

# A sharp Bogomolov-type bound

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ABSTRACT. We prove a sharp lower bound for the essential minimum of a nontranslate variety in certain abelian varieties. This uses and generalises a result of Galateau. Our bound is a new step in the direction of an abelian analogue by David and Philippon of a toric conjecture of Amoroso and David and has applications in the framework of anomalous intersections.

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## 1. Introduction

In this paper, by variety we mean an algebraic variety defined over the algebraic numbers. Let  $A$  be an abelian variety; with a symmetric ample line bundle  $\mathcal{L}$  on  $A$  we associate an embedding  $i_{\mathcal{L}} : A \rightarrow \mathbb{P}_m$  defined by the minimal power of  $\mathcal{L}$  which is very ample. Heights and degrees corresponding to  $\mathcal{L}$  are computed via this embedding. More precisely,  $\hat{h}_{\mathcal{L}}$  will denote the

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$\mathcal{L}$ -canonical Néron–Tate height, and the degree  $\deg_{\mathcal{L}}$  of a subvariety of  $A$  is defined as the degree of its image in  $\mathbb{P}_m$  under  $i_{\mathcal{L}}$ .

If  $A$  is a product of abelian varieties, we fix on each simple nonisogenous factor a symmetric very ample line bundle. On  $A$  we consider the line bundle  $\mathcal{L}$  obtained as the tensor product of the pullbacks of the natural projections of  $A$  onto its factors.

A subvariety  $Y$  of an abelian variety is *translate* if it is the union of translates of algebraic subgroups. We also say that a subvariety  $Y \subseteq A$  is *torsion* if it is the union of components of algebraic subgroups. An irreducible  $Y$  is called *transverse* (resp. *weak-transverse*) if it is not contained in any proper translate (resp. in any proper torsion variety).

The torsion is a dense subset of  $A$  and, in view of the abelian analogue of Kronecker’s theorem, it is exactly the set of points of height 0. This statement motivates further questions about points of small height on  $A$  and on its subvarieties.

In the context of the Lehmer problem, effective lower bounds for the height of nontorsion points of  $A$  have been studied for example in [Sil84], [Mas87], [DH00] and [BS04].

One is then led to study the height function on an algebraic subvariety  $Y$  of  $A$ . For instance, by the Manin–Mumford conjecture (proved by Raynaud [Ray83a], [Ray83b]), the points of height zero are dense in  $Y$  if and only if  $Y$  is torsion.

More in general, setting

$$Y(\theta) = \{x \in Y(\overline{\mathbb{Q}}) \mid \hat{h}_{\mathcal{L}}(x) \leq \theta\}$$

and denoting by  $\overline{Y(\theta)}$  its Zariski closure, we have the following result, known as the Bogomolov Conjecture, proved by Ullmo and Zhang (see for instance [Ull98] and [Zha98]).

**Theorem 1.1** (Bogomolov Conjecture). *The essential minimum*

$$\mu_{\mathcal{L}}(Y) = \sup\{\theta \mid \overline{Y(\theta)} \subsetneq Y\}$$

*is strictly positive if and only if  $Y$  is nontorsion.*

The problem of giving explicit bounds for  $\mu_{\mathcal{L}}(Y)$  in terms of geometrical invariants of  $Y$  and of the ambient variety has been studied in several deep works, for instance in [DP98], [DP02], [DP07] in the abelian case and in [BZ96], [Sch96], [AD03], [AV09] in the corresponding toric case.

An effective Bogomolov Conjecture (up to  $\eta$ ) for transverse varieties in an abelian variety  $A$  states the following.

*Effective Bogomolov Bound:* For any abelian variety and for any real  $\eta > 0$  there exists a positive effective constant  $C(A, \mathcal{L}, \eta)$  such that for every transverse subvariety  $Y$  of  $A$

$$(1) \quad \mu_{\mathcal{L}}(Y) \geq C(A, \mathcal{L}, \eta) \frac{1}{(\deg_{\mathcal{L}} Y)^{\frac{1}{\text{codim } Y} + \eta}}.$$

Galateau, in [Gal10], proves this lower bound for varieties transverse in an abelian variety with a positive density of ordinary primes (Hypothesis H in [Gal10]). According to a conjecture of Serre, this shall always be true (see [Pin98, §7]). Some cases are proved: CM abelian varieties (see [BS04, part 5]), powers of elliptic curves (see [Ser89, chap. IV]) and abelian surfaces (see [DMOyS82, part VI, 2.7]) have a positive density of ordinary primes. Thus the bound (1) holds in all such abelian varieties. The constant of Galateau is effective, for instance, for powers of elliptic curves with the canonical embedding.

In this paper we prove a strong result for nontranslates, in the following sense. For an irreducible variety  $Y$ , we can consider the minimal translate  $H$  which contains  $Y$ . It is then natural to give the following definition.

**Definition 1.2.** An irreducible variety  $Y$  is *transverse in a translate  $H$*  if  $Y \subseteq H$  is not contained in any translate strictly contained in  $H$ . The relative codimension of  $Y$  is then the codimension of  $Y$  in  $H$  denoted  $\text{codim}_H Y$ .

In our main theorem, we prove that an effective Bogomolov bound as in (1) implies an  $H$ -effective Bogomolov bound, in which  $\text{deg } H$  works in our favour.

**Theorem 1.3.** *Let  $A$  be an abelian variety. Assume that an effective Bogomolov bound (1) holds. Then, for every variety  $Y$  transverse in a translate  $H \subset A$ ,*

$$\mu_{\mathcal{L}}(Y) \geq c(A, \mathcal{L}, \eta) \frac{(\text{deg}_{\mathcal{L}} H)^{\frac{1}{\text{codim}_H Y} - \eta}}{(\text{deg}_{\mathcal{L}} Y)^{\frac{1}{\text{codim}_H Y} + \eta}},$$

where  $c(A, \mathcal{L}, \eta)$  is a positive effective constant.

As described in Section 2.3, from the result of Galateau we immediately deduce:

**Theorem 1.4.** *Let  $A$  be an abelian variety with a positive density of ordinary primes. Let  $Y$  be a variety transverse in a translate  $H \subset A$ . Then, for any  $\eta > 0$ , there exists a positive constant  $c'(A, \mathcal{L}, \eta)$  such that*

$$\mu_{\mathcal{L}}(Y) \geq c'(A, \mathcal{L}, \eta) \frac{(\text{deg}_{\mathcal{L}} H)^{\frac{1}{\text{codim}_H Y} - \eta}}{(\text{deg}_{\mathcal{L}} Y)^{\frac{1}{\text{codim}_H Y} + \eta}}.$$

In our bound, as in the effective Bogomolov bound, the degree of  $Y$  appears in the denominator, but here the degree of  $H$  appears at the numerator. This last dependence is crucial for applications in the context of torsion anomalous intersections and of the Zilber–Pink conjecture, enabling a significant simplification in some classical uses of such Bogomolov-type bounds. For instance, using a special case of Theorem 1.4, we obtain some new results. In particular, in [CVV] we show that: If  $V$  is a weak-transverse variety in a product of elliptic curves with CM, then the  $V$ -torsion anomalous varieties of relative codimension one are finitely many. In addition, their degree

and normalised height are effectively bounded; see [CVV, Theorem 1.3] for a precise statement.

Our theorems are analogues, up to the logarithmic correction factor, of a toric conjecture of Amoroso and David in [AD03]; see [DP07] for a suitable conjecture in the abelian case. In the context of the Lehmer problem, a lower bound of similar nature to ours for translates of tori is given by Philippon and Sombra in [PS08] and for points in CM abelian varieties is given by Carrizosa in [Car09].

The main point in our proof is to compare the line bundle  $\mathcal{L}$  on  $A$  with its restriction  $\mathcal{L}|_H$  to  $H$ . To overcome this difficulty we prove an equivalence of line bundles which enables us to describe the restriction of the line bundle to  $H$  in terms of pull-backs through several isogenies. From the bound (1), we immediately deduce the good behaviour of essential minima under isogenies. We notice that our constant depends explicitly on the constant in (1). In addition, our method applies to any lower bound of the kind (1) where the hypothesis on  $Y$  are preserved by isogenies. We finally remark that the hypothesis of irreducibility for  $Y$  is not restrictive, indeed the essential minimum of a reducible variety is the maximum of the essential minima of its components.

