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# On the local residue symbol in the style of Tate and Beilinson

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Abstract. Tate gave a famous construction of the residue symbol on curves by using some non-commutative operator algebra in the context of algebraic geometry. We explain Beilinson's multidimensional generalization, which is not so well-documented in the literature. We provide a new approach using Hochschild homology.

Contents



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Suppose $X/k$ is a smooth proper algebraic curve over a perfect field. One can define the residue of a rational 1-form $\omega$ at a closed point $x$ as

$$\operatorname{res}_x \omega = \operatorname{Tr}_{\kappa(x)/k} a_{-1}, \qquad \text{where} \qquad \omega = \sum a_i t^i \, \mathrm{d}t \tag{0.1}$$

in terms of a local coordinate $t$, i.e. by picking an isomorphism $\operatorname{Frac} \widehat{\mathcal{O}}_{X,x} \simeq \kappa(x)((t))$. This works, but is unwieldy since it depends on the choice of the isomorphism and one needs to prove that it is well-defined, cf. Serre [Ser97, Ch. II]. One could ask for a bit more:

**Aim:** Construct the local residue symbol without ever needing to choose coordinates.

J. Tate [Tat68] has pioneered an approach which circumvents choices of coordinates at all times by employing ideas in the style of functional analysis: The local field

$$\widehat{\mathcal{K}}_{X,x} := \operatorname{Frac} \widehat{\mathcal{O}}_{X,x} = \varinjlim_{s \in \mathcal{O}_{X,x} \setminus \{0\}} \varprojlim_{i} \left( \frac{1}{s} \mathcal{O}_{X,x} / \mathfrak{m}_x^i \right) \tag{0.2}$$

carries a canonical topology, defined by viewing it as an ind-pro limit of finite-dimensional discrete $k$-vector spaces. This topology needs no assumptions on the base field, e.g. it could be just a finite field. We get a non-commutative algebra of continuous $k$-vector space endomorphisms $E$. Via multiplication operators $x \mapsto f \cdot x$ the functions $f \in \widehat{\mathcal{K}}_{X,x}$ embed into $E$. Using the ideal of compact operators, Tate shows that $E$ has a canonical central extension $\widehat{E}$ as a Lie algebra by a formal element $\mathsf{c}$ such that

$$[f, g]_{\widehat{E}} = \operatorname{res}_x f \mathrm{d}g \cdot \mathsf{c}. \tag{0.3}$$

Tate now uses the left-hand side as an intrinsically coordinate-independent definition for the residue (R. Hartshorne advertises this as 'clever' in his textbook [Har77, Ch. III, §7]). For an $n$-dimensional smooth proper algebraic variety $X/k$, the global residue

$$H^n(X, \Omega^n_{X/k}) \longrightarrow k$$

is induced from $n$-dimensional local residue symbols. There is the conventional approach to this using A. Grothendieck's residue symbol [Har66], however A. Beilinson [Beĭ80] has shown that one can also describe this map by a beautiful multidimensional generalization of Tate's approach. He interprets the commutators which appear in Tate's theory as low-degree avatars of the differential in Lie homology. As such, one can give an explicit formula for

the higher residue in terms of cascading commutators, roughly generalizing Equation 0.3.

**0.1. Overview.** Let us give a little orientation regarding the different viewpoints on residues. We begin with the very general perspective of Grothendieck, [Har66]. If $f : X \to Y$ is a proper morphism of varieties, the derived pushforward $\mathbf{R}f_*$ has a right adjoint denoted by "$\mathbf{R}f^!$". As for any such adjunction, we get a co-unit transformation of functors

$$\mathbf{R}f_* \circ \mathbf{R}f^! \to \mathrm{id}\,.$$

There are many different methods establishing this adjunction, with large variations in generality. We will have nothing to say about such a general setup, see [Con00], [Har66], [Lip09], [LipH09], [Nee96] for example.

We will only be concerned with the special case that $Y = \mathrm{Spec}(k)$ is a point and $f$ is proper smooth of relative dimension $n$. Then $\mathbf{R}f^!\mathcal{O}_Y \simeq \Omega^n_{X/k}[n]$ and the above co-unit becomes a functorial map

$$H^n(X, \Omega^n_{X/k}) \longrightarrow k. \tag{0.4}$$

The theory of residues essentially aims at making this map explicit using local data.

