New York Journal of Mathematics

New York J. Math. **24** (2018) 389–403.

Power bounded composition operators on weighted Dirichlet spaces

Hamzeh Keshavarzi and Bahram Khani-Robati

Abstract. In this paper, we study power bounded composition operators on weighted Dirichlet spaces $\mathcal{D}_\alpha$. As applications, we give the necessary and sufficient conditions for the composition operators to be Riesz operator on $\mathcal{D}_\alpha$, when C_φ is power bounded on $\mathcal{D}_\beta$, for some $0 < \beta < \alpha$. For $\alpha > 1$, we completely characterize the Riesz composition operators on $\mathcal{D}_\alpha$. Moreover, we investigate the functions $f \in \mathcal{D}_\alpha$, when $f \circ \varphi_n$ is convergent or $\lim_{n\to\infty} f \circ \varphi_n = 0$, in $\mathcal{D}_\alpha$. Some of the techniques developed in the paper are not new but lead to new results.

Contents

1. Introduction

Let $\mathbb{D}$ be the unit disk in the complex plane and $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. Let φ be a function analytic on the unit disk such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. A composition operator on $H(\mathbb{D})$ is defined by $C_\varphi f = f \circ \varphi$ for every $f \in H(\mathbb{D})$.

An operator T, on a Hilbert space H, is called power bounded if $\{T^n\}$ is a bounded sequence in $B(H)$, the space of all bounded operators on H. Many authors studied the power bounded composition operators on different spaces, see [1, 2, 3, 4, 8, 15, 16]. In this paper, we study these operators on weighted Dirichlet spaces $\mathcal{D}_\alpha$, when $-1 < \alpha < 1$.

Received October 19, 2017.

2010 *Mathematics Subject Classification.* Primary: 47B33; Secondary: 47B38, 47B06.

Key words and phrases. power bounded, composition operators, weighted Dirichlet space, Riesz operator.

ISSN 1076-9803/2018

The operator $T : H \longrightarrow H$ is said to be a Riesz operator if

$$\lim_{n\to\infty} \|T^n\|_e^{1/n} = 0.$$

Where $\|.\|_e$ denotes the essential norm on H. J. H. Shapiro and P. D. Taylor in [14] have shown that if C_φ is compact on H^2, then φ cannot have an angular derivative at any point of the boundary of the unit disk. Using Carleson measure techniques, MacCluer and Shapiro [11] proved the Shapiro-Taylor result in the more general setting of the weighted Dirichlet spaces, $\mathcal{D}_\alpha$, and showed that, for composition operators C_φ acting on $A^p_\alpha(\alpha > -1)$, the non-existence of the angular derivative for φ is also sufficient condition for compactness of the composition operator C_φ. In this paper, we show that the Riesz composition operators, also, have a straight relationship with the angular derivative. Indeed, we prove that if $0 < \beta < \alpha$ and C_φ is power bounded on $\mathcal{D}_\beta$, then

$$C_\varphi \text{ is a Riesz operator on } \mathcal{D}_\alpha \iff \lim_{n\to\infty} \Big(\min_{\zeta\in\partial\mathbb{D}} d(\zeta,\varphi_n)\Big)^{\frac{1}{n}} = \infty,$$

where φ_n denotes the n-th iterate of φ and $d(\zeta,\varphi_n)$ is the angular derivative of φ_n at ζ. Moreover, we show that when $\alpha > 1$, the above statement holds without assuming the power boundedness of C_φ. In [5] and [13], some results about Riesz composition operators have been given.

Our manuscript is organized as follows: In section 3, we give the necessary and sufficient conditions for the power boundedness of composition operators on $\mathcal{D}_\alpha$. In Theorem 3.2, the characterization is done by using Carleson measure. In Theorem 3.4 we give another characterization for the power boundedness of composition operators on $\mathcal{D}_\alpha$ when $0 < \alpha < 1$. In section 4, we investigate the Riesz composition operators on $\mathcal{D}_\alpha$. As an another application, for a power bounded composition operator C_φ on $\mathcal{D}_\alpha$, we characterize the following sets

$$\mho_{c,\alpha}(\varphi) = \{f \in \mathcal{D}_\alpha : \ C_{\varphi_n} f \ \textit{is convergent}\}$$

and

$$\mho_{0,\alpha}(\varphi) = \{f \in \mathcal{D}_\alpha : \ \lim_{n\to\infty} \|C_{\varphi_n} f\| = 0\}.$$

Finally in section 5, we present several examples related to our results.

Throughout this paper, $A(z) \lesssim B(z)$ on a set S means that there exists some positive constant C such that for each $z \in S$, we have $A(z) \leq CB(z)$. Also we use the notation $A(z) \asymp B(z)$ on S, to say that there are some positive constants C and D such that $CB(z) \leq A(z) \leq DB(z)$ for each $z \in S$.

2. Preliminaries

Let $\alpha > -1$, the weighted Bergman space A_α is the space of all $f \in H(\mathbb{D})$ for which

$$||f||^2_{A_\alpha} = \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^\alpha dA(z) < \infty,$$

where A is the normalized area measure on $\mathbb{D}$. Also, the space of all analytic functions on the unit disk $\mathbb{D}$, whose derivatives are in A_α with the norm given by

$$||f||_\alpha^2 = |f(0)|^2 + ||f'||_{A_\alpha}^2,$$

is called the weighted Dirichlet space and is denoted by $\mathcal{D}_\alpha$. These spaces with the above norms are Hilbert spaces. The space $\mathcal{D}_\alpha$ is a reproducing kernel Hilbert space with kernel functions

$$K_w(z) = \sum_{k=0}^{\infty} \frac{\overline{w}^k z^k}{(k+1)^{1-\alpha}} \quad \textit{and} \quad \|K_w\|_\alpha^2 = \sum_{k=0}^{\infty} \frac{|w|^{2k}}{(k+1)^{1-\alpha}}.$$