This paper is structured in the following way. In Section 2, we give some preliminaries. Then, as a straightforward consequence of bound (1), we show the good behaviour of the essential minimum under the action of an isogeny.

In Section 3 we present the main technical ingredients of the proof, which is an equivalence of line bundles.

In Section 4 we finally prove our main theorem in two steps: we first consider the case when  $H$  is an abelian subvariety, and then we reduce to this case the general case of  $H$  a translate.

## 2. Preliminaries

Let  $A$  be an abelian variety. We consider irreducible subvarieties  $Y$  of  $A$  and we shall analyse the sets of points of small height on  $Y$  with respect to different line bundles; for this reason, when talking about heights, degrees and essential minima, we always indicate which line bundle we are using.

An abelian variety is isogenous to a product of simple abelian varieties. We suppose

$$(2) \quad A = A_1^{N_1} \times \cdots \times A_r^{N_r},$$

where the  $A_i$ 's are pairwise nonisogenous simple abelian varieties, and we fix embeddings  $A_i \hookrightarrow \mathbb{P}^{m_i}$ , given by symmetric ample line bundles on  $A_i$ . On any product variety we consider the line bundle  $\mathcal{L}$  obtained as the tensor product of the pullbacks of the natural projections on its factors.

If  $A$  is as in (2) and  $B$  is an abelian subvariety of  $A$ , then

$$B = B_1 \times \cdots \times B_r,$$

where for every  $i$ ,  $B_i$  is an abelian subvariety of  $A_i^{N_i}$ ; this follows, for instance, from [MW85, Lemma 7, p. 262] because  $A_i^{N_i}$  and  $A_j^{N_j}$  do not have any nontrivial isomorphic subquotients, when  $i \neq j$ .

We denote by  $\text{End}(-)$  the ring of endomorphisms of abelian varieties. Recall that  $\text{End}(A_i) \otimes_{\mathbb{Z}} \mathbb{R}$  is a real, complex or quaternionic algebra, and in each of these cases we have a standard euclidean norm.

We first prove several results which hold for a power of a simple abelian variety. Then we extend them to a general abelian variety.

In the following, we write  $\ll$  and  $\gg$  to indicate inequalities up to constants depending only on the ambient variety, the fixed line bundles and a positive real number  $\eta$ , but not on the variety  $Y$  or the isogenies involved, which may vary.

**2.1. Basic relations of degrees.** We recall some basic properties of degrees of subvarieties. In this section, we assume that  $A$  is an abelian variety of dimension  $D$ .

Let  $m$  be a positive integer. Let  $\mathcal{L}$  be a symmetric ample line bundle on  $A$  and let  $\mathcal{L}^m$  be the tensor product of  $m$  copies of  $\mathcal{L}$ . Let  $Y$  be an irreducible algebraic subvariety of  $A$  of dimension  $d$ . Then,

$$(3) \quad \deg_{\mathcal{L}^m} Y = m^d \deg_{\mathcal{L}} Y.$$

We are interested in how the degree changes under the action of the multiplication morphism. For  $a \in \mathbb{Z}$ , we denote by  $|a|$  its absolute value and by  $[a]$  the multiplication by  $a$  on  $A$ .

From [Mum70, Corollary 3, p. 59], we have

$$[a]^* \mathcal{L} = \mathcal{L}^{a^2}.$$

Hindry [Hin88, Lemma 6] proves

$$\deg_{\mathcal{L}} [a]^{-1} Y = |a|^{2(D-d)} \deg_{\mathcal{L}} Y$$

and

$$(4) \quad \deg_{\mathcal{L}} [a] Y = \frac{|a|^{2d}}{|\text{Stab } Y \cap \ker [a]|} \deg_{\mathcal{L}} Y.$$

Let  $\phi : A \rightarrow A$  be an isogeny; then the Projection Formula gives

$$(5) \quad \deg_{\phi^* \mathcal{L}} Y = \deg_{\mathcal{L}} \phi_*(Y),$$

where  $\phi_*(Y)$  is the cycle with support  $\phi(Y)$  and multiplicity  $\deg(\phi|_Y)$ .

Furthermore, by [BL04, Corollary 3.6.6] we have

$$\deg_{\phi^* \mathcal{L}} A = |\ker \phi| \deg_{\mathcal{L}} A,$$

and more in general:

**Lemma 2.1.** *Let  $\phi : A \rightarrow A$  be an isogeny. Let  $Y$  be an irreducible algebraic subvariety of  $A$ . Then*

$$\deg_{\mathcal{L}} \phi_*(Y) = |\text{Stab } Y \cap \ker \phi| \deg_{\mathcal{L}} \phi(Y).$$

We also have:

**Lemma 2.2.** *Let  $\phi : A \rightarrow A$  be an isogeny. Then,*

- (i)  $\text{Stab } \phi^{-1}(Y) = \phi^{-1}(\text{Stab } Y)$ .
- (ii) *Let  $\hat{\phi}$  be an isogeny such that  $\hat{\phi}\phi = \phi\hat{\phi} = [a]$ . Then*

$$|\text{Stab } \hat{\phi}^{-1}(Y) \cap \ker[a]| = |\ker \hat{\phi}| |\text{Stab } Y \cap \ker \phi|.$$

**Proof.** Part (i): Let  $t \in \text{Stab } \phi^{-1}(Y)$  then

$$\phi^{-1}(Y) + t = \phi^{-1}(Y).$$

Taking the image,  $Y + \phi(t) = Y$  and  $\phi(t) \in \text{Stab } Y$ , giving  $t \in \phi^{-1}(\text{Stab } Y)$ . Conversely, let  $t \in \text{Stab } Y$ , then

$$Y + t = Y$$

and taking the preimage  $\phi^{-1}(Y + t) = \phi^{-1}(Y)$ . Then  $\phi^{-1}(t) \in \text{Stab } \phi^{-1}(Y)$ .

Part (ii): By part (i) applied to  $\hat{\phi}$ , we have  $\text{Stab } \hat{\phi}^{-1}(Y) = \hat{\phi}^{-1}(\text{Stab } Y)$ . As  $\hat{\phi}\phi = [a]$ ,  $\ker[a] = \hat{\phi}^{-1}(\ker \phi)$ . Then

$$\text{Stab } \hat{\phi}^{-1}(Y) \cap \ker[a] = \hat{\phi}^{-1}(\text{Stab } Y \cap \ker \phi). \quad \square$$

**2.2. Basic relations of essential minima.** We now investigate useful relations for the essential minimum.

Recall that, by definition, for every  $x \in A$  and isogeny  $\phi : A \rightarrow A$ , we have  $\hat{h}_{\phi^*\mathcal{L}}(x) = \hat{h}_{\mathcal{L}}(\phi(x))$ ; then

$$(6) \quad \mu_{\phi^*\mathcal{L}}(Y) = \mu_{\mathcal{L}}(\phi(Y)).$$

In addition

$$(7) \quad \mu_{\mathcal{L}^m}(Y) = m\mu_{\mathcal{L}}(Y).$$

Another easy remark is stated in the following lemma.

**Lemma 2.3.** *Let  $\mathcal{L}_1, \mathcal{L}_2$  be two ample line bundles on  $A$ . Then for every irreducible subvariety  $Y \subseteq A$ ,*

$$\mu_{\mathcal{L}_1 \otimes \mathcal{L}_2}(Y) \geq \mu_{\mathcal{L}_1}(Y) + \mu_{\mathcal{L}_2}(Y).$$

**Proof.** This lemma is proved by contradiction, and it relies on the height relation  $\hat{h}_{\mathcal{L}_1 \otimes \mathcal{L}_2}(x) = \hat{h}_{\mathcal{L}_1}(x) + \hat{h}_{\mathcal{L}_2}(x)$  for every  $x \in A$ .