How to do this? The approach in Grothendieck and Hartshorne [Har66] is to express the cohomology group on the left by working with the Cousin resolution of the sheaf $\Omega^n_{X/k}$:

$$\Omega^n_{X/k} \simeq \left[ \coprod_{x \in X^0} H^0_x(X, \Omega^n_{X/k}) \to \cdots \to \coprod_{x \in X^n} H^n_x(X, \Omega^n_{X/k}) \right]_{0,n}, \tag{0.5}$$

where $X^i$ denotes the set of points of $X$ such that $\overline{\{x\}}$ has codimension $i$, and $H^i_x$ denotes the $i$-th local cohomology group with support $\{x\}$. See Ch. IV op. cit. for details. Describing the morphism in Equation 0.4 thus reduces to describing it on the degree $n$ term of this resolution, i.e., to give a $k$-valued map on the local cohomology group. This leads to the *local residue in the Grothendieck–Hartshorne approach*

$$H^n_x(X, \Omega^n_{X/k}) \to k, \tag{0.6}$$

which is attached to any closed point $x \in X$. How this local map can be identified is explained in various places, e.g., [Har66], or by Sastry-Yekutieli [SaY95]. To be truly explicit, one has to make the local cohomology group $H^n_x(-,-)$ explicit. We will not go into this matter here.

The approach of Tate and Beilinson differs as follows: We return to Equation 0.4, but this time we resolve the sheaf $\Omega^n_{X/k}$ using the so-called adèle resolution:

$$\Omega^n_{X/k} \simeq \left[ \cdots \to \text{subspace of } \prod_{\triangle} A(\triangle, \mathcal{O}_X) \otimes \Omega^n_{X/k} \right]_{0,n}, \tag{0.7}$$

where $\triangle = (\eta_0 > \cdots > \eta_n)$ runs through chains of scheme points such that $\operatorname{codim}_X \overline{\{\eta_i\}} = i$. Each $A(\triangle, \mathcal{O}_X)$, whose actual definition we shall discuss only later, is a finite direct sum of fields, so we deal with a rather concrete object and no cohomology is left.

Concretely, each of these fields is isomorphic to

$$k'((t_1))((t_2))\cdots((t_n))$$

for $k'/k$ a finite field extension. However, the Tate–Beilinson approach does *not* use such isomorphisms since their choice is non-canonical and as we had explained at the beginning of the introduction, the whole point is to avoid any unnatural choices. Instead, just like Tate's 1968 approach studies ideals of compact operators acting on $k'((t))$, Beilinson studies a non-commutative algebra of (certain well-behaved) operators acting on $A(\triangle, \mathcal{O}_X)$. This leads to the so-called "cubically decomposed algebra" $E_\triangle$. Beilinson then constructs a functional on Lie homology

$$\phi_{Beil} : \quad H_{n+1}((E_\triangle)_{Lie}, k) \longrightarrow k. \tag{0.8}$$

We shall discuss how to do this below. But assuming this has been dealt with, the Lie homology functional $\phi_{Beil}$ gives rise to the *local residue in the Tate–Beilinson approach*

$$\begin{aligned} A(\triangle, \mathcal{O}_X) \otimes \Omega^n_{X/k} &\longrightarrow k \\ g \otimes f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n &\longmapsto \phi_{Beil}(g \cdot f_0 \wedge f_1 \wedge \cdots \wedge f_n) \end{aligned}$$

and this is the analogue of the map in Equation 0.6 of the Grothendieck viewpoint. Instead of a closed point, each local residue map is attached to the choice of a chain $\triangle$ here. For $n = 1$, the functional $\phi_{Beil}$ defines a Lie *co*homology class in $H^2$, describing a central extension, and this is the one of Equation 0.3.

This summary leaves two issues open:

(1) How to construct $\phi_{Beil}$?
(2) Why *Lie* homology?

We will discuss (1) below in §0.3. Regarding (2), we propose in this paper an alternative approach based on Hochschild or cyclic homology, and carefully study the relation to Lie homology.