Which means that the functions K_w are in $\mathcal{D}_\alpha$ for all $w \in \mathbb{D}$ and $\langle f, K_w \rangle = f(w)$. Also evaluation of the derivative of functions in $\mathcal{D}_\alpha$ at w is a bounded linear functional and $\langle f, K'_w \rangle = f'(w)$, where by [6, Theorem 2.16]

$$K'_w(z) = \sum_{k=1}^{\infty} k \frac{\overline{w}^{k-1} z^k}{(k+1)^{1-\alpha}} \quad \textit{and} \quad \|K'_w\|_\alpha^2 = \sum_{k=1}^{\infty} k^2 \frac{|w|^{2(k-1)}}{(k+1)^{1-\alpha}}.$$

For $\alpha > 0$ we can see that

$$\|K_w\|_\alpha^2 \asymp \frac{1}{(1-|w|^2)^\alpha} \quad \textit{and} \quad \|K'_w\|_\alpha^2 \asymp \frac{1}{(1-|w|^2)^{\alpha+2}}.$$

The pseudohyperbolic distance between the points z and a in $\mathbb{D}$ is defined as $\rho(z,a) = |\varphi_a(z)|$, where $\varphi_a(z) = \frac{a-z}{1-\overline{a}z}$. The pseudohyperbolic disk with center a and radius $r \in (0,1)$ is

$$\Delta(a,r) = \{z : \ \rho(z,a) < r\} = \varphi_a(\Delta(0,r)) = \varphi_a(\{z : \ |z| < r\}).$$

For φ an analytic self-map of the unit disk and $w \neq \varphi(0)$, a point of the plane, let $z_j(w)$ be the points of the disk for which $\varphi(z_j(w)) = w$, with their multiplicities. Let $\alpha > -1$, the generalized Nevanlinna counting function is

$$N_{\varphi,\alpha}(w) = \sum_j (1 - |z_j(w)|^2)^\alpha,$$

where we understand $N_{\varphi,\alpha}(w) = 0$ for w which is not in $\varphi(\mathbb{D})$. For convenience, we introduce two notations:

- $\frac{u.c}{f_n \longrightarrow f}$, that is, the sequence $\{f_n\}$ converges to f uniformly on compact subsets of $\mathbb{D}$.
- $\frac{\mathcal{D}_\alpha}{f_n \longrightarrow f}$, that is, the sequence $\{f_n\}$ converges to f in the norm of $\mathcal{D}_\alpha$.

The following theorems are key theorems of this paper, for the proofs see [6, Theorem 2.35, Theorem 2.44 and Theorem 2.51].

Theorem 2.1. *(Julia-Carathéodory Theorem) For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and ζ in $\partial\mathbb{D}$, the following are equivalent:*

1. $d(\zeta, \varphi) = \liminf_{z \to \zeta} (1 - |\varphi(z)|)/(1 - |z|) < \infty$,
2. *φ has finite angular derivative $\varphi'(\zeta)$ at ζ.*
3. *Both φ and φ' have finite nontangential limits at ζ, with $|\eta| = 1$ for $\eta = \lim_{r \to 1} \varphi(r\zeta)$.*

Moreover, when these conditions hold, we have $\lim_{r\to 1}\varphi'(r\zeta) = \varphi'(\zeta) = d(\zeta,\varphi)\overline{\zeta}\eta$ *and* $d(\zeta,\varphi)$ *is the nontangential limit* $\lim_{z\to\zeta}(1-|\varphi(z)|)/(1-|z|)$.

Theorem 2.2 (Denjoy-Wolff Theorem)**.** *If* φ*, not the identity and not an elliptic automorphism of* $\mathbb{D}$*, is an analytic map of unit disk into itself, then there is a point* w *in* $\overline{\mathbb{D}}$ *so that* $\frac{u.c}{\varphi_n \longrightarrow w}$.

The point in the above theorem is called the Denjoy-Wolff point of φ. Indeed, the Denjoy-Wolff point of φ can be described as the unique fixed point of φ in $\overline{\mathbb{D}}$ with $|\varphi'(a)| \le 1$, see [6, page 59] .

Theorem 2.3 (Change of Variable Theorem)**.** *If* g *and* W *are non-negative measurable functions on* $\mathbb{D}$ *and* φ *is a holomorphic self-map of* $\mathbb{D}$*, then*

$$\int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^2 W(z)dA(z) = \int_{\varphi(\mathbb{D})} g(w)N_{\varphi,W}(w)dA(w).$$

3. Conditions for Power Boundedness

In this section, we characterize the power bounded composition operators on weighted Dirichlet spaces $\mathcal{D}_\alpha$, when $-1 < \alpha < 1$. When $\alpha \ge 1$, the result is obvious. Indeed, if $\alpha \ge 1$, then $\mathcal{D}_\alpha = A_{\alpha-2}$ and their norms are equivalent and

$$\left(\frac{1}{1-|\varphi_n(0)|^2}\right)^\alpha \le \|C_{\varphi_n}\|^2_{A_{\alpha-2}} \le \left(\frac{1+|\varphi_n(0)|^2}{1-|\varphi_n(0)|^2}\right)^\alpha. \tag{3.1}$$

Therefore, if φ has a Denjoy-Wolff point, then C_φ is power bounded on $\mathcal{D}_\alpha$, for $\alpha \ge 1$, if and only if the Denjoy-Wolff point of φ is in $\mathbb{D}$. It is clear that if φ is the identity or an elliptic automorphism of $\mathbb{D}$ and $\alpha > -1$, then C_φ is power bounded. Indeed there are some $\lambda \in \partial\mathbb{D}$ and some disk automorphism φ_a such that $\psi = \varphi_a \circ \varphi \circ \varphi_a(z) = \lambda z$. So $\psi_n = \varphi_a \circ \varphi_n \circ \varphi_a(z) = \lambda^n z$. Throughout this paper, φ is an analytic self-map of $\mathbb{D}$ which is not the identity and not an elliptic automorphism, so φ has a Denjoy-Wolff point. Now we are going to prove our main results. First, we need the following lemma.