Suppose, by contradiction, that  $\mu_{\mathcal{L}_1 \otimes \mathcal{L}_2}(Y) < \mu_{\mathcal{L}_1}(Y) + \mu_{\mathcal{L}_2}(Y)$ . Then there exist reals  $k_1, k_2$  such that  $0 < k_i < \mu_{\mathcal{L}_i}(Y)$  and  $\mu_{\mathcal{L}_1 \otimes \mathcal{L}_2}(Y) < k_1 + k_2$ , and a dense subset  $U$  of  $Y$  such that

$$(8) \quad \hat{h}_{\mathcal{L}_1 \otimes \mathcal{L}_2}(x) \leq k_1 + k_2 \quad \forall x \in U.$$

From the definition of  $\mu_{\mathcal{L}_i}(Y)$ , the set of points of  $Y$  such that  $\hat{h}_{\mathcal{L}_i}(x) \leq k_i$  is contained in a closed subset  $V_i \subsetneq Y$ . Since  $U$  is dense,  $U' = U \setminus \cup_i V_i$  is also dense in  $Y$ . In addition, for every  $x \in U'$ ,  $\hat{h}_{\mathcal{L}_i}(x) > k_i$ . Then

$$\hat{h}_{\mathcal{L}_1 \otimes \mathcal{L}_2}(x) = \hat{h}_{\mathcal{L}_1}(x) + \hat{h}_{\mathcal{L}_2}(x) > k_1 + k_2 \quad \forall x \in U'$$

which contradicts (8). □

**2.3. Theorem 1.3 implies Theorem 1.4.** The implication ‘Theorem 1.3 implies Theorem 1.4’ is straight forward from a result of Galateau [Gal10]. For convenience we recall his theorem.

For  $Y$  a subvariety of a (semi)abelian variety, with an ample line bundle  $\mathcal{L}$ , define

$$\omega_{\mathcal{L}}(Y) = \min_Z \{\deg_{\mathcal{L}} Z\}$$

where the minimum is taken over all the hypersurfaces (not necessarily irreducible) containing  $Y$ .

**Theorem 2.4** ([Gal10, Theorem 1.1]). *Let  $B$  be an abelian variety with a positive density of ordinary primes, and  $\mathcal{L}$  be an ample and symmetric line bundle on  $B$ . Let  $Y \subseteq B$  be a transverse variety. Then*

$$\mu_{\mathcal{L}}(Y) \geq \frac{C_0(B, \mathcal{L})}{\omega_{\mathcal{L}}(Y)} (\log(3 \deg_{\mathcal{L}} Y))^{-\lambda(Y)}$$

where  $C_0(B, \mathcal{L})$  is a positive real depending on the variety  $B$  and the line bundle  $\mathcal{L}$ , and  $\lambda(Y) = (5 \dim B(1 + \text{codim}_B Y))^{1+\text{codim}_B Y}$ . In particular, for every  $\eta > 0$ , there exists a constant  $C(B, \mathcal{L}, \eta)$  such that

$$\mu_{\mathcal{L}}(Y) \geq \frac{C(B, \mathcal{L}, \eta)}{(\deg_{\mathcal{L}} Y)^{\frac{1}{\text{codim}_B Y} + \eta}}.$$

Note that if  $A$  has a positive density of ordinary primes, then also all its abelian subvarieties have it.

As mentioned in the introduction the hypothesis on a positive density of ordinary primes always holds for CM abelian varieties, powers of elliptic curves and abelian surfaces and is conjectured to hold for every abelian variety.

**2.4. Essential minimum and isogenies.** In this section we derive from the effective Bogomolov bound (1) a theorem which shows the good behaviour of the essential minimum under the action of an isogeny. This result will be among the ingredients used, in Section 4, to prove the main theorem.

**Theorem 2.5.** *Let  $B$  be abelian variety such that an effective Bogomolov bound (1) holds. Let  $Y$  be a transverse subvariety of  $B$  and  $\phi : B \rightarrow B$  an isogeny. Then for every  $\eta > 0$  we have*

$$\mu_{\phi^* \mathcal{L}}(Y) \gg \frac{(\deg_{\phi^* \mathcal{L}} B)^{\frac{1}{\text{codim}_B Y} - \eta}}{(\deg_{\phi^* \mathcal{L}} Y)^{\frac{1}{\text{codim}_B Y} + \eta}}.$$

**Proof.** Let  $g = \dim B$ , and let  $a$  be an integer of minimal absolute value such that there exists an isogeny  $\hat{\phi}$  with  $\phi \hat{\phi} = \hat{\phi} \phi = [a]$ . By definition of dual we have  $|a| \leq \deg \phi$ .

Let  $W$  be an irreducible component of  $\hat{\phi}^{-1}(Y)$ . Note that isogenies preserve dimensions and transversality, so  $\dim W = \dim Y = d$  and  $W$  is transverse.

We have

$$[a]W = \phi\hat{\phi}W = \phi\hat{\phi}\hat{\phi}^{-1}(Y) = \phi(Y).$$

Then

$$(9) \quad \mu_{\phi^*\mathcal{L}}(Y) = \mu_{\mathcal{L}}(\phi(Y)) = \mu_{\mathcal{L}}([a]W) = |a|^2\mu_{\mathcal{L}}(W).$$

We now need to estimate  $\deg_{\mathcal{L}} W$ .

Since  $Y$  is transverse,  $d > 0$ ; then by formula (4),

$$\deg_{\mathcal{L}} \phi(Y) = \deg_{\mathcal{L}} [a]W = \frac{|a|^{2d}}{|\text{Stab } W \cap \ker[a]|} \deg_{\mathcal{L}} W$$

or equivalently

$$\deg_{\mathcal{L}} W = \frac{|\text{Stab } W \cap \ker[a]|}{|a|^{2d}} \deg_{\mathcal{L}} \phi(Y).$$

Using Lemma 2.2 (ii) and Lemma 2.1, we obtain

$$\begin{aligned} \deg_{\mathcal{L}} W &= \frac{|\ker \hat{\phi}|}{|a|^{2d}} |\text{Stab } Y \cap \ker \phi| \deg_{\mathcal{L}} \phi(Y) \\ &= \frac{|\ker \hat{\phi}|}{|a|^{2d}} \deg_{\mathcal{L}} \phi_*(Y). \end{aligned}$$

Since  $W$  is transverse, applying bound (1) we have

$$\begin{aligned} \mu_{\mathcal{L}}(W) &\gg (\deg_{\mathcal{L}} W)^{-\frac{1}{g-d}-\eta} \\ &= \left( \frac{|a|^{2d}}{|\ker \hat{\phi}| \deg_{\mathcal{L}} \phi_*(Y)} \right)^{\frac{1}{g-d}+\eta} \\ &= \left( \frac{|a|^{2d} \deg_{\mathcal{L}} B}{|\ker \hat{\phi}| \deg_{\mathcal{L}} \phi_*(Y)} \right)^{\frac{1}{g-d}+\eta} (\deg_{\mathcal{L}} B)^{\frac{-1}{g-d}-\eta} \\ (10) \quad &\gg \left( \frac{|a|^{2d} \deg_{\mathcal{L}} B}{|\ker \hat{\phi}| \deg_{\mathcal{L}} \phi_*(Y)} \right)^{\frac{1}{g-d}+\eta} (\deg_{\mathcal{L}} B)^{-\eta}, \end{aligned}$$

where line (10) follows absorbing the appropriate power of  $\deg_{\mathcal{L}} B$  in the implicit constant.

