Essential normality and the decomposability of algebraic varieties

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ABSTRACT. We consider the Arveson–Douglas conjecture on the essential normality of homogeneous submodules corresponding to algebraic subvarieties of the unit ball. We prove that the property of essential normality is preserved by isomorphisms between varieties, and we establish a similar result for maps between varieties that are not necessarily invertible. We also relate the decomposability of an algebraic variety to the problem of establishing the essential normality of the corresponding submodule. These results are applied to prove that the Arveson–Douglas conjecture holds for submodules corresponding to varieties that decompose into linear subspaces, and varieties that decompose into components with mutually disjoint linear spans.

CONTENTS

1. Introduction

In this paper, we consider a conjecture of Douglas and Arveson that implies a correspondence between algebraic varieties and C*-algebras of essentially normal operators. In the papers [Sha11] and [Ken12], we showed that this conjecture can be viewed as a problem of finding certain nice decompositions of submodules of $\mathbb{C}[z_1, \ldots, z_d]$. In the present paper, we take a slightly different perspective, and relate the conjecture to the geometry of the variety in question.

Let d be a fixed positive integer, and let $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_d]$ denote the algebra of complex polynomials in d variables. The Drury–Arveson space H_d^2 is the reproducing kernel Hilbert space on the unit ball $\mathbb{B}_d$ generated by the kernel functions

$$k_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle}, \quad \lambda \in \mathbb{B}_d.$$

Equivalently, H_d^2 is the completion of $\mathbb{C}[z]$ with respect to the inner product

$$\langle z^\alpha, z^\beta \rangle = \delta_{\alpha,\beta} \frac{\alpha_1! \cdots \alpha_d!}{(\alpha_1 + \cdots + \alpha_d)!}, \quad \alpha, \beta \in \mathbb{N}_0^d,$$

where we have used the notation $z^\alpha = z_1^{\alpha_1} \cdots z_d^{\alpha_d}$ for $\alpha = (\alpha_1, \ldots, \alpha_d)$ in $\mathbb{N}_0^d$.

The d-shift $S = (S_1, \ldots, S_d)$ is the d-tuple of multiplication operators on H_d^2 corresponding to the coordinate functions $z_1, \ldots, z_d$. They act by

$$(S_i f)(z) = z_i f(z), \quad f \in H_d^2.$$

We will be particularly interested in these operators, which were introduced and extensively studied in [Arv98]. Together with the d-shift S, the space H_d^2 forms a Hilbert module over $\mathbb{C}[z]$, with the module action given by

$$pf = p(S_1, \ldots, S_d)f, \quad p \in \mathbb{C}[z], \ f \in H_d^2.$$

Endowed with this module structure, H_d^2 is known as the *d-shift Hilbert module*.

For an ideal I of $\mathbb{C}[z]$, we define

$$\mathcal{F}_I = H_d^2 \ominus I.$$

Note that since the closure of I in H_d^2 is an invariant subspace for each S_j, the space $\mathcal{F}_I$ is coinvariant for each S_j. We let S_j^I denote the compression of S_j to $\mathcal{F}_I$, i.e.,

$$S_j^I = P_{\mathcal{F}_I} S_j \mid_{\mathcal{F}_I}.$$

Then as a Hilbert module, $\mathcal{F}_I$ is equivalent to the quotient of H_d^2 by the closure of I in H_d^2.

We will require the following correspondence between ideals of $\mathbb{C}[z]$ and subsets of the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$. For an ideal I of $\mathbb{C}[z]$, we define

$$V(I) = \{z \in \mathbb{B}_d \mid p(z) = 0 \ \forall p \in I\},$$

and for a subset V of $\mathbb{B}_d$, we define

$$I(V) = \{p \in \mathbb{C}[z] \mid p(z) = 0 \ \forall z \in V\}.$$

For a homogeneous ideal I we shall call the set $V(I)$ a *homogeneous variety in* $\mathbb{B}_d$. All the varieties in this paper will be homogeneous varieties in $\mathbb{B}_d$.

If the ideal I is radical, then the space $\mathcal{F}_I$ is a reproducing kernel Hilbert space over $V(I)$. More generally, it was established in [DRS11, Lemma 5.5] that in this case we have the equality

$$\mathcal{F}_I = \overline{\mathrm{span}}\{k_\lambda \mid \lambda \in V(I)\}.$$

Arveson's conjecture is that for every homogeneous ideal I of $\mathbb{C}[z]$, the quotient operators $S_1^I, \ldots, S_d^I$ satisfy

$$(1.1) \qquad [S_i^I, S_j^{I*}] := S_i^I S_j^{I*} - S_j^{I*} S_i^I \in \mathcal{L}^p, \quad p > d, \ 1 \le i, j \le d,$$

where for $1 \le p \le \infty$, $\mathcal{L}^p$ denotes the set of Schatten p-class operators on H_d^2. The general version of Arveson's conjecture includes multiplicity, but we are not worrying about that for now, and in fact, by [Sha11, Section 5], the full conjecture is equivalent to the scalar case (up to a small modification of the range of p).