Lemma 3.1. (i) [7, Lemma 4, page 42] *In each pseudohyperbolic disk* $\Delta(a,r)$*, the function* $k_a(z) = (1-\overline{a}z)^{-2}$ *satisfies the sharp inequalities*

$$\left(\frac{1-r|a|}{1-|a|^2}\right)^2 \le |k_a(z)| \le \left(\frac{1+r|a|}{1-|a|^2}\right)^2, \qquad \textit{for all } z \textit{ in } \Delta(a,r).$$

(ii) [17, Proposition 4.5] *If* $r \in (0,1)$ *is fixed and* $z \in \Delta(a,r)$*, then*

$$A(\Delta(z,r)) \asymp (1-|z|^2)^2 \asymp (1-|a|^2)^2 \asymp A(\Delta(a,r)).$$

(iii) [7, Lemma 12, page 62] *For each pseudohyperbolic radius* $r \in (0,1)$*, there exists a sequence* $\{a_k\}$ *of points in* $\mathbb{D}$ *and an integer* N *such* *that*

$$\bigcup_{k=1}^{\infty} \Delta(a_k, r) = \mathbb{D}$$

and no point $z \in \mathbb{D}$ *belong to more than* N *of the dilated disk* $\Delta(a_k, R)$*, where* $R = \frac{1}{2}(1 + r)$*.*

(*iv*) [7, Lemma 13, page 63] *If* $0 < r < 1$*, and* f *is analytic in* $\mathbb{D}$*, then for arbitrary* $a \in \mathbb{D}$ *and for all* $z \in \Delta(a, r)$*,*

$$|f(z)|^2 \leq \frac{4(1-R)^{-4}}{|\Delta(a,R)|} \int_{\Delta(a,R)} |f(\zeta)|^2 dA(\zeta), \qquad where\ R = \frac{1}{2}(1+r).$$

(*v*) [9, Theorem 1.7] *Independently of* a *in* $\mathbb{D}$*,*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^c dA(z)}{|1-\overline{a}z|^{2+c+d}} \asymp \frac{1}{(1-|a|^2)^d}, \qquad if\ d > 0,\ c > -1.$$

Theorem 3.2. *Let* φ *be an analytic self-map of the unit disk which is not the identity or an elliptic automorphism. Then*

(*i*) *if* $0 \leq \alpha < 1$*, then* C_φ *is power bounded on* $\mathcal{D}_\alpha$ *if and only if* φ *has its Denjoy-Wolff point in* $\mathbb{D}$ *and for every* $0 < r < 1$*,*

$$\sup_{n\in\mathbb{N}, a\in\mathbb{D}} \frac{\int\limits_{\Delta(a,r)} N_{\varphi_n,\alpha}(z) dA(z)}{(1-|a|^2)^{\alpha+2}} < \infty; \tag{3.2}$$

(*ii*) *if* $-1 < \alpha < 0$*, then* C_φ *is power bounded on* $\mathcal{D}_\alpha$ *if and only if for all* $0 < r < 1$*, Equation* 3.2 *holds.*

Proof. (*i*): Let C_φ be power bounded on $\mathcal{D}_\alpha$. Hence, there is some positive constant C such that for any f in the unit ball of $\mathcal{D}_\alpha$ and $n \in \mathbb{N}$, $|f(\varphi_n(0))| < C$. Thus, if $n \in \mathbb{N}$, then $\| K_{\varphi_n(0)} \| \leq C$. But we know that $\lim_{|z|\to 1} \|K_z\| = \infty$, hence there exists some $0 < r < 1$ such that $\varphi_n(0) \in r\mathbb{D}$, $n \in \mathbb{N}$. If $w \in \overline{D}$ is the Denjoy-Wolff point of φ, then $\lim_{n\to\infty} \varphi_n(0) = w$. Therefore, w must be in $\mathbb{D}$. Now we show that Equation 3.2 holds. Let

$$f_a(z) = (1-|a|^2)^{1+\frac{\alpha}{2}} \int_0^z \frac{d\zeta}{(1-\overline{a}\zeta)^{2+\alpha}}.$$

So $f_a'(z) = \dfrac{(1-|a|^2)^{1+\frac{\alpha}{2}}}{(1-\overline{a}z)^{2+\alpha}}$. By using power boundedness of C_φ and Lemma 3.1, part (v),

$$\begin{aligned}\frac{\int\limits_{\Delta(a,r)} N_{\varphi_n,\alpha}(z)dA(z)}{(1-|a|^2)^{\alpha+2}} &\lesssim \int\limits_{\Delta(a,r)} \frac{(1-|a|^2)^{\alpha+2}}{|1-\overline{a}z|^{4+2\alpha}} N_{\varphi_n,\alpha}(z)dA(z)\\ &\le \int\limits_{\mathbb{D}} \frac{(1-|a|^2)^{\alpha+2}}{|1-\overline{a}z|^{4+2\alpha}} N_{\varphi_n,\alpha}(z)dA(z) \le \|f_a\circ\varphi_n\|_\alpha^2\\ &\lesssim \|f_a\|_\alpha^2 = \int\limits_{\mathbb{D}} \frac{(1-|a|^2)^{\alpha+2}}{|1-\overline{a}z|^{4+2\alpha}}(1-|z|^2)^\alpha dA(z) \approx 1.\end{aligned}$$