We substitute this last estimate in (9), then

$$\begin{aligned} \mu_{\phi^*\mathcal{L}}(Y) &= |a|^2 \mu_{\mathcal{L}}(W) \\ &\gg |a|^2 \left( \frac{|a|^{2d} \deg_{\mathcal{L}} B}{|\ker \hat{\phi}| \deg_{\mathcal{L}} \phi_*(Y)} \right)^{\frac{1}{g-d} + \eta} (\deg_{\mathcal{L}} B)^{-\eta} \\ &= \left( \frac{|a|^{2g-2d} |a|^{2d} |\ker \phi| \deg_{\mathcal{L}} B}{|\ker \hat{\phi}| |\ker \phi| \deg_{\mathcal{L}} \phi_*(Y)} \right)^{\frac{1}{g-d} + \eta} (\deg_{\mathcal{L}} B)^{-\eta} |a|^{-2(g-d)\eta} \\ &= \left( \frac{|a|^{2g} \deg_{\phi^*\mathcal{L}} B}{|a|^{2g} \deg_{\mathcal{L}} \phi_*(Y)} \right)^{\frac{1}{g-d} + \eta} (\deg_{\mathcal{L}} B)^{-\eta} |a|^{-2(g-d)\eta} \\ &= \left( \frac{\deg_{\phi^*\mathcal{L}} B}{\deg_{\phi^*\mathcal{L}} Y} \right)^{\frac{1}{g-d} + \eta} (\deg_{\mathcal{L}} B)^{-\eta} |a|^{-2(g-d)\eta}, \end{aligned}$$

where  $|\ker \hat{\phi}| |\ker \phi| = |a|^{2g}$  because  $\phi \hat{\phi} = \hat{\phi} \phi = [a]$ . In addition

$$\deg_{\phi^*\mathcal{L}} B = |\ker \phi| \deg_{\mathcal{L}} B = \deg \phi \deg_{\mathcal{L}} B.$$

Since  $|a| \leq \deg \phi$ , we have

$$(\deg_{\mathcal{L}} B)^{-\eta} |a|^{-2(g-d)\eta} \geq (|a| \deg_{\mathcal{L}} B)^{-2(g-d)\eta} \geq (\deg_{\phi^*\mathcal{L}} B)^{-2(g-d)\eta}.$$

Then

$$\mu_{\phi^*\mathcal{L}}(Y) \gg \frac{(\deg_{\phi^*\mathcal{L}} B)^{\frac{1}{g-d} - 2(g-d)\eta + \eta}}{(\deg_{\phi^*\mathcal{L}}(Y))^{\frac{1}{g-d} + \eta}},$$

which easily implies the wished bound, after changing  $\eta$ . □

**2.5. Morphisms.** Let  $A$  be a simple abelian variety of dimension  $D$  and let  $M \leq N$  be positive integers.

We can associate a matrix  $\mathcal{A} = (\psi_{ij})_{ij} \in \text{Mat}_{M \times N}(\text{End}(A))$  with a morphism

$$\begin{aligned} \psi_{\mathcal{A}} : A^N &\rightarrow A^M \\ (x_1, \dots, x_N) &\mapsto (\psi_{11}x_1 + \dots + \psi_{1N}x_N, \dots, \psi_{M1}x_1 + \dots + \psi_{MN}x_N). \end{aligned}$$

We also associate a morphism  $\psi : A^N \rightarrow A^M$  with a matrix

$$\mathcal{A}_{\psi} = \begin{pmatrix} \psi_{11} & \dots & \psi_{1N} \\ \vdots & \ddots & \vdots \\ \psi_{M1} & \dots & \psi_{MN} \end{pmatrix}$$

such that all components of  $\ker \psi$  are components of

$$(11) \quad \begin{cases} \psi_{11}x_1 + \dots + \psi_{1N}x_N = 0 \\ \vdots \\ \psi_{M1}x_1 + \dots + \psi_{MN}x_N = 0 \end{cases}$$

and such that the product of the norms of the rows of  $\mathcal{A}_\psi$  is minimised. As a norm for the rows we take the euclidean norm in  $(\text{End}(A) \otimes_{\mathbb{Z}} \mathbb{R})^N$ .

For notation's sake, sometimes we identify the morphism with the matrix.

We associate a morphism  $\psi : A^N \rightarrow A^M$  with an abelian subvariety  $A'$  of  $A^N$  given by the connected component of  $\ker \psi$  passing through the zero of  $A^N$ .

Finally, we associate an abelian subvariety  $A' \subseteq A^N$  of codimension  $MD$  with the projection morphism

$$\psi : A^N \rightarrow A^M = A^N/A'.$$

Then  $A' = \ker \psi$  is a component of the variety defined by the system (11), where  $\mathcal{A}_\psi = (\psi_{ij})_{ij}$  is the matrix associated with  $\psi$ , as above.

We call the morphism (resp. the matrix) just defined the *associated morphism* (resp. *associated matrix*) of  $A'$ .

We finally define the norm of a morphism in the usual way.

### 3. An equivalence of line bundles

In this section we let  $A$  be a simple abelian variety of dimension  $D$ .

**3.1. Definitions.** Let now  $B$  be an abelian subvariety of  $A^N$  of dimension  $nD$ .

**Lemma 3.1.** *Let  $B$  be an abelian subvariety of  $A^N$  of dimension  $nD$ . Then there exists an isogeny  $\phi : A^N \rightarrow A^N$  defined by a matrix*

$$\mathcal{A}_\phi = \begin{pmatrix} \phi_B \\ \phi_{B'} \end{pmatrix}$$

such that:

- (i)  $\phi_B \in \text{Mat}_{(N-n) \times N}(\text{End}(A))$  and  $\phi_{B'} \in \text{Mat}_{n \times N}(\text{End}(A))$  are both of full rank.
- (ii)  $\phi(B) = \{0\}^{N-n} \times A^n$ .
- (iii)  $\deg \phi \leq c(A, N)$ , where  $c(A, N)$  is a constant depending only on  $A$  and  $N$ .

**Proof.** By a lemma of Bertrand, (see [Ber87, Proposition 2, p. 15]) we can find a supplement  $B'$  of  $B$  such that  $B + B' = A^N$  and the cardinality of  $B \cap B'$  is bounded by a constant depending on  $A$  and  $N$ .

Then, there exists a matrix

$$\phi_B \in \text{Mat}_{(N-n) \times N}(\text{End}(A))$$

of rank  $N - n$  such that  $\ker \phi_B = B + \tau$  for  $\tau$  a torsion group contained in  $B'$  of cardinality bounded by a constant depending only on  $N$ . Similarly, let  $\phi_{B'} \in \text{Mat}_{n \times N}(\text{End}(A))$  be a matrix of rank  $n$  such that  $\ker \phi_{B'} = B' + \tau'$  for  $\tau'$  a torsion group contained in  $B$  of cardinality bounded only in terms of  $N$  (see also Masser and Wüstholz [MW93, Lemma 1.3]).

If we define the isogeny  $\phi$  as follows:

$$\phi = \begin{pmatrix} \phi_B \\ \phi_{B'} \end{pmatrix} : A^N \rightarrow A^N,$$

then properties (i) and (ii) are satisfied by construction, and we have that

$$|\ker \phi| = |(B + \tau) \cap (B' + \tau')| = |(B \cap B') + \tau + \tau'| \ll 1. \quad \square$$

It is an exercise in linear algebra to show that there exists an isomorphism  $T$  of norm bounded only in terms of  $N$ , such that all  $n \times n$  minors of the matrix consisting of the last  $n$  columns of  $(\phi T)^{-1}$  have determinant different from zero.

Clearly, for a point  $P$ , we have

$$(12) \quad \hat{h}(T(P)) \ll \|T\|^2 h(P).$$

Moreover, as in [Via09, Proposition 4.2] for elliptic curves and in [Via10, Lemma 4.5] for abelian varieties, for any subvariety  $X$  we get

$$(13) \quad \deg T(X) \ll \|T\|^{2 \dim X} \deg X.$$

We deduce the following lemma.

**Lemma 3.2.** *Let  $T$ ,  $P$  and  $X$  be as above. Then:*

- (i)  $\|T\|^{-2(N-1)} \hat{h}(P) \ll \hat{h}(T(P)) \ll \|T\|^2 \hat{h}(P).$
- (ii)  $\|T\|^{-2N} \deg X \ll \deg T(X) \ll \|T\|^{2 \dim X} \deg X.$

**Proof.** The upper bounds are immediate from (12) and (13), while the lower bounds are obtained applying (12) to  $\hat{T}T(P)$  and (13) to  $\hat{T}T(X)$ . In particular, using (13), one has

$$\deg(\hat{T}T(X)) \leq \|\hat{T}\|^{2 \dim T(X)} \deg T(X).$$

On the other hand

$$\deg(\hat{T}T(X)) = \deg([\deg T]X);$$

which, combined with [Hin88, Lemma 6], give the desired bound.  $\square$

Hence  $T$  changes degrees and heights by a constant depending only on  $N$  and  $D$ .