Douglas conjectured further that (1.1) should hold for all $p > \dim I$. Note that $\dim I$ is defined in the following way. It is known that there is a polynomial $h_I(x)$, called the *Hilbert polynomial*, such that for sufficiently large n, the dimension of $\mathbb{H}_n \ominus I_n$ is equal to $h_I(n)$. The dimension $\dim I$ is defined to be $\deg(h_I(x)) + 1$ (see, e.g., [CLS92, Chapter 9]). If V is the affine variety determined by I then $\dim I = \dim V$. For example, when the variety V is a union of subspaces this is just the maximal dimension of the subspaces.

In this note we will be concerned with the Arveson–Douglas conjecture for radical homogeneous ideals. To express our ideas in the clearest way, we are led to introduce the following notation. If X is a subspace of $\mathbb{C}^d$, then we write X^n for the n-th symmetric tensor power of X with itself. If $V \subseteq X$ is a homogeneous variety in the ball, i.e., if V is of the form $V = V(I)$, for some radical homogeneous ideal I of $\mathbb{C}[z]$, then we define V^n to be the subspace of X^n spanned by elements of the form

$$\lambda^n = \underbrace{\lambda \otimes \cdots \otimes \lambda}_{n \text{ times}}, \quad \lambda \in V.$$

Thus, if $V = V_1 \cup \cdots \cup V_k$ is a union of varieties, then we have that

$$V^n = \sum_{i=1}^k V_i^n.$$

Using the natural identification of $\mathbb{C}[z]$ with symmetric Fock space gives the decomposition

$$\mathcal{F}_I = \oplus_{n=0}^\infty V^n.$$

With this identification, the kernel functions k_λ of $\mathcal{F}_I$ are of the form

$$k_\lambda = \sum_{n=0}^\infty \overline{\lambda}^n, \quad \lambda \in V.$$

We remark that (for sufficiently large n) the dimension of V^n is bounded by n^{d-1}, because it is a subspace of $(\mathbb{C}^d)^n$, which has dimension $\frac{(n+d-1)!}{n!(d-1)!}$.

When we consider $\mathcal{F}_I$ as a reproducing kernel Hilbert space over $V(I)$, then the operators S_i^I correspond to multiplication operator M_{f_i} defined by

$$(M_{f_i}g)(z) = (f_i g)(z), \quad g \in \mathcal{F}_I,$$

where $f_i = z_i\big|_{V(I)}$. The algebra $\mathcal{A}_I$ is defined to be the normed closed unital algebra generated by $(S_1^I, \ldots, S_d^I)$. This algebra is a normed closed subalgebra of the multiplier algebra of $\mathcal{F}_I$. If p belongs to $\mathbb{C}[z]$, then it will be convenient to identify $p(S_1^I, \ldots, S_d^I)$ with the multiplication operator M_p.

For $p \geq 1$, we will say that the quotient module $\mathcal{F}_I$ is *p-essentially normal* if

$$[S_i^I, S_j^{I*}] \in \mathcal{L}^p, \quad 1 \leq i, j \leq d.$$

Recall that this is equivalent to $|[S_i^I, S_j^{I*}]|^p$ being trace class for $1 \leq i, j \leq d$.

If $V = V(I)$ and $I = I(V)$, which is the case whenever I is a radical ideal, then we will write $S_1^V, \ldots, S_d^V$ for $S_1^I, \ldots, S_d^I$. Similarly, we will write $\mathcal{F}_V$ for $\mathcal{F}_I$, and $\mathcal{A}_V$ for $\mathcal{A}_I$. Using this notation, we now state for reference the form of the Arveson–Douglas conjecture that we consider in this paper.

Conjecture 1.1 (Geometric Arveson–Douglas Conjecture)**.** *Let V be a homogenous variety in $\mathbb{B}_d$. Then the submodule $\mathcal{F}_V$ is p-essentially normal for every $p > \dim V$.*

Note that the essential normality of $\mathcal{F}_V$ is independent of the ambient space $\mathbb{C}^d$ (and in particular of the dimension d) in which we choose to (isometrically) embed V (see [DRS11, Remark 8.1]).

Conjecture 1.1 originated with Arveson's investigation of the curvature invariant of a commuting d-tuple [Arv00, Arv02]. In the past decade, it has drawn a lot of attention, for example in the papers [Arv05, Arv07, Dou06a, Dou06b, DS11, DW12, Esc11, GW08, Ken12, Sha11], which deal directly with this conjecture. We also wish to mention two recent papers, [DW11] and [FX12], which treat the essential normality of a principal ideal generated by a (not necessarily homogeneous) polynomial. These papers are worth mentioning, not only because the problem they treat is closely related, but also because they introduce promising analytic techniques that are quite different from previous approaches to the general problem of essential normality.

