariant bigon train track.

As stated in Theorem 2.3 the spectral radius of a Perron–Frobenius matrix is bounded by the largest column sum of the matrix. So now we would like to compute the matrix defining the action on the transverse measures. This matrix will be Perron–Frobenius by Lemma 3.5 and can be computed using Lemma 3.4.

The map ρ permutes the curves of J_m and so the induced map on the space of weights spaned by $\mu_1,\dots,\mu_{mN}$, is defined by an $m \times m$ block permutation matrix

$$M_{\rho_m} = \begin{pmatrix} 0 & 0 & . & 0 & I \\ I & 0 & . & 0 & 0 \\ 0 & I & . & 0 & 0 \\ . & . & . & . & . \\ 0 & 0 & . & I & 0 \end{pmatrix}$$

where each block is $N \times N$, all block matrices will have the same dimensions.

The Dehn twist about the connecting curve gives the map with transition matrix defined by

$$M_{d_\gamma} = \begin{pmatrix} U & V & 0 & . & 0 \\ 0 & I & 0 & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & I \end{pmatrix}$$

since γ intersects only elements of $\pi_m(D) \cup \rho_m(\pi_m(D))$.

Lastly the transition matrix for the map induced by the word ω is given by

$$M_\omega = \begin{pmatrix} W & 0 & 0 & . & X \\ 0 & I & 0 & . & 0 \\ 0 & 0 & I & . & 0 \\ . & . & . & . & . \\ 0 & 0 & 0 & . & I \end{pmatrix},$$

since the curves we perform Dehn twists about may only intersect the curves $\pi_m(C \cup D \cup \gamma \cup \rho_m^{-1}(\gamma))$.

Then the matrix for the map ϕ_m is given below by matrix multiplication after making the identifications $UW = Y$ and $UX = Z$:

$$M_{\phi_m} = \begin{pmatrix} 0 & 0 & 0 & 0 & . & 0 & I \\ Y & V & 0 & 0 & . & 0 & Z \\ 0 & I & 0 & 0 & . & 0 & 0 \\ 0 & 0 & I & 0 & . & 0 & 0 \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & . & I & 0 \end{pmatrix}.$$

The matrix Y depends on the word ω. The matrix V may have nonzero entries only in the last row except that the last column must be zero since $\gamma \cap \rho(\gamma) = \emptyset$. The observation $V^2 = 0$ will be important later.

$$V = \begin{pmatrix} 0 & . & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & . & . & . & 0 \\ v_1 & v_2 & . & v_{r+s} & 0 \end{pmatrix}.$$

The matrix Z will depend on ω as well but only has nonzero entries in the last column. Now we want to consider the matrix $M^m_{\phi_m}$. Inductively we see that for $1 < k < m-1$ the matrix $M^k_{\phi_m}$ has the form given by the following matrix with the first column starting with $k-1$ zero entries and the last $k-1$ entries of the first row equal to zero. Depicted is $M^5_{\phi_m}$. Here we use the fact that V^2 is the zero matrix.

$$M^k_{\phi_m} = \begin{pmatrix} 0 & 0 & 0 & . & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 & Z & Y+VZ & VY & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 & 0 & Z & Y+VZ & VY & 0 \\ 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & Z & Y+VZ & VY \\ VY & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 & Z & Y+VZ \\ Y & V & 0 & . & 0 & 0 & 0 & 0 & 0 & 0 & Z \\ 0 & I & 0 & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & . & 0 & I & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we can find the transition matrix for the mth iterate:

$$M^m_{\phi_m} = \begin{pmatrix} Y & V & 0 & 0 & 0 & . & 0 & 0 & Z \\ ZY & Y+VZ+ZV & VY & 0 & 0 & . & 0 & 0 & Z^2 \\ 0 & Z & Y+VZ & VY & 0 & . & 0 & 0 & 0 \\ 0 & 0 & Z & Y+VZ & VY & . & 0 & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & Z & Y+VZ & VY \\ VY & 0 & 0 & 0 & 0 & . & 0 & Z & Y+VZ \end{pmatrix}.$$

Then this matrix is Perron–Frobenius by Lemma 3.5 and by Theorem 2.3 the spectral radius is bounded by the largest column sum. A block column sum is either equal to a column sum of $Y+ZY+VY$, $V+Y+VZ+ZV+Z$, $VY+Y+VZ+Z$, or $Z+Z^2+VY+Y+VZ$. Therefore the dilatation of the mth iterate is bounded by a constant, say P. This tells us that

$$\log(\lambda(\phi_m)) \le \frac{P}{m}.$$

Then Theorem 1.2 with the upper bound just given finishes the proof. □

Next we use this to prove Theorem 1.5.