Our Hochschild variant will be a map

$$\phi_C : HH_n(E_\triangle) \longrightarrow k \tag{0.9}$$

replacing Equation 0.8. Using tools which are specific to Hochschild homology and have no true Lie algebra analogue (Wodzicki Excision), we will then give a construction of this map $\phi_C$ which is completely different from the

way how $\phi_{Beil}$ is constructed in [Beĭ80]. This is the principal novelty of this paper.

**0.2. Other methods.** One may also use the adèle resolution, but still define the local residue in a more classical way. We refer to Yekutieli [Yek92] for a complete treatment along these lines.

**0.2.1. Lipman's method.** In the literature one can also find a different approach to the residue symbol based on Hochschild homology due to Lipman [Lip87]. It belongs to the Grothendieck–Hartshorne viewpoint. Let us briefly explain the connection: Let $i : \operatorname{Spec} \kappa(x) \hookrightarrow X$ be the closed immersion of a closed point $x$. Then there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} \kappa(x) & \xhookrightarrow{\ i\ } & X \\ & \searrow^{g} & \downarrow{\scriptstyle f} \\ & & \operatorname{Spec} k, \end{array}$$

and the composition $g$ is a finite morphism. Since $\mathbf{R}f^!$ arises from the adjunction with $\mathbf{R}f_*$, the composition of morphisms gives rise to a canonical natural equivalence of functors

$$\operatorname{Comp} : \mathbf{R}g^! \xrightarrow{\sim} \mathbf{R}i^! \circ \mathbf{R}f^!. \tag{0.10}$$

If $h : Z_1 \to Z_2$ denotes a finite morphism of schemes, there is the general formula

$$\mathbf{R}h^! = h^* \mathbf{R}\operatorname{Hom}_{Z_2}(h_* \mathcal{O}_{Z_1}, -).$$

Since both $g$ and $i$ are finite morphisms and $\mathbf{R}f^! \mathcal{O}_Y \simeq \Omega^n_{X/k}[n]$, one can make both sides of the map 'Comp' explicit. Identifying this map explicitly is another approach to residue theory.

Let us quickly explain why: Note that the local cohomology for a closed subscheme $Z$ defined by the ideal sheaf $\mathcal{I}_Z$ is given by

$$\mathbf{R}\Gamma_Z(X, -) = \varinjlim_{m \geq 1} \mathbf{R}\operatorname{Hom}_X(\mathcal{O}_X / \mathcal{I}_Z^m, -),$$

which except for the colimit is of course essentially $\mathbf{R}i^!$ for the nil-thickened immersions. This way the local cohomology group $H^n_x(X, \Omega^n_{X/k})$ of the Grothendieck–Hartshorne approach is linked to the right-hand side in Equation 0.10.

Now, Lipman's theory, see [Lip87, §0, Introduction], describes the map 'Comp' as arising from a pairing of a Hochschild cohomology class with the Hochschild homology class of a differential form. The reader will see that this is a quite different method in comparison to our definition in §6. Besides the homological differences, Lipman uses affine locally the Hochschild homology of the commutative ring $\mathcal{O}_X(X)$ of functions, whereas the Tate–Beilinson method is based on the non-commutative algebra $E_\triangle$ of endomorphisms of higher local fields.

**0.2.2. Boundary maps in the Hochschild localization sequence.** Finally, in joint work with J. Wolfson [BW], we provide another viewpoint: We replace the Cousin resolution of Equation 0.5 by its counterpart coming from the coniveau filtration of the Hochschild homology of the scheme $X$. The resulting complex, up to some "Hochschild–Kostant–Rosenberg isomorphism with supports", a posteriori turns out to agree with the Cousin resolution.

However, the boundary maps in this complex by construction stem from boundary maps in the localization sequence of Hochschild homology. This can be related to Tate objects and gives yet a different viewpoint. The appearance of Tate objects is explained by joint work with Groechenig and Wolfson [BGW16b] which shows that the endomorphism algebras of Tate objects are essentially the cubically decomposed algebras as they appear in Beilinson's work [Beĭ80], and which also play a prominent rôle in the present paper. This also closes the circle with Yekutieli's approach to residues, see [Yek15].

**0.3. The results.** We still have not explained how to construct the maps in Equation 0.8 (Lie homology) or Equation 0.9 (Hochschild homology) and how to get $E_\triangle$.