The main result of [DRS11, Section 7.3] is that if V and W are "tractable" homogeneous varieties, and if A is an invertible linear map that maps W onto V that is isometric on W, then the map $f \mapsto f \circ A$ is an isomorphism

between the algebras $\mathcal{A}_V$ and $\mathcal{A}_W$ [DRS11, Theorem 7.17]. Furthermore, it was shown that this isomorphism is implemented by a similarity $\tilde{A}^*$, i.e.,

$$\varphi(M_f) = \tilde{A}^* M_f (\tilde{A}^*)^{-1},$$

where $\tilde{A} : \mathcal{F}_W \to \mathcal{F}_V$ is an invertible bounded linear map satisfying

$$\tilde{A} k_\lambda = k_{A\lambda}.$$

Recently, in [Har12], Hartz was able to prove a stronger version of this result that does not require the varieties to be tractable. We will require this result for what follows.

In this paper, we study the Arveson–Douglas conjecture for submodules of the form $\mathcal{F}_V$, where V is a homogeneous variety in $\mathbb{B}_d$. In Section 2, we prove that if W is a homogeneous variety in $\mathbb{B}_{d'}$, for some positive integer d', and if $\mathcal{A}_V$ is isomorphic to $\mathcal{A}_W$, then $\mathcal{F}_V$ is p-essentially normal if and only if $\mathcal{F}_W$ is p-essentially normal. We also establish a similar result for maps between varieties that are not necessarily isomorphic.

In Section 3, we consider when it is possible to decompose V as $V = V_1 \cup \cdots \cup V_n$, where $V_1, \ldots, V_n$ are homogeneous varieties in $\mathbb{C}^d$ with the property that the algebraic sum $\mathcal{F}_{V_1} + \cdots + \mathcal{F}_{V_n}$ is closed. This is a geometric analogue of the notion of the decomposability of a submodule that was introduced in [Ken12]. We relate this geometric notion of decomposability to the problem of establishing the p-essential normality of the submodule $\mathcal{F}_V$.

Finally, in Section 4, we apply the results from Section 2 and Section 3 to establish the Arveson–Douglas conjecture for two new classes of examples. Using Hartz's result from [Har12], we prove that $\mathcal{F}_V$ satisfies the Arveson–Douglas conjecture when V decomposes as the union of linear subspaces. We also prove that $\mathcal{F}_V$ satisfies the Arveson–Douglas conjecture when V decomposes into varieties $V_1, \ldots, V_n$ such that each $\mathcal{F}_{V_i}$ satisfies the conjecture, and $\mathrm{span}(V_i) \cap \mathrm{span}(V_j) = 0$ whenever $i \neq j$. These are perhaps the simplest classes of examples for which the conjecture was not previously known to be true.

2. Linear maps between varieties and essential normality

2.1. Invertible maps.

Theorem 2.1. *Let V and W be homogeneous varieties in $\mathbb{B}_d$ and $\mathbb{B}_{d'}$ respectively. Suppose the algebras $\mathcal{A}_V$ and $\mathcal{A}_W$ are algebraically isomorphic. Then for $p \geq 1$, $\mathcal{F}_V$ is p-essentially normal if and only if $\mathcal{F}_W$ is p-essentially normal.*

Proof. Since $\mathcal{A}_V$ and $\mathcal{A}_W$ are isomorphic, by results of [DRS11] (Proposition 7.1, Theorem 7.4 and Proposition 8.3) there is a linear transformation $A : \mathbb{C}^d \to \mathbb{C}^{d'}$ that maps V bijectively onto W. It now follows from [Har12]

(Proposition 2.5 and Corollary 5.8) that there is an invertible linear map $\tilde{A} : \mathcal{F}_V \to \mathcal{F}_W$ defined by

$$\tilde{A}k_\lambda = k_{A\lambda}, \quad \lambda \in V.$$

It follows that if f is a polynomial in $\mathcal{F}_V$ then

$$\tilde{A}f = f \circ A^*.$$

The adjoint $\tilde{A}^* : \mathcal{F}_W \to \mathcal{F}_V$ is defined by

$$\tilde{A}^* f = f \circ A, \quad f \in \mathcal{F}_W.$$

Note that for a polynomial f in $\mathcal{F}_V$, $\tilde{A}M_f = M_{f \circ A^*}\tilde{A}$, and similarly for a polynomial f in $\mathcal{F}_W$, $\tilde{A}^* M_f = M_{f \circ A}\tilde{A}^*$.

Fix polynomials f and g in $\mathcal{F}_V$. Then using the identities $M_{g \circ A^*} = \tilde{A}M_g\tilde{A}^{-1}$, $M_{f \circ A^*}^*\tilde{A} = \tilde{A}M_{f \circ A^* A}^*$ and $M_{f \circ A^*}^* = \tilde{A}M_{f \circ A^* A}^*\tilde{A}^{-1}$, we have

$$
\begin{aligned}
M_{f \circ A^*}^* M_{g \circ A^*} &= M_{f \circ A^*}^*\tilde{A}M_g\tilde{A}^{-1} \\
&= \tilde{A}M_{f \circ A^* A}^* M_g\tilde{A}^{-1} \\
&= \tilde{A}M_g M_{f \circ A^* A}^*\tilde{A}^{-1} + \tilde{A}[M_{f \circ A^* A}, M_g]\tilde{A}^{-1} \\
&= \tilde{A}M_g\tilde{A}^{-1}\tilde{A}M_{f \circ A^* A}^*\tilde{A}^{-1} + \tilde{A}[M_{f \circ A^* A}, M_g]\tilde{A}^{-1} \\
&= M_{g \circ A^*} M_{f \circ A^*}^* + \tilde{A}[M_{f \circ A^* A}, M_g]\tilde{A}^{-1}.
\end{aligned}
$$