Conversely, let w in $\mathbb{D}$ be the Denjoy-Wolff point of φ and Equation (3.1) holds. So, $\lim_{n\to\infty}\varphi_n(0) = w$. Thus, there is some $0 < r < 1$ such that $\{\varphi_n(0)\}_{n\in\mathbb{N}} \subseteq r\mathbb{D}$. Therefore, for f in the unit ball of $\mathcal{D}_\alpha$

$$|f(\varphi_n(0))|^2 \le \|K_{\varphi_n(0)}\|_\alpha^2 \le \|K_r\|_\alpha^2.$$

Let $\{a_k\}$ be the sequence in Lemma 3.1, part (ii). By using Lemma 3.1, Fubini's theorem and Equation 3.2,

$$\int\limits_{\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \le \sum_{k=1}^{\infty} \int\limits_{\Delta(a_k,r)} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z)$$

$$\begin{aligned}&\lesssim \sum_{k=1}^{\infty} \frac{1}{(1-|a_k|^2)^2} \int\limits_{\Delta(a_k,r)}\int\limits_{\Delta(a_k,R)} |f'(\zeta)|^2 N_{\varphi_n,\alpha}(z)dA(\zeta)dA(z)\\ &\lesssim \sum_{k=1}^{\infty} \int\limits_{\Delta(a_k,R)} |f'(\zeta)|^2 \Big(\frac{\int\limits_{\Delta(a_k,r)} N_{\varphi_n,\alpha}(z)dA(z)}{(1-|a_k|^2)^{\alpha+2}}\Big)(1-|\zeta|^2)^\alpha dA(\zeta)\\ &\lesssim \sum_{k=1}^{\infty} \int\limits_{\Delta(a_k,R)} |f'(\zeta)|^2(1-|\zeta|^2)^\alpha dA(\zeta) \le N.\end{aligned}$$

Therefore, there is some $C > 0$ such that

$$\|f\circ\varphi_n\|_\alpha^2 = |f(\varphi_n(0))|^2 + \int\limits_{\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \le \| K_r \|_\alpha^2 + CN.$$

(ii): Let f be in the unit ball of $\mathcal{D}_\alpha$. Then

$$|f\circ\varphi_n(0)|^2 \le \|K_{\varphi_n(0)}\|^2 = \sum_{j=0}^{\infty} \frac{|\varphi_n(0)|^{2j}}{(j+1)^{1-\alpha}} \le \sum_{j=0}^{\infty} \frac{1}{(j+1)^{1-\alpha}}.$$

Since $\alpha < 0$

$$\sum_{j=0}^{\infty} \frac{1}{(j+1)^{1-\alpha}} < \infty.$$

Therefore, C_φ is power bounded if and only if

$$\sup_{n\in\mathbb{N}, f\in Ball\mathcal{D}_\alpha} \int_{\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z) dA(z) < \infty.$$

Similar to the proof of part (i) we can show that the above inequality is equivalent to Inequality 3.2. □

By using the following proposition, we give a better characterization, Theorem 3.4, for the power boundedness of composition operators on $\mathcal{D}_\alpha$, when $0 < \alpha < 1$.

Proposition 3.3. [12, Proposition 2.1] *Let $0 < \alpha < 1$ and $0 < p < \infty$. Suppose that φ be an analytic self-map of the unit disk. Then there is a positive constant $C = C_p < \infty$ such that*

$$N_{\varphi,\alpha}(\zeta)^p \le \frac{C}{|B|} \int_B N_{\varphi,\alpha}(w)^p dA(w),$$

where $\zeta \in \mathbb{D} \setminus \{\varphi(0)\}$ and B is any Euclidean disk centered at ζ contained in $\mathbb{D} \setminus \{\varphi(0)\}$. Moreover, one can take $C = 1$ if $p \ge 1$.

Theorem 3.4. *Let $0 < \alpha < 1$ and φ be an analytic self-map of the unit disk which is not the identity or an elliptic automorphism with w as its Denjoy-Wolff point. Then C_φ is power bounded on $\mathcal{D}_\alpha$ if and only if*

- *w is in $\mathbb{D}$,*
- *$\{\varphi_n\}$ is a bounded sequence in $\mathcal{D}_\alpha$,*
- *there exists some $C > 0$ such that if $n \in \mathbb{N}$ and $|a| \ge \frac{1+|\varphi_n(0)|}{2}$, then $\frac{N_{\varphi_n,\alpha}(a)}{(1-|a|^2)^\alpha} < C$.*

Proof. Let C_φ be power bounded. By using the preceding theorem, w must be in $\mathbb{D}$. Since $\varphi_n = C_{\varphi_n} z$, the second condition also holds. For the third condition, suppose that $|a| > \frac{1+|\varphi_n(0)}{2}$ and $D(a) = \{z : \ |z-a| < \frac{1}{2}(1-|a|)\}$. Easily we can see that every point in $D(a)$ has modulus greater than $|\varphi_n(0)|$.

Therefore, by Proposition 3.3 and Lemma 3.1,

$$\begin{aligned}
\frac{N_{\varphi_n,\alpha}(a)}{(1-|a|^2)^{\alpha}} &\leq \frac{\int\limits_{D(a)} N_{\varphi_n,\alpha}(z)dA(z)}{(1-|a|^2)^{\alpha+2}} \\
&\lesssim \int\limits_{D(a)} \frac{(1-|a|^2)^{\alpha+2}}{|1-\overline{a}z|^{4+2\alpha}} N_{\varphi_n,\alpha}(z)dA(z) \\
&\leq \int\limits_{\mathbb{D}} \frac{(1-|a|^2)^{\alpha+2}}{|1-\overline{a}z|^{4+2\alpha}} N_{\varphi_n,\alpha}(z)dA(z) \\
&\lesssim \int\limits_{\mathbb{D}} \frac{(1-|a|^2)^{\alpha+2}}{|1-\overline{a}z|^{4+2\alpha}} (1-|z|^2)^{\alpha}dA(z) \asymp 1.
\end{aligned}$$

Conversely, let the above conditions hold. Let f be in the unit ball of $\mathcal{D}_\alpha$. Then

$$\begin{aligned}
\int\limits_{\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) &= \int\limits_{|z|\geq \frac{1+|\varphi_n(0)|}{2}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \\
&+ \int\limits_{|z|\leq \frac{1+|\varphi_n(0)|}{2}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \\
&\leq C \int\limits_{|z|\geq \frac{1+|\varphi_n(0)|}{2}} |f'(z)|^2 (1-|z|^2)^{\alpha}dA(z) \\
&+ ||K'_{\frac{1+|\varphi_n(0)|}{2}}||_{\alpha}^2 \int\limits_{|z|\leq \frac{1+|\varphi_n(0)|}{2}} N_{\varphi_n,\alpha}(z)dA(z) \\
&\leq C + ||K'_{\frac{1+|\varphi_n(0)|}{2}}||_{\alpha}^2 ||\varphi_n||_{\alpha}^2.
\end{aligned}$$