For $E_\triangle$ and $\phi_{Beil}$, we refer to Beilinson's paper [Beĭ80], or to our review of his paper in §2 and §3 in the main body of the paper.

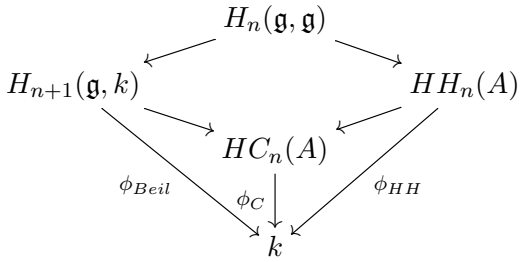
We propose a new path which one may follow as an alternative and which seems quite efficient: Under mild assumptions, we exhibit $A^n := E_\triangle$ as an iterated algebra extension of simpler cubically decomposed algebras $A^i$. Now define

$$\phi_C : \quad HH_n(A^n) \xrightarrow{\delta\Lambda} HH_{n-1}(A^{n-1}) \xrightarrow{\delta\Lambda} \cdots \xrightarrow{\delta\Lambda} HH_0(A^0) \longrightarrow k, \tag{0.11}$$

where $\Lambda : A^n \to A^n/A^{n-1}$ is a kind of Toeplitz operator mechanism, and $\delta$ the connecting homomorphism $\delta : HH_*(A^n/A^{n-1}) \to HH_{*-1}(A^{n-1})$ coming from the algebra extension $A^{n-1} \hookrightarrow A^n \twoheadrightarrow A^n/A^{n-1}$. Modulo maps relating Lie with Hochschild homology and identifying differential forms along $\Omega^n_{R/k} \simeq HH_n(R)$, this construction produces the same map as $\phi_{Beil}$. The main idea is to view Beilinson's use of Lie homology as the Hodge $n$-part of Hochschild homology and get rid of relative Lie homology by Hochschild excision, see §4 for a detailed explanation. Concretely, one can state these comparisons as the following result, which might appear a little technical at first.

**Theorem** (Lie-to-Hochschild Comparison)**.** *Suppose $A$ is a unital $n$-fold cubically decomposed algebra over a field $k$ which has local units on all levels. Let $\mathfrak{g}$ denote its Lie algebra. Then there are canonical maps, making the*

*diagram*




*commutative. Here $\phi_{Beil}$ is Beilinson's construction in* [Beĭ80]*, and $\phi_{HH}$ and $\phi_C$ are constructed in this paper; $\phi_C$ being as in Line 0.11 (the map will turn out to factor over cyclic homology, Lemma 31).*

This is glued from the triangles of Corollary 24 and Corollary 33. Applied to the concrete task of describing residues, this leads to our Local Formula, Theorem 26, unravelling all these maps in concrete terms, once local coordinates are chosen. This is our multidimensional generalization of Equation 0.3.

We also give our own interpretation of the Lie homology mechanism in Beilinson's [Beĭ80] in §4. No such attempt of an explanation seems to exist in the literature, and we hope that future readers of [Beĭ80] will find this helpful.

Moreover, we propose a way to phrase reciprocity-like vanishing theorems in a new way. We tried to find the correct formulation of such a result on the level of cubically decomposed algebras. The 'abstract residue formula' of Arbarello, de Concini and Kac [ADCK89, §2] may be regarded as its ancestor.

**Theorem** (Cube Reciprocity Law)**.** *Let $A$ be a unital $n$-fold cubically decomposed algebra with local units on all levels. Let $P^{\pm} \in A$ be idempotents such that*

$$P^+ + P^- = 1 \qquad \text{and} \qquad P^{\pm} A \in I_1^{\pm}.$$

*If $R \subseteq A$ is a subalgebra such that $P^+A$ (or $P^-A$) is a left $R$-submodule of $A$, then for all $r \in HH_n(R)$:*

$$\phi_C(r) = 0.$$

See Theorem 35. It is a possible abstraction and generalization of the corresponding vanishing theorem in Tate's paper [Tat68].

We will only see few applications of this result in this text since this text is about the *local* situation only. The global situation will be treated elsewhere.