Note that the isogeny  $\phi : A^N \rightarrow A^N$  sends  $B$  to the last  $n$  factors,

$$\phi(B) = 0 \times \cdots \times 0 \times A^n.$$

We denote the immersion of  $A^n$  in the last  $n$  coordinates by

$$i : A^n \rightarrow A^N; \quad (x_1, \dots, x_n) \mapsto (0, \dots, 0, x_1, \dots, x_n).$$

Let us denote by  $\alpha$  the minimal positive integer such that there exists an isogeny  $\hat{\phi}$  satisfying  $\hat{\phi}\hat{\phi} = \hat{\phi}\phi = [\alpha]$ . Note that,  $|\alpha| \leq \deg \phi$ . Of course, we may take  $\hat{\phi}$  to be the dual of  $\phi$ , but, eventually, for explicit computations, the above definition could be more convenient.

We decompose  $\hat{\phi} = (\mathcal{A}|\mathcal{B})$  with  $\mathcal{A}$  a matrix in  $\text{Mat}_{N \times (N-n)}(\text{End}(A))$  and  $\mathcal{B}$  a matrix in  $\text{Mat}_{N \times n}(\text{End}(A))$ . We denote by  $a_i$  the  $i$ -th row of  $\mathcal{A}$ ; similarly

$$\mathcal{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}$$

and  $(a_i, b_i)$  is the  $i$ -th row of  $\hat{\phi}$ .

**Lemma 3.3.** *For  $I \in \mathbb{I} = \{(i_1, \dots, i_n) \in \{1, \dots, N\}^n \text{ and } i_j < i_{j+1}\}$  the morphism*

$$\varphi_I = \begin{pmatrix} b_{i_1} \\ \vdots \\ b_{i_n} \end{pmatrix} : A^n \rightarrow A^n$$

is an isogeny.

**Proof.** We are assuming that the last  $n$  columns of  $\phi^{-1}$  have  $n \times n$  minors with nonzero determinant. Together with  $\phi\hat{\phi} = \alpha\text{Id}_N$ , this implies that all  $n \times n$  minors of  $\mathcal{B}$  have nonzero determinant. Then  $\det \varphi_I \neq 0$ . This is equivalent to say that  $\varphi_I$  is an isogeny, as  $\varphi_I$  is a morphism associated with a square matrix of full rank.  $\square$

We remark that the map  $\varphi_I : A^n \rightarrow A^n$  may be described as the composition

$$A^n \xrightarrow{i} A^N \xrightarrow{\hat{\phi}} A^N \xrightarrow{\pi_I} A^n,$$

where  $\pi_I$  is the projection on the  $n$  coordinates appearing in the multi-index  $I$ .

Indeed, we can sum up the situation with the following commutative diagram:

$$\begin{array}{ccccc} & & & & [\alpha] \\ & & & & \curvearrowright \\ A^N & \xrightarrow{\phi} & A^N & \xrightarrow{\hat{\phi}} & A^N \\ \uparrow j & \nearrow \phi|_B & \uparrow i & \searrow \phi_I & \downarrow \pi_I \\ B & \xrightarrow{\varphi_B} & A^n & \xrightarrow{\varphi_I} & A^n \end{array}$$

where  $j$  is the inclusion of  $B$  in  $A^N$ ,  $\phi_I$  is defined as the composition  $\pi_I \circ \hat{\phi}$ , and  $\varphi_B$  is the restriction to  $B$  of the morphism associated with the matrix  $\phi_{B'}$  in Lemma 3.1. The fact that  $\phi(B) = \{0\}^{N-n} \times A^n$  allows us to say that all horizontal arrows are isogenies.

With this notation, we remark a consequence of Lemmas 2.1 and 3.1. For  $\phi$  as above we have

$$(14) \quad \deg_{\mathcal{L}} \varphi_{B^*}(B) = \deg \varphi_B \deg_{\mathcal{L}} A^n \leq \deg \phi \deg_{\mathcal{L}} A^n \ll \deg_{\mathcal{L}} A^n.$$

In fact  $\deg \varphi_B = |B \cap \ker \phi| \leq |\ker \phi|$ .

Note that by our definitions of degree in  $A^N$  and  $A^n$ , the map  $i$  preserves the degree of subvarieties of  $\{0\}^{N-n} \times A^n$ .

**3.2. The equivalence.** In this section  $A$  is a simple abelian variety. We recall that on  $A$  we fixed a symmetric ample line bundle  $\mathcal{L}_1$ ; for any integer  $m$ , we denote by  $\mathcal{L}_m$  the bundle on  $A^m$  obtained as the tensor product of the pullbacks of the natural projections of  $A^m$  on its simple factors. We now study  $\mathcal{L}_{N|B}$  for  $B$  an abelian subvariety of  $A^N$ . We can express a power of  $\mathcal{L}_{N|B}$  as a tensor product of pull-backs via different morphisms of the bundle  $\mathcal{L}_n$  on  $A^n$ . Of course, even if  $B$  is isogenous to  $A^n$ , in general  $\mathcal{L}_{N|B}$  is not the natural bundle  $\mathcal{L}_n$ .

**Theorem 3.4.** *The following equivalence of line bundles holds*

$$\mathcal{L}_N^{\alpha^2 \binom{N-1}{n-1}} \Big|_B \cong \phi_{|B}^* \bigotimes_I \phi_I^* \mathcal{L}_n \cong \varphi_B^* \bigotimes_I \varphi_I^* \mathcal{L}_n,$$

where  $\phi_{\hat{\phi}} = \hat{\phi}\phi = [\alpha]$  is as above and  $\varphi_I$  is defined in Lemma 3.3.

**Proof.** We denote by  $c_1(\mathcal{L})$  a representative of the first Chern-class of  $\mathcal{L}$ . By  $\boxplus$  we mean the sum of cycles and by  $\mathcal{L}^m$  we mean as before the tensor product of  $m$  copies of  $\mathcal{L}$ . We recall that, for  $f$  a morphism, we have

$$(15) \quad c_1(f^* \mathcal{L}) = f^* c_1(\mathcal{L}).$$

Let  $e_i : A^n \rightarrow A$  and  $f_i : A^N \rightarrow A$  be the projections on the  $i$ -th factor. Note that  $e_i$ 's (resp. the  $f_i$ 's) generate a free  $\mathbb{Z}$ -module of rank  $n$  (resp. rank  $N$ ) of  $\text{Hom}(A^n, A)$  (resp.  $\text{Hom}(A^N, A)$ ). By definition of standard line bundle,

$$(16) \quad \begin{aligned} c_1(\mathcal{L}_n) &= \boxplus_{i=1}^n e_i^*(\mathcal{L}), \\ c_1(\mathcal{L}_N) &= \boxplus_{i=1}^N f_i^*(\mathcal{L}). \end{aligned}$$

Note that, for any integer  $\alpha$ ,  $(\alpha f_i)^* = (f_i[\alpha])^* = [\alpha]^* f_i^*$ , and more in general, for  $\psi \in \text{Mat}_{N \times m}(\text{End}(A))$  and  $\psi' \in \text{Mat}_{m' \times N}(\text{End}(A))$  with  $m, m' \in \mathbb{N}^*$ , we have

$$(17) \quad \psi^*(\psi')^* = (\psi'\psi)^*.$$

In addition, by [BL04, Corollary 3.6, p. 34] for  $\mathcal{M}$  a symmetric ample line bundle on  $A^n$ ,

$$c_1([\alpha]^* \mathcal{M}) = c_1(\mathcal{M}^{\alpha^2}),$$

and then,

$$(18) \quad c_1(\mathcal{L}_N^{\alpha^2}) = \boxplus_{i=1}^N (\alpha f_i)^*(\mathcal{L}).$$

Two line bundles are equivalent if and only if they have the same Chern-class; the proof of the theorem is based on this remark.