Where the first two conditions of the theorem show that the last quantity is bounded above. □

Remark 3.5. *In Example 5.3, we present an analytic self-map of the unit disk which has its Denjoy-Wolff point in $\mathbb{D}$, but is not power bounded on $\mathcal{D}_\alpha$, when $0 < \alpha < 1$. Also, we give another analytic self-map of the unit disk in Example 5.5 whose Denjoy-Wolff point is in the unit circle, however, it is power bounded on $\mathcal{D}_\alpha$, for $-1 < \alpha < 0$.*

Remark 3.6. *By Lemmas* [10, Lemma 2.2 and Lemma 2.3], *if $\alpha > 0$ and $\varphi(0) = 0$, then*

$$N_{\varphi,\alpha}(\zeta) \leq \frac{2\pi}{|B|}\int_B N_{\varphi,\alpha}(w)dA(w),$$

where $\zeta \in \mathbb{D}\setminus\{0\}$ and B is any Euclidean disk centered at ζ contained in $\mathbb{D}\setminus\{0\}$. Now if $\varphi(0) \neq 0$, then by [10, Lemma 2.1], *there exists some positive* *constant $C(\alpha)$ depending only on α such that*

$$N_{\varphi,\alpha}(\zeta) \leq \frac{C(\alpha)}{|B|(1-|\varphi(0)|^2)^\alpha} \int_B N_{\varphi,\alpha}(w) dA(w).$$

Therefore, by using an argument similar to the proof of Theorem 3.4, we can show that if C_φ is power bounded on $\mathcal{D}_\alpha$, $\alpha > 0$, then there exists some $C > 0$ such that if $n \in \mathbb{N}$ and $|a| \geq \frac{1+|\varphi_n(0)|}{2}$, then $\frac{N_{\varphi_n,\alpha}(a)}{(1-|a|^2)^\alpha} < C$.

4. Applications

In this section, we give some applications of our results obtained from the preceding section.

4.1. Riesz composition operators. We denote by $\|.\|_{e,\alpha}$ the essential norm of operators on $\mathcal{D}_\alpha$. Pau and Perez in [12, Theorem 3.2], for $0 < \alpha < 1$, independently of φ, showed that

$$\|C_\varphi\|_{e,\alpha}^2 \asymp \limsup_{|z|\to 1} \frac{N_{\varphi,\alpha}(z)}{(1-|z|)^\alpha}.$$

By using [6, page 136] and Remark 3.6 and with an argument similar to the proof of [12, Theorem 3.2] we can show that the above inequality is also true for $\alpha \geq 1$. Thus, if $\alpha > 0$, then C_φ is a Riesz operator on $\mathcal{D}_\alpha$ if and only if

$$\lim_{n\to\infty} \left(\limsup_{|z|\to 1} \frac{N_{\varphi_n,\alpha}(z)}{(1-|z|)^\alpha} \right)^{\frac{1}{2n}} = 0. \tag{4.1}$$

Theorem 4.1. *Let $0 < \beta < \alpha$ and C_φ be power bounded on $\mathcal{D}_\beta$. Then C_φ is a Riesz operator on $\mathcal{D}_\alpha$ if and only if*

$$\lim_{n\to\infty} \left(\min_{\zeta\in\partial\mathbb{D}} d(\zeta, \varphi_n) \right)^{\frac{1}{n}} = \infty. \tag{4.2}$$

Proof. ($\Leftarrow$): Let $r = \sup_{n\in\mathbb{N}} \frac{1+|\varphi_n(0)}{2}$, so $\frac{1}{2} \leq r < 1$. By using Remark 3.6, there is some $C > 0$ such that if $a \in \mathbb{D} \setminus r\mathbb{D}$, then

$$\frac{N_{\varphi_n,\beta}(a)}{(1-|a|^2)^\beta} \leq C.$$

Also, let $z(a)$ be a point in $\mathbb{D}$ with minimum modulus where $\varphi(z(a)) = a$. Hence

$$\begin{aligned} \limsup_{|a|\to 1} \frac{N_{\varphi_n,\alpha}(a)}{(1-|a|)^\alpha} &\leq \limsup_{|a|\to 1} \Big(\frac{(1-|z(a)|^2}{1-|a|^2}\Big)^{\alpha-\beta} \frac{N_{\varphi_n,\beta}(a)}{(1-|a|^2)^\beta} \\ &\leq C\Big(\frac{(1-|z(a)|^2}{1-|a|^2}\Big)^{\alpha-\beta} \leq C \limsup_{|z|\to 1} \Big(\frac{1-|z|^2}{1-|\varphi_n(z)|^2}\Big)^{\alpha-\beta} \\ &= \frac{C}{\min_{\zeta\in\partial\mathbb{D}} d(\zeta, \varphi_n)^{\alpha-\beta}}. \end{aligned}$$

Therefore,

$$\lim_{n\to\infty}\Big(\limsup_{|a|\to 1}\frac{N_{\varphi_n,\alpha}(a)}{(1-|a|)^{\alpha}}\Big)^{\frac{1}{2n}} \le \lim_{n\to\infty}\Big(\min_{\zeta\in\partial\mathbb{D}} d(\zeta,\varphi_n)\Big)^{-\frac{\alpha-\beta}{n}} = 0.$$

($\Rightarrow$): it is trivial by the known estimate

$$\|C_{\varphi_n}\|_{e,\alpha} \ge \limsup_{|z|\to 1}\Big(\frac{1-|z|^2}{1-|\varphi_n(z)|^2}\Big)^{\frac{\alpha}{2}}.$$

□

Corollary 4.2. *Let $\alpha > 1$ and φ be an analytic self-map of the unit disk which is not the identity or an elliptic automorphism. Then C_φ is a Riesz operator on $\mathcal{D}_\alpha$ if and only if Equation 4.2 holds.*