Proof of Theorem 1.5. With Penner's lower bound we only need the upper bound to prove the asymptotic behavior. Suppose a ray has slope $\frac{p}{q}$ with $\gcd(p,q)=1$ then let $S_{g,n,1}$ have $(n,g)=(q,p)$. Create any Penner sequence with a^+ being an arc on the boundary component and a^- another disjoint arc on the boundary component such that the complement of the two arcs in the boundary component is two points. This sequence of mapping classes then has the required upper bound on dilatation. This gives sequences on gn-rays through $(n,g)=(2,0)$. Further if we choose our curves

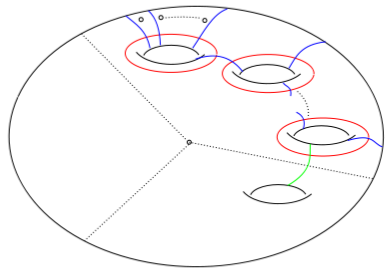

FIGURE 2. Choice for curves

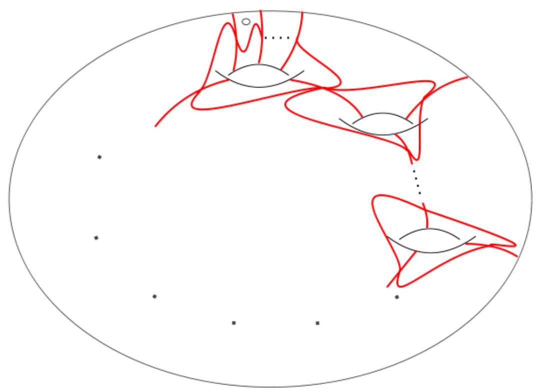

FIGURE 3. Train track

as in Figure 2, which is shown with a chosen connecting curve as well, then an invariant train track for ϕ_m is given in Figure 3.

From this we can see by Theorem 2.5 that the two fixed punctures are not 1-pronged. Filling in both fixed punctures we obtain sequences of mapping classes with the same dilatation and two fewer punctures, the sequences for the lines passing through the origin. □

References

[1] ARNOUX, PIERRE; YOCCOZ, JEAN-CHRISTOPHE. Construction de difféomorphismes pseudo-Anosov. *C. R. Acad. Sci. Paris Sér. I Math.* **292** (1981), no. 1, 75–78. MR0610152 (82b:57018), Zbl 0478.58023.

[2] CHO, JIN-HWAN; HAM, JI-YOUNG. The minimal dilatation of a genus-two surface. *Experiment. Math.* **17** (2008), no. 3, 257–267. MR2455699 (2009i:37096), Zbl 1153.37375, doi: 10.1080/10586458.2008.10129045.

[3] DEHN, M. Die gruppe der abbildungsklassen. Das arithmetische Feld auf Flächen. *Acta Math.* **69** (1938), no. 1, 135–206. MR1555438, Zbl 0019.25301, doi: 10.1007/BF02547712.

[4] FATHI, ALBERT; LAUDENBACH, FRANÇOIS; POÉNARU, VALENTIN. Thurston's work on surfaces. Mathematical Notes, 48. *Princeton University Press, Princeton, NJ*, 2012. Zbl 06023011.

[5] Ham, Ji-Young; Song, Won Taek. The minimum dilatation of pseudo-Anosov 5-braids. *Experiment. Math.*, **16** (2007), no. 2, 167–179. MR2339273 (2008e:37043), Zbl 1151.37037, arXiv:math/0506295, doi: 10.1080/10586458.2007.10129000.

[6] Hironaka, Eriko; Kin, Eiko. A family of pseudo-Anosov braids with small dilatation. *Algebr. Geom. Topol.* **6** (2006), 699–738. MR2240913 (2008h:57027), Zbl 1126.37014, arXiv:0904.0594, doi: 10.2140/agt.2006.6.699.

[7] Ivanov, N. V. Stretching factors of pseudo-Anosov homeomorphisms. *Journal of Mathematical Sciences*, **52**, (1990), no. 1, 2819–2822. MR0964259 (89i:32047), Zbl 0707.57007, doi: 10.1007/BF01099245.

[8] Penner, Robert C. A construction of pseudo-Anosov homeomorphisms. *Trans. Amer. Math. Soc.* **310** (1988), no. 1, 179–197. MR0930079 (89k:57026), Zbl 0706.57008, doi: 10.1090/S0002-9947-1988-0930079-9.

[9] Penner, R. C. Bounds on least dilatations. *Proc. Amer. Math. Soc.* **113** (1991), no. 2, 443–450. MR1068128 (91m:57010), Zbl 0726.57013, doi: 10.1090/S0002-9939-1991-1068128-8.

[10] Penner, R. C.; Harer, J. L. Combinatorics of train tracks. Annals of Mathematics Studies, 125. *Princeton University Press, Princeton, NJ*, 1992. xii+216 pp. ISBN: 0-691-08764-4; 0-691-02531-2. MR1144770 (94b:57018), Zbl 0765.57001.

[11] Song, Won Taek; Ko, Ki Hyoung; Los, Jérôme E. Entropies of braids. *J. Knot Theory Ramifications* **11** (2002), no. 4, 647–666. MR1915500 (2003e:57013), Zbl 1010.57004, doi: 10.1142/S021821650200186X.

[12] Tsai, Chia-Yen. The asymptotic behavior of least pseudo-Anosov dilatations. *Geom. Topol.* **13** (2009), no. 4, 2253–2278. MR2507119 (2010d:37081), Zbl 1204.37043, arXiv:0810.0261, doi: 10.2140/gt.2009.13.2253.

Florida Southern College, 111 Lake Hollingsworth Drive, Lakeland, FL 33801-5698

aaron.david.valdivia@gmail.com

This paper is available via http://nyjm.albany.edu/j/2012/18-32.html.