For any morphism  $\psi : A^N \rightarrow A^n$ , applying (15), (16) and (17), we have

$$c_1(\psi^* \mathcal{L}_n) = \psi^* c_1(\mathcal{L}_n) = \boxplus_{i=1}^n (e_i \psi)^*(\mathcal{L}) = \boxplus_{i=1}^n \psi_i^*(\mathcal{L}),$$

where  $\psi_i : A^N \rightarrow A$  is the  $i$ -th row of  $\psi$ .

Apply this formula to each  $\phi_I$ . Then, for  $I = (i_1, \dots, i_n)$ ,

$$(19) \quad c_1(\phi_I^* \mathcal{L}_n) = (a_{i_1}, b_{i_1})^*(\mathcal{L}) \boxplus \dots \boxplus (a_{i_n}, b_{i_n})^*(\mathcal{L}).$$

Since the Chern class of the tensor product is the sum of the Chern classes, we obtain

$$(20) \quad c_1 \left( \bigotimes_I \phi_I^* \mathcal{L}_n \right) = \boxplus_{I \in \mathbb{I}} ((a_{i_1}, b_{i_1})^*(\mathcal{L}) \boxplus \dots \boxplus (a_{i_n}, b_{i_n})^*(\mathcal{L})) \\ = \binom{N-1}{n-1} \boxplus_{i=1}^N (a_i, b_i)^*(\mathcal{L}),$$

where the last equality is justified from the fact that each multi-index  $I$  consists of  $n$  coordinates and each of the  $N$  indices appears the same number of times, so each row  $(a_i, b_i)$  appears  $\binom{N-1}{n-1}$  times, once for each possible choice of the other  $n-1$  rows.

Recall that  $\hat{\phi} = [\alpha]$ ; by (15)

$$c_1 \left( \phi^* \bigotimes_I \phi_I^* \mathcal{L}_n \right) = \phi^* c_1 \left( \bigotimes_I \phi_I^* \mathcal{L}_n \right) \\ = \binom{N-1}{n-1} \boxplus_{i=1}^N ((a_i, b_i)\phi)^*(\mathcal{L}) \\ = \binom{N-1}{n-1} \boxplus_{i=1}^N (\alpha f_i)^*(\mathcal{L}).$$

Hence, using (18) and restricting to  $B$  we conclude

$$c_1 \left( \phi_{|B}^* \bigotimes_I \phi_I^* \mathcal{L}_n \right) = \binom{N-1}{n-1} \boxplus_{i=1}^N (\alpha f_i)^*(\mathcal{L})_{|B} \\ = \binom{N-1}{n-1} c_1 \left( \mathcal{L}_N^{\alpha^2} \right)_{|B} \\ = c_1 \left( \mathcal{L}_N^{\alpha^2 \binom{N-1}{n-1}} \right)_{|B}.$$

The last isomorphism in the statement of the theorem follows from

$$\phi_I \circ \phi_{|B} = \varphi_I \circ \varphi_B. \quad \square$$

The following relation has been pointed out by Gaël Rémond.

**Proposition 3.5.** *The following equality holds:*

$$\deg \bigotimes_{I \in \mathbb{I}} \phi_I^* \mathcal{L}_n = \binom{N-1}{n-1} \sum_{I \in \mathbb{I}} \deg \phi_I^* \mathcal{L}_n.$$

**Proof.** We compute the degrees as intersection numbers. By relation (19), we have

$$\deg \phi_I^* \mathcal{L}_n = n! \deg \prod_{i_j \in I} (a_{i_j}, b_{i_j})^*(\mathcal{L}).$$

Similarly, by formula (20), we obtain

$$\begin{aligned} \deg \bigotimes_I \phi_I^* \mathcal{L}_n &= n! \binom{N-1}{n-1}^n \sum_{i_1 < \dots < i_n} \deg \prod_{j=1}^n (a_{i_j}, b_{i_j})^*(\mathcal{L}) \\ &= \binom{N-1}{n-1}^n \sum_I \deg \phi_I^* \mathcal{L}_n. \end{aligned} \quad \square$$

#### 4. The proof of the main theorem

Recall that

$$A = A_1^{N_1} \times \dots \times A_r^{N_r},$$

where  $A_i$  are pairwise nonisogenous simple abelian varieties. We fixed embeddings  $A_i \rightarrow \mathbb{P}_{m_i}$  given by symmetric ample line bundles on  $A_i$ . The bundles  $\mathcal{L}$  on  $A$  and  $\mathcal{L}_{N_i}$  on  $A_i^{N_i}$  are obtained as the tensor product of the pullbacks of the natural projections.

We first prove a weak form of Theorem 1.3, and then we remove the more restrictive hypothesis.

**Theorem 4.1.** *Theorem 1.3 holds for  $H$  an abelian subvariety of  $A$ . In particular, if  $Y$  is a transverse subvariety of  $H$ , then for any  $\eta > 0$ , there exists a positive constant  $c_1(A, \mathcal{L}, \eta)$  such that*

$$\mu_{\mathcal{L}}(Y) \geq c_1(A, \mathcal{L}, \eta) \frac{(\deg_{\mathcal{L}} H)^{\frac{1}{\text{codim}_H Y} - \eta}}{(\deg_{\mathcal{L}} Y)^{\frac{1}{\text{codim}_H Y} + \eta}}.$$

**Proof.** Unless specified otherwise, in this proof we keep the same notation as in the previous section. Recall that

$$H = H_1 \times \dots \times H_r$$

where  $H_i$  is an abelian subvariety of  $A_i^{N_i}$ , set  $n_i = \frac{\dim H_i}{\dim A_i}$  and  $d = \dim Y$ . We set

$$\Phi_H = \varphi_{H_1} \times \dots \times \varphi_{H_r} : H \rightarrow A',$$

where  $A'$  is the abelian variety  $A_1^{n_1} \times \dots \times A_r^{n_r}$ .

Denoting by

$$\pi_i : A \rightarrow A_i^{N_i}$$

the projection on  $A_i^{N_i}$ , we have

$$\mathcal{L} = \bigotimes_{i=1}^r \pi_i^* \mathcal{L}_{N_i}.$$

Recall that

$$\mathbb{I}_i = \{(i_1, \dots, i_{n_i}) \in \{1, \dots, N_i\}^{n_i}\};$$

for every

$$\mathbf{I} = (I_1, \dots, I_r) \in \mathbb{I}_1 \times \dots \times \mathbb{I}_r$$

let

$$\Phi_{\mathbf{I}} = \varphi_{I_1} \times \dots \times \varphi_{I_r} : A_1^{n_1} \times \dots \times A_r^{n_r} \rightarrow A_1^{n_1} \times \dots \times A_r^{n_r},$$

which is an isogeny by Lemma 3.3.

We also define

$$\alpha = \max_i \alpha_i^2 \binom{N_i - 1}{n_i - 1},$$

where  $\alpha_i$  is the minimal positive integer such that  $[\alpha_i] = \phi_i \hat{\phi}_i = \hat{\phi}_i \phi_i$ , with the notations and definitions of Section 3.