Proof. We show that both of our conditions imply that φ has its Denjoy-Wolff point in $\mathbb{D}$. So C_φ is power bounded on every $\mathcal{D}_\beta$, where $\beta > 1$. Then by using Theorem 4.1, the proof is complete. Let C_φ be a Riesz operator on $\mathcal{D}_\alpha$ and w, the Denjoy-Wolff point of φ, be in the unit circle. We can easily see that w is the Denjoy-Wolff point of any iterate function φ_n and $d(\zeta,\varphi_n) \le 1$. Hence,

$$\begin{aligned}\|C_{\varphi_n}\|_{e,\alpha} &\ge \limsup_{|z|\to 1}\Big(\frac{1-|z|^2}{1-|\varphi_n(z)|^2}\Big)^{\frac{\alpha}{2}}\\ &= \Big(\frac{1}{\min_{\zeta\in\partial\mathbb{D}} d(\zeta,\varphi_n)}\Big)^{\frac{\alpha}{2}} \ge \Big(\frac{1}{d(w,\varphi_n)}\Big)^{\frac{\alpha}{2}} \ge 1.\end{aligned}$$

This contradicts the assumption that C_φ is a Riesz operator. Thus, w is in $\mathbb{D}$. Now let Equation 4.2 hold. Hence, the angular derivative of φ_n at any point of unit circle converges to infinity as $n \to \infty$. Thus, again the Denjoy-Wolff point of φ cannot be in $\partial\mathbb{D}$. □

4.2. Characterization of sets $\mho_{c,\alpha}(\varphi)$ and $\mho_{0,\alpha}(\varphi)$. For a positive constant δ and an analytic function f on $\mathbb{D}$ we define

$$\Omega_\delta(f) = \{z \in \mathbb{D} : \ |f(z)|^2(1-|z|^2)^{\alpha+2} \ge \delta\}.$$

Theorem 4.3. *Let $\alpha > 0$, φ be an analytic self-map of $\mathbb{D}$ with Denjoy-Wolff point w and let C_φ be power bounded on $\mathcal{D}_\alpha$. Then f is in $\mho_{c,\alpha}(\varphi)$ if and only if for each $\delta > 0$,*

$$\lim_{n\to\infty}\int_{\Omega_\delta(f')}\frac{N_{\varphi_n,\alpha}(z)dA(z)}{(1-|z|^2)^{\alpha+2}} = 0. \tag{4.3}$$

Moreover, f is in $\mho_{0,\alpha}(\varphi)$ if and only if $f(w) = 0$ and equation 4.3 holds.

Proof. Let f be in $\mathcal{D}_\alpha$. Since w is the Denjoy-Wolff point of φ, we have $\overline{f\circ\varphi_n \overset{u.c}{\longrightarrow} f(w)}$. Thus, f is in $\mho_{c,\alpha}(\varphi)$ if and only if

$$\lim_{n\to\infty}\int_{\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) = 0.$$

If for some $\delta > 0$, Equation 4.3 does not hold, then there is a sequence $\{n_k\}$ in $\mathbb{N}$ and some positive constant ε such that for any $k \in \mathbb{N}$ we have

$$\int_{\Omega_\delta(f')} \frac{N_{\varphi_{n_k},\alpha}(z)dA(z)}{(1-|z|^2)^{\alpha+2}} > \varepsilon.$$

Thus

$$\begin{aligned}\int_{\mathbb{D}} |f'(z)|^2 N_{\varphi_{n_k},\alpha}(z)dA(z) &\geq \int_{\Omega_\delta(f')} |f'(z)|^2 N_{\varphi_{n_k},\alpha}(z)dA(z)\\ &\geq \delta \int_{\Omega_\delta(f')} \frac{N_{\varphi_{n_k},\alpha}(z)}{(1-|z|^2)^{\alpha+2}}dA(z) > \delta\varepsilon.\end{aligned}$$

Conversely, let f be in $\mathcal{D}_\alpha$ such that Equation 4.3 holds. Let $\varepsilon > 0$ be arbitrary. We choose $0 < \delta < \varepsilon$ sufficiently small such that

$$\int_{\Omega_\delta(f')^c} |f'(z)|^2(1-|z|^2)^\alpha dA(z) < \varepsilon.$$

Now for this δ, there is some $N \in \mathbb{N}$ such that for each $n \geq N$

$$\int_{\Omega_\delta(f')} \frac{N_{\varphi_n,\alpha}(z)}{(1-|z|^2)^{\alpha+2}}dA(z) < \varepsilon.$$

Thus,

$$\int_{\Omega_\delta(f')} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \lesssim \|f\|^2 \int_{\Omega_\delta(f')} \frac{N_{\varphi_n,\alpha}(z)}{(1-|z|^2)^{\alpha+2}}dA(z) < \varepsilon\|f\|^2.$$

Also,

$$\begin{aligned}
\int\limits_{\Omega_\delta(f')^c} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) &= \int\limits_{\Omega_\delta(f')^c\cap r\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \\
&+ \int\limits_{\Omega_\delta(f')^c\setminus r\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \\
&< \delta \int\limits_{\Omega_\delta(f')^c\cap r\mathbb{D}} \frac{N_{\varphi_n,\alpha}(z)}{(1-|z|^2)^{\alpha+2}} dA(z) \\
&+ C \int\limits_{\Omega_\delta(f')^c\setminus r\mathbb{D}} |f'(z)|^2(1-|z|^2)^\alpha dA(z) \\
&\le \varepsilon \frac{\|\varphi_n\|^2}{(1-r^2)^{\alpha+2}} + C\varepsilon.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int\limits_{\mathbb{D}} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) &= \int\limits_{\Omega_\delta(f')} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \\
&+ \int\limits_{\Omega_\delta(f')^c} |f'(z)|^2 N_{\varphi_n,\alpha}(z)dA(z) \\
&\le (\|f\|_\alpha^2 + \frac{\|\varphi_n\|^2}{(1-r^2)^{\alpha+2}} + C)\varepsilon.
\end{aligned}$$

□

5. Examples

A well-known fact is that if C_φ is compact on $\mathcal{D}_\alpha$ then φ has its Denjoy-Wolff point w in $\mathbb{D}$. So for $\alpha \ge 1$, if C_φ is compact on $\mathcal{D}_\alpha$ then it is power bounded.