## 1. Tate's original construction

**1.1. Operator ideals and the snake lemma.** We shall quickly recall the classical construction of Tate [Tat68], from a perspective which points naturally to the multidimensional generalization. Let $X/k$ be a smooth algebraic curve. For every closed point $x \in X$, the completed stalk of the structure sheaf is a complete discrete valuation ring with residue field $\kappa(x)$. By Cohen's Structure Theorem there is an isomorphism

$$\widehat{\mathcal{K}}_{X,x} := \operatorname{Frac} \widehat{\mathcal{O}}_{X,x} \simeq \kappa(x)((t)), \tag{1.1}$$

however there is no canonical isomorphism.

Without needing to choose such an isomorphism, $\widehat{\mathcal{K}}_{X,x}$ has a canonical topology coming from the presentation $\widehat{\mathcal{K}}_{X,x} = \varinjlim_s \varprojlim_i \frac{1}{s}\mathcal{O}_{X,x}/\mathfrak{m}_x^i$, where we regard each $\mathcal{O}_{X,x}/\mathfrak{m}_x^i$ as a *discrete* $k$-vector space. This turns the inner pro-limit into a linearly compact $k$-vector space and the ind-limit over all finitely generated $\widehat{\mathcal{O}}_{X,x}$-submodules of $\widehat{\mathcal{K}}_{X,x}$ into a linearly locally compact $k$-vector space.

We can now regard $\widehat{\mathcal{K}}_{X,x}$ as a infinite-dimensional topological $k$-vector space. The topology differs from the ones conventionally used in functional analysis over $\mathbf{R}$ or $\mathbf{C}$ because it is generated from an open neighbourhood basis of 0 which consists of linear subspaces; they are called *lattices*:

**Definition 1.** *A* lattice *in a finite-dimensional $\widehat{\mathcal{K}}_{X,x}$-vector space $V$ is a finitely generated $\widehat{\mathcal{O}}_{X,x}$-submodule $L \subseteq V$ so that $\widehat{\mathcal{K}}_{X,x} \cdot L = V$.*

Using the topology, we get the associative operator algebra of continuous $k$-linear endomorphisms

$$E := \{\phi : \widehat{\mathcal{K}}_{X,x} \to \widehat{\mathcal{K}}_{X,x} \mid \phi \text{ is } k\text{-linear and continuous}\}. \tag{1.2}$$

**Definition 2.** *We call an operator $\phi \in E$*

1. compact *if there is a lattice $L$ with $\operatorname{im}\phi \subseteq L$;*
2. discrete *if there is a lattice $L$ with $L \subseteq \ker\phi$.*

These classes of operators form two-sided ideals $I^+, I^-$ in $E$. Moreover, we have $I^+ + I^- = E$. Write $I_{tr} := I^+ \cap I^-$ for their intersection. Thus, we get a short exact sequence of $E$-bimodules,

$$0 \longrightarrow I_{tr} \longrightarrow I^+ \oplus I^- \longrightarrow E \longrightarrow 0. \tag{1.3}$$

We may formally "exterior tensor" this with another copy of $E$, giving a commutative diagram with exact rows:

(1.4)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (I^+ \wedge E) \cap (I^- \wedge E) & \longrightarrow & (I^+ \wedge E) \oplus (I^- \wedge E) & \longrightarrow & E \wedge E & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle [-,-]} & & \downarrow{\scriptstyle [-,-]} & & \downarrow{\scriptstyle [-,-]} & & \\
0 & \longrightarrow & I_{tr} & \longrightarrow & I^+ \oplus I^- & \longrightarrow & E & \longrightarrow & 0
\end{array}$$

(for $V \subseteq W$ a subspace of a vector space, $V \wedge W$ denotes the subspace of $\bigwedge^2 W$ generated by vectors $v \wedge w$ with $v \in V, w \in W$.) The snake lemma gives us a canonical morphism, call it $(*)$, and thus

$$\phi : \widehat{\mathcal{K}}_{X,x} \wedge \widehat{\mathcal{K}}_{X,x} \longrightarrow \ker(E \wedge E \to E) \xrightarrow{(*)} \operatorname{coker}([\ldots] \to I_{tr}) \xrightarrow{\operatorname{tr}} k. \tag{1.5}$$

The local rational functions $\widehat{\mathcal{K}}_{X,x} \subset E$ are viewed as the respective multiplication operator $x \mapsto f \cdot x$, which is clearly continuous. Functions commute, i.e. $[f, g] = 0$, so the left-hand side arrow indeed exists. On the other hand, traces satisfy $\operatorname{tr}([X, Y]) = 0$, so the trace on the right-hand side factors through the cokernel. Tate now proves that $\phi(f \wedge g) = \operatorname{res}_x f \mathrm{d} g$. See Lemma 4 for the proof. [Tat68, §2].