Therefore,

$$[M_{f \circ A^*}^*, M_{g \circ A^*}] = \tilde{A}[M_{f \circ A^* A}, M_g]\tilde{A}^{-1}.$$

and hence $[M_{f \circ A^*}^*, M_{g \circ A^*}]$ belongs to $\mathcal{L}^p$ if and only if $[M_{f \circ A^* A}, M_g]$ belongs to $\mathcal{L}^p$. Letting f and g be suitable linear combinations of the coordinate functions one sees that $\mathcal{F}_W$ is p-essentially normal if and only if $\mathcal{F}_V$ is p-essentially normal. $\qquad\square$

2.2. Maps that are not necessarily invertible.

Proposition 2.2. *Let V and W be homogeneous varieties in $\mathbb{B}_d$ and $\mathbb{B}_{d'}$, respectively, with decompositions into (not necessarily irreducible) subvarieties $V = V_1 \cup \cdots \cup V_k$ and $W = W_1 \cup \cdots \cup W_k$ with the property that $\mathrm{span}(W_i) \cap \mathrm{span}(W_j) = \{0\}$ whenever $i \neq j$. Suppose that there is a linear map $A : \mathbb{C}^d \to \mathbb{C}^{d'}$ such that $A(V_i) = W_i$ and such that the restriction of A to $\mathrm{span}(V_i)$ is isometric for all i, $1 \leq i \leq k$. Then the map defined by*

$$\tilde{A}k_\lambda = k_{A\lambda}, \quad \lambda \in V$$

extends to a bounded linear map $\tilde{A} : \mathcal{F}_V \to \mathcal{F}_W$. Moreover, $\tilde{A}$ is the sum of a unitary operator and a trace class operator.

Remark. If V_i is irreducible, and if A is *isometric* on V_i, then it follows from [DRS11, Proposition 7.6] that the restriction of A to $\mathrm{span}(V_i)$ is automatically isometric.

Proof. It suffices to prove the lemma for the case when V and W are unions of nontrivial subspaces (see the first paragraph of [DRS11, Theorem 7.16]). Hence we can suppose that $\{0\} \neq V_i = \mathrm{span}(V_i)$ and $\{0\} \neq W_i = \mathrm{span}(W_i)$.

The fact that the operator $\tilde{A}$ is bounded follows from the results in [Har12]. However, in order to prove that $\tilde{A}$ is the sum of a unitary operator and a trace class operator, we will need to obtain quantitative estimates. If M and N are two subspaces of a Hilbert space then we denote (following [Fri37])

$$\cos(M, N)$$
$$= \sup\{|\langle x, y\rangle| : x \in M \ominus (M \cap N), y \in N \ominus (M \cap N), \|x\| = \|y\| = 1\}.$$

By the finite-dimensionality of $V_1, \ldots, V_n$, $\cos(V_i, V_j) < 1$ and $\cos(W_i, W_j) < 1$ whenever $i \neq j$. Let

$$c = \max(\{\cos(V_i, V_j) \mid i \neq j\} \cup \{\cos(W_i, W_j) \mid i \neq j\}).$$

Then $0 \leq c < 1$. For v in V^n and w in W^n, write $v = \sum_{i=1}^{k} v_i$ and $w = \sum_{i=1}^{k} w_i$, where each w_i belongs to W_i^n. Then as in the proof of [DRS11, Lemma 7.10], for sufficiently large n,

$$(2.1) \qquad (1 - kc^n)\|v\|^2 \leq \sum_{i=1}^{k} \|v_i\|^2 \leq (1 + kc^n)\|v\|^2$$

and

$$(2.2) \qquad (1 - kc^n)\|w\|^2 \leq \sum_{i=1}^{k} \|w_i\|^2 \leq (1 + kc^n)\|w\|^2.$$

The space $\mathcal{F}_V$ decomposes as $\mathcal{F}_V = \oplus_{n=0}^{\infty} V^n$, and $\tilde{A}$ is defined on V^n by setting

$$\tilde{A}\lambda^n = (A\lambda)^n, \quad \lambda \in V,$$

and extending by linearity. Since $W^n = W^{\otimes n}$, the operator $\tilde{A}$ can also be realized as

$$\tilde{A} = \oplus_{n=0}^{\infty} A^{\otimes n}.$$

Therefore, by the hypothesis that A is isometric on each V_i, the restriction of $\tilde{A}$ to V_i^n is a unitary from V_i^n to W_i^n.