Example 5.1. *Let $-1 < \alpha < 0$ and φ be an analytic self-map of the unit disk. If C_φ is compact on $\mathcal{D}_\alpha$, then it is power bounded.*

Proof. Since C_φ is compact, we have $\overline{\varphi(\mathbb{D})} \subset \mathbb{D}$. Thus, there is some positive constant C such that

$$\|K'_{\varphi_n(z)}\|_\alpha^2 \le C, \qquad \forall z \in \mathbb{D},\ \forall n \in \mathbb{N}.$$

Also, $\overrightarrow{\varphi_n \quad w}^{u.c}$, so $\overrightarrow{\varphi_n' \quad 0}^{u.c}$. Hence, there exists a $D > 0$ such that

$$|\varphi_n'(\varphi(z))|^2 \le D, \qquad \forall z \in \mathbb{D},\ \forall n \in \mathbb{N}.$$

Finally, if f is in the unit ball of $\mathcal{D}_\alpha$, then

$$\int_{\mathbb{D}} |f'(\varphi_{n+1}(z))|^2 |\varphi_{n+1}'(z)|^2 (1-|z|^2)^\alpha dA(z)$$

$$\leq \int_{\mathbb{D}} \|K'_{\varphi_{n+1}(z)}\|_\alpha^2 |\varphi_n'(\varphi(z))|^2 |\varphi'(z)|^2 (1-|z|^2)^\alpha dA(z)$$
$$\leq CD\|\varphi\|_\alpha^2.$$

□

Example 5.2. *Let $\alpha > -1$, φ be an analytic self-map of the unit disk, and w be its Denjoy-Wolff point. If C_φ is compact and power bounded on $\mathcal{D}_\alpha$, then for any f in $\mathcal{D}_\alpha$, we have $\overline{f\circ\varphi_n \overset{\mathcal{D}_\alpha}{\longrightarrow} f(w)}$. Moreover, $\mho_{c,\alpha}(\varphi) = \mathcal{D}_\alpha$ and $\mho_{0,\alpha}(\varphi) = \{f \in \mathcal{D}_\alpha : \ f(w) = 0\}$. Indeed, if f is in $\mathcal{D}_\alpha$, then the sequence $\{f\circ\varphi_n\}$ is bounded and $\overline{f\circ\varphi_n \overset{u.c}{\longrightarrow} f(w)}$. Therefore, by the compactness of C_φ, $\overline{f\circ\varphi_n \overset{\mathcal{D}_\alpha}{\longrightarrow} f(w)}$.*

Example 5.3. *Let $\varphi(z) = z^2$, so for any n in $\mathbb{N}$, $\varphi_n(z) = z^{2^n}$ and*

$$\|\varphi_n\|^2 = (1+2^n)^{1-\alpha}.$$

So C_φ is not power bounded on $\mathcal{D}_\alpha$, for $-1 < \alpha < 1$, however, since zero is the Denjoy-Wolff point of φ, C_φ is power bounded on each $\mathcal{D}_\alpha$, for $\alpha \geq 1$. Also, by using Schwartz Lemma

$$2^n|z|^{2^n-1} = |\varphi_n'(z)| \leq \frac{1-|\varphi_n(z)|^2}{1-|z|^2}.$$

Thus, if ζ is in unit circle, then $d(\zeta,\varphi_n) \geq 2^n$. Therefore, Corollary 4.2 implies that C_φ is a Riesz operator on every $\mathcal{D}_\alpha$, when $\alpha > 1$.

Example 5.4. *Let φ be a univalent self-map of $\mathbb{D}$ with Denjoy-Wolff point in $\mathbb{D}$. Then for any $\alpha \geq 0$, C_φ is power bounded on $\mathcal{D}_\alpha$. We can easily see that there is a positive constant C, independently of φ, such that if $z \in \mathbb{D}$, then*

$$1-|z|^2 \leq \frac{C}{1-|\varphi(0)|^2}(1-|\varphi(z)|^2).$$

Thus, there is a $D > 0$ such that for any $z \in \mathbb{D}$ and any $n \in \mathbb{N}$

$$1-|z|^2 \leq D(1-|\varphi_n(z)|^2).$$

Thus,

$$\frac{\int_{\Delta(a,r)} (1-|\varphi_n^{-1}(z)|^2)^\alpha dA(z)}{(1-|a|^2)^{2+\alpha}} \leq \frac{D^\alpha \int_{\Delta(a,r)} (1-|z|^2)^\alpha dA(z)}{(1-|a|^2)^{2+\alpha}} \leq D^\alpha C_r.$$

Example 5.5. *Consider $\varphi(z) = \frac{1}{2}(1+z)$. Then for any $-1 < \alpha < 0$, C_φ is power bounded on $\mathcal{D}_\alpha$. However, the Denjoy-Wolff point of φ is $1 \in \partial\mathbb{D}$.*

Proof. We can see that

$$\varphi_n(z) = \sum_{i=1}^{n} \frac{1}{2^i} + \frac{1}{2^n}z.$$

Hence, $\varphi_n(0) = 1 - \frac{1}{2^n}$ and $\varphi_n'(z) = \frac{1}{2^n}$. Let f be in the ball of $\mathcal{D}_\alpha$ so f' is in the ball of A_α. Therefore,

$$\begin{aligned}
&\int_{\mathbb{D}} |f'(\varphi_n(z))|^2 |\varphi_n'(z)|^2 (1-|z|)^\alpha dA(z) \\
&= \frac{1}{2^{2n}} \int_{\mathbb{D}} |f'(\varphi_n(z))|^2 (1-|z|)^\alpha dA(z) \leq \frac{1}{2^{2n}} \Big(\frac{1+|\varphi_n(0)|^2}{1-|\varphi_n(0)|^2} \Big)^{\alpha+2} \\
&\leq \frac{2^{\alpha+2}}{2^{2n}} \Big(\frac{1}{1-|\varphi_n(0)|^2} \Big)^2 = 2^{\alpha+2} \Big(\frac{1}{2^n - 2^n |1-\frac{1}{2^n}|^2} \Big)^2 \\
&= 2^{\alpha+2} \Big(\frac{1}{1-\frac{1}{2^n}} \Big)^2 \leq 2^{\alpha+2}.
\end{aligned}$$

□

References

[1] Beltrán-Meneu, María J.; Gómez-Collado, M. Carmen; Jordá, Enrique; Jornet, David. Mean ergodicity of weighted composition operators on spaces of holomorphic functions. *J. Math. Anal. Appl.* **444** (2016), no. 2, 1640–1651. MR3535780, Zbl 1351.47019, doi: 10.1016/j.jmaa.2016.07.039.