*Remark* 1. Tate's original paper [Tat68] actually defines $I^+, I^-$ (called $E_1, E_2$ in *loc. cit.*) slightly differently. He fixes a special open, the ring of integers $\widehat{\mathcal{O}}_{X,x} \subset \widehat{\mathcal{K}}_{X,x}$, and instead of compactness he demands an operator to map the entire space into this open, up to a finite-dimensional discrepancy. See also Definition 10. But this comes down to the same as the topological definition we use here. The presentation using a topological language is taken from [BeĭFM91, §1.2] ($I^+, I^-$ are called $Hom_+, Hom_-$ in *loc. cit.*).

**1.2. Finite-potent trace.** We have tacitly swept a detail under the rug: Since $E$ is infinite-dimensional, a general operator in $E$ will not have a well-defined trace. Clearly finite-rank operators will still have a trace, but in Tate's construction the operators in $I_{tr}$ a priori need not be of finite rank. In functional analysis one would now hope for the ideal of nuclear operators, but the ind-pro type topologies are not rich enough to give a convergence condition on the operator spectrum any interesting content. Instead, Tate uses the philosophy that any nilpotent operator should have trace zero, even if it is not of finite rank. We briefly summarize Tate's operator trace [Tat68] as we will also need it later:

Let $F_0$ be a field and $V$ an $F_0$-vector space. Call an endomorphism $g \in \operatorname{End}_{F_0}(V)$ *finite-potent* if there is some $n \geq 1$ such that the image $g^n V$ is finite-dimensional over $F_0$. An $F_0$-vector subspace $\Gamma \subseteq \operatorname{End}_{F_0}(V)$ is called a *finite-potent family* if there is some $n \geq 1$ such that $(g_1 \circ \cdots \circ g_n)V$ is finite-dimensional for any choice of $g_1, \ldots, g_n \in \Gamma$.

**Proposition 3** ([Tat68])**.** *(Tate) For every $F_0$-vector space $V$ and every finite-potent $g \in \operatorname{End}_{F_0}(V)$ there is a unique element, denoted $\operatorname{tr}_V g \in F_0$ (and called* Tate trace*), such that the following rules hold:*

- **T1:** *If $V$ is finite-dimensional, $\operatorname{tr}_V g$ is the usual trace.*
- **T2:** *If $W \subseteq V$ is any $F_0$-vector subspace and $gW \subseteq W$, we have $\operatorname{tr}_V g = \operatorname{tr}_W g + \operatorname{tr}_{V/W} g$.*
- **T3:** *If $g$ is nilpotent, $\operatorname{tr}_V g = 0$.*

**T4:** *Suppose* $\Gamma \subseteq \mathrm{End}_{F_0}(V)$ *is a finite-potent family. Then* $\mathrm{tr}_V|_\Gamma$ *is* $F_0$*-linear, i.e.* $\mathrm{tr}_V(af+bg) = a\,\mathrm{tr}_V f + b\,\mathrm{tr}_V g$ *for all* $a,b \in F_0$ *and* $f,g \in \Gamma$. *([1])*

**T5:** *Suppose* $f : V \to V'$ *and* $g : V' \to V$ *are* $F_0$*-vector space homomorphisms and the composition* $f \circ g$ *is finite-potent on* $V'$. *Then the reverse composition* $g \circ f$ *is finite-potent on* $V$ *and* $\mathrm{tr}_{V'}(f \circ g) = \mathrm{tr}_V(g \circ f)$.