Finally denote by  $\mathcal{M}$  the bundle on  $A_1^{n_1} \times \dots \times A_r^{n_r}$  given by

$$\mathcal{M} = \bigotimes_{i=1}^r \pi_i^* \mathcal{L}_{n_i}.$$

By Lemma 2.3, we know that

$$\mu_{\bigotimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}}(\Phi_H(Y)) \geq \sum_{\mathbf{I}} \mu_{\Phi_{\mathbf{I}}^* \mathcal{M}}(\Phi_H(Y)).$$

We apply Theorem 2.5 to each  $\Phi_{\mathbf{I}}$  on  $A'$  and  $\mathcal{M}$ . We deduce that for each  $\mathbf{I}$ ,

$$\mu_{\Phi_{\mathbf{I}}^* \mathcal{M}}(\Phi_H(Y)) \gg \frac{\left(\deg_{\Phi_{\mathbf{I}}^* \mathcal{M}} A'\right)^{\frac{1}{\text{codim}_H Y - \eta}}}{\left(\deg_{\Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_H(Y)\right)^{\frac{1}{\text{codim}_H Y + \eta}}}.$$

We obtain

$$\begin{aligned} \mu_{\bigotimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}}(\Phi_H(Y)) &\geq \sum_{\mathbf{I}} \mu_{\Phi_{\mathbf{I}}^* \mathcal{M}}(\Phi_H(Y)) \\ &\gg \sum_{\mathbf{I}} \frac{\left(\deg_{\Phi_{\mathbf{I}}^* \mathcal{M}} A'\right)^{\frac{1}{\text{codim}_H Y - \eta}}}{\left(\deg_{\Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_H(Y)\right)^{\frac{1}{\text{codim}_H Y + \eta}}}. \end{aligned}$$

Since each bundle is ample, for every variety  $X$ , we have

$$\deg_{\Phi_{\mathbf{I}}^* \mathcal{M}} X \leq \deg_{\bigotimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} X.$$

Note also that, for  $x_i \geq 1$ ,  $(\sum x_i)^{1/m} \leq \sum x_i^{1/m}$ .

Recall that, by definition, the degree of the abelian variety is the degree of the line bundle. Using then Proposition 3.5, we deduce

$$\begin{aligned} & \sum_{\mathbf{I}} \frac{\left(\deg_{\Phi_{\mathbf{I}}^* \mathcal{M}} A'\right)^{\frac{1}{\operatorname{codim}_H Y} - \eta}}{\left(\deg_{\Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_H(Y)\right)^{\frac{1}{\operatorname{codim}_H Y} + \eta}} \\ & \geq \frac{\left(\sum_{\mathbf{I}} \deg_{\Phi_{\mathbf{I}}^* \mathcal{M}} A'\right)^{\frac{1}{\operatorname{codim}_H Y} - \eta}}{\left(\deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_H(Y)\right)^{\frac{1}{\operatorname{codim}_H Y} + \eta}} \\ & \geq \left(\max_i \binom{N_i - 1}{n_i - 1}\right)^{\frac{-1}{\operatorname{codim}_H Y} + \eta} \frac{\left(\deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} A'\right)^{\frac{1}{\operatorname{codim}_H Y} - \eta}}{\left(\deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_H(Y)\right)^{\frac{1}{\operatorname{codim}_H Y} + \eta}}. \end{aligned}$$

Therefore

$$\mu_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}}(\Phi_H(Y)) \gg \frac{\left(\deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} A'\right)^{\frac{1}{\operatorname{codim}_H Y} - \eta}}{\left(\deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_H(Y)\right)^{\frac{1}{\operatorname{codim}_H Y} + \eta}}.$$

Note that with our notation  $A' = \Phi_H(H)$ . Moreover, by (14),

$$\deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} A' \gg \deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_{H^*}(H).$$

We deduce

$$(21) \quad \mu_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}}(\Phi_H(Y)) \gg \frac{\left(\deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_{H^*}(H)\right)^{\frac{1}{\operatorname{codim}_H Y} - \eta}}{\left(\deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_H(Y)\right)^{\frac{1}{\operatorname{codim}_H Y} + \eta}}.$$

By (5) and Theorem 3.4 we obtain

$$(22) \quad \deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} \Phi_{H^*}(H) = \deg_{\Phi_H^* \otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} H \geq \deg_{\mathcal{L}} H.$$

From Lemma 2.1 and relation (5) we have

$$(23) \quad \deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} \Phi|_H(Y) = \frac{1}{|\ker \Phi_H \cap \operatorname{Stab} Y|} \deg_{\Phi_H^* \otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} Y$$

and from Theorem 3.4 and relation (3) we get

$$(24) \quad \frac{1}{|\ker \Phi_H \cap \operatorname{Stab} Y|} \deg_{\Phi_H^* \otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} Y \leq \alpha^{\dim Y} \deg_{\mathcal{L}} Y.$$

Thus

$$(25) \quad \deg_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}} \Phi|_H(Y) \leq \alpha^{\dim Y} \deg_{\mathcal{L}} Y.$$

Finally

$$(26) \quad \mu_{\otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}}(\Phi_H(Y)) = \mu_{\Phi_H^* \otimes_{\mathbf{I}} \Phi_{\mathbf{I}}^* \mathcal{M}}(Y) \leq \alpha \mu_{\mathcal{L}}(Y),$$

where the first equality comes from relation (6), while the second inequality by Theorem 3.4 and relation (7).

Notice that, by Lemma 3.1,  $\alpha \ll 1$ . Plugging inequalities (22), (25), (26) into (21), we deduce

$$\mu_{\mathcal{L}}(Y) \gg \frac{(\deg_{\mathcal{L}} H)^{\frac{1}{\operatorname{codim}_H Y} - \eta}}{(\deg_{\mathcal{L}} Y)^{\frac{1}{\operatorname{codim}_H Y} + \eta}}. \quad \square$$

We can now weaken the hypothesis on  $H$  and prove Theorem 1.3 for  $Y$  transverse in a translate of an abelian subvariety.

**Proof of Theorem 1.3.** We write the translate  $H$  as  $H = B + p$ , with  $B$  an abelian subvariety. Let  $c_1$  be the constant in Theorem 4.1; define

$$\theta = c_1 \frac{(\deg_{\mathcal{L}} B)^{\frac{1}{\operatorname{codim}_B Y} - \eta}}{(\deg_{\mathcal{L}} Y)^{\frac{1}{\operatorname{codim}_B Y} + \eta}}.$$

If the set of points of  $Y$  of height at most  $\frac{1}{4}\theta$  is empty then  $\mu_{\mathcal{L}}(Y) \geq \frac{1}{4}\theta$ . If not, choose a point  $q \in Y$  such that  $\hat{h}_{\mathcal{L}}(q) \leq \frac{1}{4}\theta$ . We now translate by  $-q$ , so to have that  $Y - q \subseteq B$ . Translations preserve transversality and degrees, and so by Theorem 4.1 for  $Y - q$ ,

$$\mu_{\mathcal{L}}(Y - q) \geq \theta.$$

If  $x \in Y$  and  $\hat{h}_{\mathcal{L}}(x) \leq \frac{1}{4}\theta$ , then  $x - q \in Y - q$  and

$$\hat{h}_{\mathcal{L}}(x - q) \leq 2\hat{h}_{\mathcal{L}}(x) + 2\hat{h}_{\mathcal{L}}(q) \leq \theta \leq \mu_{\mathcal{L}}(Y - q).$$

This shows that

$$\mu_{\mathcal{L}}(Y) \geq \frac{1}{4}\theta. \quad \square$$

We finally remark that the constant in Theorem 1.3 depends explicitly on the constant in bound (1). In particular, the constant of Theorem 1.4 depends explicitly on the one given by Galateau.

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## References

- [AD03] AMOROSO, F.; DAVID, S. Minoration de la hauteur normalisée dans un tore. *J. Inst. Math. Jussieu* **2** (2003), no. 3, 335–381. [MR1990219](#) (2004m:11101), [Zbl 1041.11048](#), doi: [10.1017/S1474748003000094](#).
- [AV09] AMOROSO, F.; VIADA, E. Small points on subvarieties of a torus. *Duke Math. J.* **150** (2009), no. 3, 407–442. [MR2582101](#) (2010m:11080), [Zbl 1234.11081](#), doi: [10.1215/00127094-2009-056](#).
- [BS04] BAKER, M. H.; SILVERMAN, J. H. A lower bound for the canonical height on abelian varieties over abelian extensions. *Math. Res. Lett.* **11** (2004), no. 2-3, 377–396. [MR2067482](#) (2005e:11083), [Zbl 1060.11041](#), [arXiv:math/0312393](#), doi: [10.1155/S1073792803212083](#).