As above, for v in V^n write $v = \sum_{i=1}^{k} v_i$, where each v_i belongs to V_i^n. Then by (2.1) and (2.2), for sufficiently large n,

$$(2.3) \qquad \|\tilde{A}v\|^2 = \left\| \sum_{i=1}^{k} A^{\otimes n} v_i \right\|^2$$

$$\leq \frac{1}{1 - kc^n} \sum_{i=1}^{k} \|A^{\otimes n} v_i\|^2$$

$$= \frac{1}{1 - kc^n} \sum_{i=1}^{k} \|v_i\|^2$$

$$\leq \frac{1 + kc^n}{1 - kc^n} \|v\|^2.$$

By a similar argument, for sufficiently large n,

(2.4)
$$\|\tilde{A}v\|^2 \geq \frac{1 - kc^n}{1 + kc^n} \|v\|^2.$$

Let $\tilde{A} = U|\tilde{A}|$ be the polar decomposition of $\tilde{A}$. Since $\tilde{A}$ is graded, i.e., $\tilde{A}(V^n) = W^n$, it follows that U and $|\tilde{A}|$ are also graded. Write $\tilde{A} = U + U(|\tilde{A}| - I)$. Since A (and hence $\tilde{A}$) is not necessarily invertible, the partial isometry U is not necessarily a unitary. However, by (2.4), the restriction of $\tilde{A}$ to V^n is invertible for sufficiently large n, so U is a finite rank perturbation of a unitary. Hence we will be done once we show that $|\tilde{A}| - I$ is a trace class operator.

The inequalities (2.3) and (2.4) are equivalent to the existence of a constant $M > 0$ such that for v in V^n,

$$(1 - Mc^n)\|v\| \leq \||\tilde{A}|v\| \leq (1 + Mc^n)\|v\|.$$

Hence the eigenvalues of the restriction of $|\tilde{A}|$ to V^n are contained in the interval $[1 - Mc^n, 1 + Mc^n]$, and it follows that the eigenvalues of the restriction of $|\tilde{A}| - I$ to V^n are contained in the interval $[-Mc^n, Mc^n]$. Therefore, since the dimension of V^n is less than n^{d-1}, it follows that $|\tilde{A}| - I$ is a trace class operator. $\qquad \square$

Theorem 2.3. *Let V and W be homogeneous varieties in $\mathbb{B}_d$ and $\mathbb{B}_{d'}$, respectively, with decompositions into (not necessarily irreducible) subvarieties $V = V_1 \cup \cdots \cup V_k$ and $W = W_1 \cup \cdots \cup W_k$ with the property that $\mathrm{span}(W_i) \cap \mathrm{span}(W_j) = \{0\}$ whenever $i \neq j$. Suppose that there is a linear map $A : \mathbb{C}^d \to \mathbb{C}^{d'}$ such that $A(V_i) = W_i$ and such that the restriction of A to $\mathrm{span}(V_i)$ is isometric for all i, $1 \leq i \leq k$. Then for $p \geq 1$, $\mathcal{F}_W$ is p-essentially normal if and only if $\mathcal{F}_V$ is.*

Proof. Let $\tilde{A} : \mathcal{F}_V \to \mathcal{F}_W$ be as in Proposition 2.2, so that we can write $\tilde{A} = U + T$, where $U : \mathcal{F}_V \to \mathcal{F}_W$ is a unitary operator and $T : \mathcal{F}_V \to \mathcal{F}_W$ is a trace class operator. Then the identity $\tilde{A}M_f = M_{f \circ A^*}\tilde{A}$ implies that for every polynomial f in $\mathcal{F}_V$,

$$M_{f \circ A^*}(U + T) = (U + T)M_f,$$

and hence that

$$M_{f \circ A^*} = UM_fU^* + TM_fU^* - M_{f \circ A}TU^*.$$

Therefore, for polynomials f and g in $\mathcal{F}_V$, we can write

$$[M_{f \circ A^*}^*, M_{g \circ A^*}] = U[M_f^*, M_g]U^* + R,$$

where R is a trace class operator. Letting f and g be coordinate functions, it immediately follows that $\mathcal{F}_V$ is p-essentially normal if $\mathcal{F}_W$ is. To obtain the converse, assume without loss of generality that $\mathbb{C}^{d'} = \mathrm{span}(W)$. Then A is surjective, hence A^* is left invertible. Let B be a left inverse of A^*. Put $f = z_i \circ B$ and $g = z_j \circ B$, where z_i and z_j are considered as coordinate functions in $\mathbb{C}^{d'}$. Then f and g are linear combinations of coordinate function in $\mathbb{C}^d$. Now if $\mathcal{F}_V$ is p-essentially normal then $[M_f^*, M_g] \in \mathcal{L}^p$, whence $[M_{z_i}^*, M_{z_j}] = [M_{f \circ A^*}^*, M_{g \circ A^*}] \in \mathcal{L}^p$. Thus $\mathcal{F}_W$ is p-essentially normal. $\qquad\square$

3. Decompositions of varieties and essential normality

3.1. A refinement of a lemma.

Lemma 3.1. *Let I be a homogeneous ideal of $\mathbb{C}[z]$ and let P denote the projection onto $\mathcal{F}_I$. Then for $p > \dim I$, $\mathcal{F}_I$ is p-essentially normal if and only if the commutator $[S_i, P]$ belongs to $\mathcal{L}^{2p}$ for each $1 \leq i \leq d$.*

Remark 3.2. A slightly weaker form of this conjecture, holding only for $p > d$ instead of $p > \dim I$, is well known (see, e.g., [Arv07, Proposition 4.2]).