*Example* 1. Consider $F_0 := k$ and $V := k[t,t^{-1}]$. Then $f \in \mathrm{End}_{F_0}(V)$ given by $t^i \mapsto t^{-i}$ for $i \geq 0$ and $t^i \mapsto 0$ for $i < 0$ is a finite-potent morphism which is *not* finite-rank, so the usual trace is not applicable. The vector $t^0$ spans a 1-dimensional $f$-stable subspace and on the vector space quotient $k[t,t^{-1}]/k\langle t^0\rangle$ the induced operator $\overline{f}$ is nilpotent, so by **T1** and **T2** we get $\mathrm{tr}_V f = 1$.

**Lemma 4** ([Tat68, Theorem 2])**.** $\phi(f \wedge g) = \mathrm{res}_x f\mathrm{d}g$.

**Proof.** We just need to follow the snake morphism in Equation 1.4. For this we need to split the surjection in the top row of Equation 1.4, i.e. pick idempotents $P^\pm$ on $E$ such that $P^\pm E \subseteq I^\pm$ so that $P^+ + P^- = \mathbf{1}$. Then unwinding the snake morphism yields

$$\begin{array}{ccc} (P^+f \wedge g) \oplus (-P^-f \wedge g) & \longrightarrow & f \wedge g \\ \big\downarrow & & \\ [P^+f, g] & \longrightarrow & [P^+f,g] \oplus -[P^-f,g] \end{array}$$

and so the composition of maps in Equation 1.5 unwinds to the concrete formula

$$\phi : \widehat{\mathcal{K}}_{X,x} \wedge \widehat{\mathcal{K}}_{X,x} \to k \qquad \phi(f \wedge g) = \mathrm{tr}[P^+f, g] \tag{1.6}$$

(or $-\mathrm{tr}[P^-f,g]$ equivalently). It follows immediately that this formula is independent of the choice of a particular $P^+$. We may pick any isomorphism $\widehat{\mathcal{K}}_{X,x} \simeq \kappa(x)((t))$. Suppose $x$ is a $k$-rational point, i.e. $\kappa(x) = k$. In order to distinguish between $t^i$ as a multiplication operator or as a topological basis element of $\widehat{\mathcal{K}}_{X,x}$, let us write $\mathbf{t}^i$ for the latter. Then take for example $P^+(\mathbf{t}^i) := \delta_{i\geq 0}\mathbf{t}^i$. This clearly lies in $I^+$, $P^- := \mathbf{1} - P^+$ lies in $I^-$ and we compute

$$[P^+t^i, t^j]\mathbf{t}^\lambda = \delta_{\lambda+i+j\geq 0}\mathbf{t}^{\lambda+i+j} - \delta_{\lambda+i\geq 0}\mathbf{t}^{\lambda+i+j} = \delta_{-j\leq\lambda+i<0}\mathbf{t}^{\lambda+i+j}.$$

$$\bullet\mathbf{t}^{-1}\ \boxed{\bullet\mathbf{t}^0\ \ \bullet\mathbf{t}^1\ \ \bullet\mathbf{t}^2\ \ \bullet\mathbf{t}^3} - \bullet\mathbf{t}^{-1}\ \ \bullet\mathbf{t}^0\ \ \bullet\mathbf{t}^1\ \ \boxed{\bullet\mathbf{t}^2\ \ \bullet\mathbf{t}^3} = \boxed{\bullet\mathbf{t}^0\ \ \bullet\mathbf{t}^1} \tag{1.7}$$

[1] Mysteriously, in general the linearity axiom **T4** fails. A concrete counter-example is given by Pablos Romo in [PabR07]. See also [AST07], [RGPR14] for a more thorough discussion. However, this need not concern us; the non-linearity will never show up in the applications of the above proposition in this paper.