- [Ber87] BERTRAND, D. Minimal heights and polarizations on abelian varieties. MSRI, preprint 06220-87, June 1987.
- [BL04] BIRKENHAKE, C.; LANGE, H. Complex abelian varieties. Second Edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 302. *Springer-Verlag, Berlin*, 2004. xii+635 pp. ISBN: 3-540-20488-1. [MR2062673](#) (2005c:14001), [Zbl 1056.14063](#).
- [BZ96] BOMBIERI, E.; ZANNIER, U. Heights of algebraic points on subvarieties of abelian varieties. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **23** (1996), no. 4, 779–792. [MR1469574](#) (98j:11043), [Zbl 0897.11020](#).
- [Car09] CARRIZOSA, M. Petits points et multiplication complexe. *Int. Math. Res. Not. IMRN* (2009), no. 16, 3016–3097. [MR2533796](#) (2011c:11102), [Zbl 1176.11025](#), doi: [10.1093/imrn/rnp042](#).
- [CVV] CHECCOLI, S.; VENEZIANO, F.; VIADA, E. On torsion anomalous intersections (with an appendix by P. Philippon). Preprint, 2012.
- [DH00] DAVID, S.; HINDRY, M. Minoration de la hauteur de Néron-Tate sur les variétés abéliennes de type C. M. *J. Reine Angew. Math.* **529** (2000), 1–74. [MR1799933](#) (2001j:11054), [Zbl 0993.11034](#).
- [DP98] DAVID, S.; PHILIPPON, P. Minorations des hauteurs normalisées des sous-variétés de variétés abéliennes. *Number theory (Tiruchirapalli, 1996)*, 333–364. *Contemp. Math.*, 210. *Amer. Math. Soc., Providence, RI*, 1998. [MR1478502](#) (98j:11044), [Zbl 0899.11027](#), doi: [10.1090/conm/210/02795](#).
- [DP02] DAVID, S.; PHILIPPON, P. Minorations des hauteurs normalisées des sous-variétés de variétés abéliennes. II. *Comment. Math. Helv.* **77** (2002), no. 4, 639–700. [MR1949109](#) (2004a:11055), [Zbl 1030.11026](#), doi: [10.1007/PL00012437](#).
- [DP07] DAVID, S.; PHILIPPON, P. Minorations des hauteurs normalisées des sous-variétés des puissances des courbes elliptiques. *Int. Math. Res. Pap. IMRP* (2007), no. 3, Art. ID rpm006, 113 pp. [MR2355454](#) (2008h:11068), [Zbl 1163.11049](#), doi: [10.1093/imrp/rpm006](#).
- [DMOyS82] DELIGNE, P.; MILNE, J. S.; OGUS, A.; SHIH, K. Hodge cycles, motives, and Shimura varieties. *Lecture Notes in Mathematics*, 900. *Springer-Verlag, Berlin-New York*, 1982. ii+414 pp. ISBN: 3-540-11174-3. [MR654325](#) (84m:14046), [Zbl 0465.00010](#).
- [Gal10] GALATEAU, A. Une minoration du minimum essentiel sur les variétés abéliennes. *Comment. Math. Helv.* **85** (2010), no. 4, 775–812. [MR2718139](#) (2011i:11110), [Zbl 05797236](#), [arXiv:0807.0171](#), doi: [10.4171/CMH/211](#).
- [Hin88] HINDRY, M. Autour d’une conjecture de Serge Lang. *Invent. Math.* **94** (1988), no. 3, 575–603. [MR969244](#) (89k:11046), [Zbl 0638.14026](#), doi: [10.1007/BF01394276](#).
- [Mas87] MASSER, D. W. Small values of heights on families of abelian varieties. *Diophantine approximation and transcendence theory (Bonn, 1985)*, 109–148. *Lecture Notes in Math.*, 1290. *Springer, Berlin*, 1987. [MR927559](#) (89g:11048), [Zbl 0639.14025](#), doi: [10.1007/BFb0078706](#).
- [MW85] MASSER, D. W.; WÜSTHOLZ, G. Zero estimates on group varieties. II. *Invent. Math.* **80** (1985), no. 2, 233–267. [MR788409](#) (86h:11054), [Zbl 0564.10041](#), doi: [10.1007/BF01388605](#).
- [MW93] MASSER, D.; WÜSTHOLZ, G. Periods and minimal abelian subvarieties. *Ann. of Math. (2)* **137** (1993), no. 2, 407–458. [MR1207211](#) (94g:11040), [Zbl 0796.11023](#), doi: [10.2307/2946542](#).

- [Mum70] MUMFORD, D. Abelian varieties. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. *Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London* 1970. viii+242 pp. [MR0282985](#) (44 #219), [Zbl 0223.14022](#).
- [PS08] PHILIPPON, P.; SOMBRA, M. Minimum essentiel et degrés d'obstruction des translatés de sous-tores. *Acta Arith.* **133** (2008), no. 1, 1–24. [MR2413362](#) (2009g:11079), [Zbl 1230.11085](#), doi: [10.4064/aa133-1-1](#).
- [Pin98] PINK, R.  $l$ -adic algebraic monodromy groups, cocharacters, and the Mumford-Tate conjecture. *J. Reine Angew. Math.* **495** (1998), 187–237. [MR1603865](#) (98m:11060), [Zbl 0920.14006](#), doi: [10.1515/crll.1998.018](#).
- [Ray83a] RAYNAUD, M. Courbes sur une variété abélienne et points de torsion. *Invent. Math.* **71** (1983), no. 1, 207–233. [MR688265](#) (84c:14021), [Zbl 0564.14020](#), doi: [10.1007/BF01393342](#).
- [Ray83b] RAYNAUD, M. Sous-variétés d'une variété abélienne et points de torsion. *Arithmetic and geometry, Vol. I*, 327–352. *Progr. Math.*, 35, *Birkhäuser Boston, Boston, MA*, 1983. [MR717600](#) (85k:14022), [Zbl 0581.14031](#).
- [Sch96] SCHMIDT, W. M. Heights of points on subvarieties of  $\mathbf{G}_m^n$ . *Number theory (Paris, 1993–1994)*, 157–187. *London Math. Soc. Lecture Note Ser.*, 235, *Cambridge Univ. Press, Cambridge*, 1996. [MR1628798](#) (99h:11070), [Zbl 0917.11023](#), doi: [10.1017/CBO9780511662003.008](#).
- [Ser89] SERRE, J-P. Abelian  $l$ -adic representations and elliptic curves. With the collaboration of Willem Kuyk and John Labute. Second edition. *Advanced Book Classics. Addison-Wesley Publishing Company, Advanced Book Program, Redwood City, CA*, 1989. xxiv+184 pp. ISBN: 0-201-09384-7. [MR1043865](#) (91b:11071), [Zbl 0709.14002](#).
- [Sil84] SILVERMAN, J. H. Lower bounds for height functions. *Duke Math. J* **51** (1984), no. 2, 395–403. [MR747871](#) (87d:11039), [Zbl 0579.14035](#), doi: [10.1215/S0012-7094-84-05118-4](#).
- [Ull98] ULLMO, E. Positivité et discrétion des points algébriques des courbes. *Ann. of Math.* (2) **147** (1998), no. 1, 167–179. [MR1609514](#) (99e:14031), [Zbl 0934.14013](#), doi: [10.2307/120987](#).
- [Via09] VIADA, E. Nondense subsets of varieties in a power of an elliptic curve. *Int. Math. Res. Not. IMRN* (2009), no. 7, 1213–1246. [MR2495303](#) (2010d:14031), [Zbl 1168.14030](#), doi: [10.1093/imrn/rnn157](#).
- [Via10] VIADA, E. Lower bounds for the normalized height and non-dense subsets of subvarieties of abelian varieties. *Int. J. Number Theory* **6** (2010), no. 3, 471–499. [MR2652892](#) (2011i:11111), [Zbl 1201.11080](#), [arXiv:0806.0487](#), doi: [10.1142/S1793042110003010](#).
- [Zha98] ZHANG, S-W. Equidistribution of small points on abelian varieties. *Ann. of Math.* (2) **147** (1998), no. 1, 159–165. [MR1609518](#) (99e:14032), [Zbl 0991.11034](#), doi: [10.2307/120986](#).

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