Proof. In [Arv98], it is shown that

$$(3.1) \qquad \| [S_i^*, S_j] \mid_{\mathbb{H}_n} \| \leq 2/(n+1).$$

It follows that $\mathrm{trace}(|[S_i^*, S_j]|^p) < \infty$ for all $p > d$, since

$$\mathrm{trace}(|[S_i^*, S_j]|^p) \leq \sum_{n=0}^{\infty} \frac{2^p \dim \mathbb{H}_n}{(n+1)^p},$$

and this is finite for $p > d$, since $\dim \mathbb{H}_n = O(n^{d-1})$.

Write $T_i = S_i^I = PS_iP$, $i = 1, \ldots, d$. Since $\mathcal{F}_I^{\perp}$ is an invariant subspace for the d-shift,

$$[T_i^*, T_j] - P[S_i^*, S_j]P = -PS_i^*(I - P)S_jP = -[P, S_i]^*[P, S_j],$$

which we can rewrite as

$$(3.2) \qquad [T_i^*, T_j] = P[S_i^*, S_j]P - [P, S_i]^*[P, S_j].$$

By (3.1) we know that $\| [S_i^*, S_j] \mid_{\mathbb{H}_n} \| = O(n^{-1})$, so it follows that there is a constant $M > 0$ such that

$$\mathrm{trace}(|P[S_i^*, S_j]P|^p) \leq M \sum_{n=0}^{\infty} \frac{\dim(\mathbb{H}_n \ominus I_n)}{n^p},$$

and this is finite for $p > \dim I$. Therefore, $P[S_i^*, S_j]P$ belongs to $\mathcal{L}^p$ for every $p > \dim I$. Furthermore, for every $p \geq 1$, $[P, S_i]$ belongs to $\mathcal{L}^{2p}$ for all i if and only if $[P, S_i]^*[P, S_i]$ belongs to $\mathcal{L}^p$ for all i, j. The desired result now follows from (3.2). $\qquad\square$

3.2. Decomposability and essential normality.

Lemma 3.3. *Let $M_1, \ldots, M_k$ be subspaces of a Hilbert space $\mathcal{H}$. For $p \geq 1$, suppose that the projections $P_{M_1}, \ldots, P_{M_k}$ each commute modulo $\mathcal{L}^p$ with an operator T in $\mathrm{B}(\mathcal{H})$. If the algebraic sum $M_1 + \cdots + M_k$ is closed, then the projection $P_{M_1 + \cdots + M_k}$ onto the subspace $M_1 + \cdots + M_k$ also commutes modulo $\mathcal{L}^p$ with T.*

Proof. The proof of this result follows the outline of the proof of [Ken12, Theorem 3.3] or [Arv07, Theorem 4.4]. $\qquad\square$

Proposition 3.4. *Let $I_1, \ldots, I_k$ be homogeneous ideals of $\mathbb{C}[z_1, \ldots, z_d]$.*

(1) *If $\mathcal{F}_{I_1}, \ldots, \mathcal{F}_{I_k}$ are p-essentially normal for*
$$p > \max\{\dim I_1, \ldots, \dim I_k\},$$
and the algebraic sum $\mathcal{F}_{I_1}^{\perp} + \cdots + \mathcal{F}_{I_k}^{\perp}$ is closed, then $\mathcal{F}_{I_1 + \cdots + I_k}$ is also p-essentially normal.

(2) *If $\mathcal{F}_{I_1}, \ldots, \mathcal{F}_{I_k}$ are p-essentially normal for $p > \dim I_1 \cap \cdots \cap I_k$, and the algebraic sum $\mathcal{F}_{I_1} + \cdots + \mathcal{F}_{I_k}$ is closed, then $\mathcal{F}_{I_1 \cap \cdots \cap I_k}$ is also p-essentially normal.*

Proof. First, note that the submodule $\mathcal{F}_{I_1 + \cdots + I_k}$ is the orthogonal complement of the algebraic sum $\mathcal{F}_{I_1}^{\perp} + \cdots + \mathcal{F}_{I_k}^{\perp}$, and the submodule $\mathcal{F}_{I_1 \cap \cdots \cap I_k}$ is the closure of the algebraic sum $\mathcal{F}_{I_1} + \cdots + \mathcal{F}_{I_k}$.