[2] Beltrán-Meneu, María J.; Gómez-Collado, M. Carmen; Jordá, Enrique; Jornet, David. Mean ergodic composition operators on Banach spaces of holomorphic functions. *J. Funct. Anal.* **270** (2016), no. 12, 4369–4385. MR3490769, Zbl 1341.47027, arXiv:1601.03353, doi: 10.1016/j.jfa.2016.03.003.

[3] Bonet, José; Domański, Paweł. Power bounded composition operators on spaces of analytic functions. *Collect. Math.* **62** (2011), no. 1, 69–83. MR2772328, Zbl 1231.47021, doi: 10.1007/s13348-010-0005-9.

[4] Bonet, José; Domański, Paweł. A note on mean ergodic composition operators on spaces of holomorphic functions. *Rev. R. Acad. Cienc. Exactas Fís Nat. Ser. A Math. RACSAM* **105** (2011), no. 2, 389–396. MR2826716, Zbl 1275.47049, doi: 10.1007/s13398-011-0009-7.

[5] Bourdon, Paul S.; Shapiro, Joel H. Riesz composition operators. *Pacific J. Math* **181** (1997), no. 2, 231–246. MR1486530, Zbl 0910.47025.

[6] Cowen, Carl C.; MacCluer, Barbara D. Composition operators on spaces of analytic functions. Studies in Advanced Mathematics. *CRC Press, Boca Raton, FL*, 1995. xii+388 pp. ISBN: 0-8493-8492-3. MR1397026, Zbl 0873.47017.

[7] Duren, Peter; Schuster, Alexander. Bergman spaces. Mathematical Surveys and Monographs, 100. *American Mathematical Society, Providence, RI*, 2004. x+318 pp. ISBN: 0-8218-0810-9. MR2033762, Zbl 1059.30001, doi: 10.1090/surv/100.

[8] Gómez-Collado, M. Carmen; Jordá, Enrique; Jornet, David. Power bounded composition operators on spaces of meromorphic functions. *Topology Appl.* **203** (2016), 141–146. MR3481078, Zbl 1334.47031, doi: 10.1016/j.topol.2015.12.082.

[9] Hedenmalm, Haakan; Korenblum, Boris; Zhu, Kehe. Theory of Bergman spaces. Graduate Texts in Mathematics, 199. *Springer-Verlag, New York*, 2000. x+286 pp. ISBN: 0-387-98791-6. MR1758653, Zbl 0955.32003, doi: 10.1007/978-1-4612-0497-8.

[10] Kellay, Karim; Lefèvre, Pascal. Compact composition operators on weighted Hilbert spaces of analytic functions. *J. Math. Anal. Appl.* **386** (2012), no. 2, 718–727. MR2834781, Zbl 1231.47024, doi: 10.1016/j.jmaa.2011.08.033.

[11] MacCluer, Barbara D.; Shapiro, Joel H. Angular derivatives and compact composition operators on the Hardy and Bergman spaces. *Canad. J. Math.* **38** (1986), no. 4, 878–906. MR0854144, Zbl 0608.30050, doi: 10.4153/CJM-1986-043-4.

[12] Pau, Jordi; Pérez, Patricio A. Composition operators acting on weighted Dirichlet spaces. *J. Math. Anal. Appl.* **401** (2013), no. 2, 682–694. MR3018017, Zbl 1293.47022, doi: 10.1016/j.jmaa.2012.12.052.

[13] Shapiro, Joel H.; Smith, Wayne. Hardy spaces that support no compact composition operators. *J. Funct. Anal.* **205** (2003), no. 1, 62–89. MR2020208, Zbl 1041.46019, doi: 10.1016/S0022-1236(03)00215-5.

[14] Shapiro, Joel H.; Taylor, Peter D. Compact, nuclear, and Hilbert–Schmidt composition operators on H^2. *Indiana Univ. Math. J.* **23** (1973/74), 471–496. MR0326472, Zbl 0276.47037, doi: 10.1512/iumj.1973.23.23041.

[15] Wolf, Elke. Power bounded composition operators. *Comput. Methods Funct. Theory* **12** (2012), no. 1, 105–117. MR2977282, Zbl 1252.47029, doi: 10.1007/BF03321816.

[16] Wolf, Elke. Power bounded weighted composition operators. *New York J. Math.* **18** (2012), 201–212. MR2920989, Zbl 1253.47017.

[17] Zhu, Kehe. Operator theory in function spaces. Second edition. Mathematical Surveys and Monographs, 138. *American Mathematical Society, Providence, RI*, 2007. xvi+348 pp. ISBN: 978-0-8218-3965-2. MR2311536, Zbl 1123.47001, doi: 10.1090/surv/138.

[18] Zorboska, Nina. Composition operators on weighted Dirichlet spaces. *Proc. Amer. Math. Soc.* **126** (1998), no. 7, 2013–2023. MR1443862, Zbl 0894.47023, doi: 10.1090/S0002-9939-98-04266-X.

(Hamzeh Keshavarzi) Department of Mathematics, Shiraz University, Shiraz, Iran
Hamzehkeshavarzi67@gmail.com

(Bahram Khani-Robati) Department of Mathematics, Shiraz University, Shiraz, Iran
Bkhani@shirazu.ac.ir

This paper is available via http://nyjm.albany.edu/j/2018/24-21.html.