If $\mathcal{F}_{I_1}, \ldots, \mathcal{F}_{I_k}$ are p-essentially normal for $p > \max\{\dim I_1, \ldots, \dim I_k\}$, then by Lemma 3.1, each of the commutators $[S_i, P_{\mathcal{F}_{I_j}^{\perp}}]$ belongs to $\mathcal{L}^{2p}$ for $1 \leq i \leq d$ and $1 \leq j \leq k$. If the algebraic sum $\mathcal{F}_{I_1}^{\perp} + \cdots + \mathcal{F}_{I_k}^{\perp}$ is closed, then since
$$\mathcal{F}_{I_1}^{\perp} + \cdots + \mathcal{F}_{I_k}^{\perp} = \mathcal{F}_{I_1 + \cdots + I_k}^{\perp},$$
Lemma 3.3 implies that the commutators $[S_i, P_{\mathcal{F}_{I_1 + \cdots + I_k}^{\perp}}]$ also belong to $\mathcal{L}^{2p}$, and hence that the commutators $[S_i, P_{\mathcal{F}_{I_1 + \cdots + I_k}}]$ belong to $\mathcal{L}^{2p}$ for $1 \leq i \leq d$. Therefore, since $\dim(I_1 + \cdots + I_k) \leq \max\{\dim I_1, \ldots, \dim I_k\}$, it follows from Lemma 3.1 that $\mathcal{F}_{I_1 + \cdots + I_k}$ is also p-essentially normal.

If $\mathcal{F}_{I_1}, \ldots, \mathcal{F}_{I_k}$ are p-essentially normal for $p > \dim I_1 \cap \cdots \cap I_k$, and the algebraic sum $\mathcal{F}_{I_1} + \cdots + \mathcal{F}_{I_k}$ is closed, then the proof that $\mathcal{F}_{I_1 \cap \cdots \cap I_k}$ is also p-essentially normal follows in the same way after noting that
$$\mathcal{F}_{I_1} + \cdots + \mathcal{F}_{I_k} = \mathcal{F}_{I_1 \cap \cdots \cap I_k},$$
and that $\dim(I_1 \cap \cdots \cap I_k) \geq \max\{\dim I_1, \ldots, \dim I_k\}$. $\qquad\square$

Proposition 3.5. *Let $V_1, \ldots, V_k$ be homogeneous varieties in $\mathbb{B}_d$.*

(1) *For $p > \max\{\dim V_1, \ldots, \dim V_k\}$, if $\mathcal{F}_{V_1}, \ldots, \mathcal{F}_{V_k}$ are p-essentially normal and the algebraic sum $\mathcal{F}_{V_1}^{\perp} + \cdots + \mathcal{F}_{V_k}^{\perp}$ is closed, then $\mathcal{F}_{V_1 \cap \cdots \cap V_k}$ is also p-essentially normal.*

(2) *For $p > \dim V_1 \cup \cdots \cup V_k$, if $\mathcal{F}_{V_1}, \ldots, \mathcal{F}_{V_k}$ are p-essentially normal and the algebraic sum $\mathcal{F}_{V_1} + \cdots + \mathcal{F}_{V_k}$ is closed, then $\mathcal{F}_{V_1 \cup \cdots \cup V_k}$ is p-essentially normal.*

Proof. The proof of this result follows immediately from Proposition 3.4 using the correspondence between ideals of $\mathbb{C}[z_1, \ldots, z_d]$ and varieties in $\mathbb{C}^d$. $\qquad\square$

3.3. Some decomposable varieties. The following theorem was proved by Michael Hartz in [Har12]. We shall say that V is a *linear subspace* in $\mathbb{B}_d$ if $V = L \cap \mathbb{B}_d$ where $L \subseteq \mathbb{C}^d$ is a subspace.

Theorem 3.6 (Hartz). *Let $V_1, \ldots, V_k$ be linear subspaces in $\mathbb{B}_d$. Then the algebraic sum $\mathcal{F}_{V_1} + \cdots + \mathcal{F}_{V_k}$ is closed.*

We can also handle the following additional case.

Theorem 3.7. *Let $V_1, \ldots, V_n$ be homogeneous varieties in $\mathbb{B}_d$. Suppose that $\mathrm{span}(V_i) \cap \mathrm{span}(V_j) = \{0\}$ whenever $i \neq j$. Then the algebraic sum $\mathcal{F}_{V_1} + \cdots + \mathcal{F}_{V_n}$ is closed.*

Proof. We can suppose that each of the varieties $V_1, \ldots, V_n$ are nonempty. For $1 \leq i \leq d$, let $L_i = \mathrm{span}(V_i)$. Then, as in the proof of Proposition 2.2, since $L_1, \ldots, L_n$ are finite dimensional and disjoint, if we let

$$c = \max\{\cos(L_i, L_j) \mid i \neq j\},$$

then $0 \leq c < 1$. Following the proof of [DRS11, Lemma 7.11], this implies that

$$\cos(V_i^k, V_j^k) \leq c^k,$$

which we can rewrite as

$$(3.3) \quad \sup\{|\langle x_i, x_j \rangle|/(\|x_i\|\|x_j\|) \mid 0 \neq x_i \in V_i^k, \ 0 \neq x_j \in V_j^k, \ i \neq j\} \leq c^k.$$

Let $V = V_1 \cup \cdots \cup V_n$, and define an operator $T : \mathcal{F}_{V_1} \oplus \cdots \oplus \mathcal{F}_{V_n} \to \mathcal{F}_V$ by

$$T(x_1, \ldots, x_n) = x_1 + \cdots + x_n, \quad (x_1, \ldots, x_n) \in \mathcal{F}_{V_1} \oplus \cdots \oplus \mathcal{F}_{V_n}.$$

Then the range of T is precisely $\mathcal{F}_{V_1} + \cdots + \mathcal{F}_{V_n}$, and hence we will be done if we can prove that T has closed range.

Note that T is graded, in the sense that it maps $V_1^k \oplus \cdots \oplus V_n^k$ to V^k. For $(x_1, \ldots, x_n)$ in $V_1^k \oplus \cdots \oplus V_n^k$, the inequality (3.3) implies that

$$\|T(x_1, \ldots, x_n)\|^2 = \|x_1 + \cdots + x_n\|^2$$

$$= \sum_{i=1}^{n} \|x_i\|^2 + \sum_{i \neq j} \langle x_i, x_j \rangle$$

$$\geq \sum_{i=1}^{n} \|x_i\|^2 - \sum_{i \neq j} |\langle x_i, x_j \rangle|$$

$$\geq \sum_{i=1}^{n} \|x_i\|^2 - c^k \sum_{i \neq j} \|x_i\| \|x_j\|$$

$$\geq (1 - c^k(n-1)) \sum_{i=1}^{n} \|x_i\|^2$$

$$= (1 - c^k(n-1)) \|(x_1, \ldots, x_n)\|^2.$$

Therefore, for sufficiently large k, T is uniformly bounded below on the subspaces $V_1^k \oplus \cdots \oplus V_n^k$. Since each of these subspaces is finite dimensional, it follows that T has closed range. $\qquad\square$

4. Applications

We now present two classes of examples for which our results imply the Arveson–Douglas conjecture.

Theorem 4.1. *Let $V_1, \ldots, V_k$ be homogeneous varieties in $\mathbb{B}_d$ such that* $\mathrm{span}(V_i) \cap \mathrm{span}(V_j) = \{0\}$ *whenever $i \neq j$, and let $V = V_1 \cup \cdots \cup V_k$. Let* $p > \dim V$, *and suppose that $\mathcal{F}_{V_1}, \ldots, \mathcal{F}_{V_k}$ are all p-essentially normal. Then* $\mathcal{F}_V$ *is also p-essentially normal.*

Proof. This result follows immediately from (2) of Proposition 3.5 and Theorem 3.7. However, we will present a different and more constructive proof as an application of Proposition 2.2 and Theorem 2.3.

Let $L_j = \mathrm{span}(V_j)$ for $j = 1, \ldots, k$ and define $d_j = \dim L_j$. Put $D = d_1 + \cdots + d_k$, and let $\{e_1, \ldots, e_D\}$ be some orthonormal basis in $\mathbb{C}^D$. Consider the subspaces of $\mathbb{C}^D$ given by $K_1 = \mathrm{span}\{e_1, \ldots, e_{d_1}\}$, $K_2 = \mathrm{span}\{e_{d_1+1} \ldots, e_{d_2}\}$, etc., up to K_k. Let $A : \mathbb{C}^D \to \mathbb{C}^d$ be a linear map that takes K_j isometrically onto L_j for all $j = 1, \ldots, k$. Now define a homogeneous variety W by

$$W = (A|_{K_1})^{-1}(V_1) \cup \cdots \cup (A|_{K_k})^{-1}(V_k).$$

For $j = 1, \ldots, k$, the variety $W_j := (A|_{K_j})^{-1}(V_j)$ is unitarily equivalent to V_j, and therefore the Hilbert module $\mathcal{F}_{W_j}$ is unitarily equivalent to $\mathcal{F}_{V_j}$. It follows from the assumptions that $\mathcal{F}_{W_j}$ is p-essentially normal for all j. If we show that $\mathcal{F}_W$ is p-essentially normal, then Theorem 2.3 will imply that so is $\mathcal{F}_V$. But in the situation where the components W_j all lie in mutually orthogonal subspaces it is straightforward to check directly that $\mathcal{F}_W$ is essentially normal, so we are done. $\qquad\square$

Finally, let us observe that the Arveson–Douglas conjecture holds for any variety which is a union of subspaces.

Theorem 4.2. *Let $V_1, \ldots, V_k$ be linear subspaces in $\mathbb{B}_d$. Then $\mathcal{F}_{V_1 \cup \cdots \cup V_k}$ is* p-essentially normal for all $p > \dim V_1 \cup \cdots \cup V_k = \max\{\dim V_1, \ldots, \dim V_k\}$.

Proof. This follows from (2) of Proposition 3.5, from Theorem 3.6, and from the known result that, for a subspace V, $\mathcal{F}_V$ is p-essentially normal

for $p > \dim V$ (this last fact is [Arv98, Proposition 5.3], together with the observation theat $\mathcal{F}_V$ is unitarily equivalent to $H^2_{\dim V}$). $\qquad\square$

Remark 4.3. A very special case of Theorems 4.1 and 4.2 is that every quotient module associated with a 1-dimensional homogeneous variety is p-essentially normal for all $p > 1$. This special case is a known result, and was obtained by different techniques in [GW08, Proposition 4.1].

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