Suppose $j = 1$, then $[P^+t^i, t]\mathbf{t}^\lambda = \delta_{-1\leq\lambda+i<0}\mathbf{t}^{\lambda+i+1}$. This has a non-trivial invariant subspace iff $i = -1$, so $\phi(t^i \wedge t) = 0$ for $i \neq -1$. For $i = -1$ we get $[P^+t^{-1}, t]\mathbf{t}^\lambda = \delta_{-1\leq\lambda-1<0}\mathbf{t}^\lambda$, so $k\left\langle\mathbf{t}^0\right\rangle$ is a 1-dimensional invariant subspace and therefore $\phi(t^{-1} \wedge t) = 1$: Just like $\operatorname{res} t^i \mathrm{d}t = \delta_{i=-1}$. If $x$ is an arbitrary closed point, $\kappa(x)/k$ is a finite field extension. The above computation still applies if we work with $\kappa(x)$-vector spaces. Writing $\kappa(x)$ itself as a $[\kappa(x) : k]$-dimensional $k$-vector space yields the formula $\operatorname{res} t^i \mathrm{d}t = [\kappa(x) : k]\delta_{i=-1}$. □

The map $\phi : \widehat{\mathcal{K}}_{X,x} \wedge \widehat{\mathcal{K}}_{X,x} \to k$ induces a functional

$$H_2((\widehat{\mathcal{K}}_{X,x})_{Lie}, k)^* \cong H^2((\widehat{\mathcal{K}}_{X,x})_{Lie}, k)$$

and the resulting Lie central extension is the one arising from pushing out Equation 1.3 by Tate's trace,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_{tr} & \longrightarrow & I^+ \oplus I^- & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & k & \longrightarrow & \widehat{E} & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

**Definition 5.** *The central extension $\widehat{E}$ in the lower row is* Tate's central extension.

## 2. Adèles

**2.1. For curves.** Let $X/k$ be an integral smooth proper algebraic curve. Tate [Tat68] uses the language of adèles of the curve − a technique borrowed from number theory. We write $\prod_{x\in U^1} \widehat{\mathcal{O}}_{X,x}$ as a shorthand for the $\mathcal{O}_X$-module sheaf

$$U \mapsto \prod_{x\in U^1} \widehat{\mathcal{O}}_{X,x} \qquad \text{for } U \text{ any Zariski open set,}$$

where $\widehat{\mathcal{O}}_{X,x}$ is the $\mathfrak{m}_x$-adically completed local ring and $U^p$ denotes the set of codimension $p$ points in $U$. The restriction map to smaller opens is the factorwise identity so that the sheaf is flasque. There is an exact sequence of $\mathcal{O}_X$-module sheaves

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\text{diag}} k(X) \oplus \prod_{x\in U^1} \widehat{\mathcal{O}}_{X,x} \xrightarrow{\text{diff}} \prod'_{x\in U^1} \widehat{\mathcal{K}}_{X,x} \longrightarrow 0, \tag{2.1}$$

where $\mathcal{O}_X$ is the structure sheaf, $k(X)$ the locally constant sheaf of rational functions, $\widehat{\mathcal{K}}_{X,x} := \operatorname{Frac}\widehat{\mathcal{O}}_{X,x}$, and the prime superscript in the rightmost sheaf abbreviates the condition that for all but finitely many $x \in U^1$ we demand sections to lie in the subspace $\widehat{\mathcal{O}}_{X,x} \subset \widehat{\mathcal{K}}_{X,x}$. It is clear that the sequence is exact and that it is actually a flasque resolution of $\mathcal{O}_X$. Moreover, the global sections of the sheaves are classically known as

| sheaf side | adèle side | |
|---|---|---|
| $H^0(X, k(X))$ | $k(X)$ | function field of the curve |
| $H^0(X, \prod_{x\in U^1} \widehat{\mathcal{O}}_{X,x})$ | $\mathbf{A}_X^0$ | integral adèle ring |
| $H^0(X, \prod'_{x\in U^1} \widehat{\mathcal{K}}_{X,x})$ | $\mathbf{A}_X$ | adèle ring |

The adèle approach to the theory of curves is due to Weil, we refer to [Ser97], [Tat68] for a presentation of this formalism. The same technique works for arbitrary quasi-coherent sheaves by tensoring. As a result of the resolution in Equation 2.1 we obtain for example

$$H^0(X,\mathcal{O}_X) = \mathbf{A}_X^0 \cap k(X) \qquad H^1(X,\mathcal{O}_X) = \mathbf{A}_X/(\mathbf{A}_X^0 + k(X)).$$

In particular, in order to describe the global residue map

$$H^1(X,\Omega^1_{X/k}) \longrightarrow k$$

we can employ such an adèle resolution of the sheaf $\Omega^1_{X/k}$ to give elements of the left-hand side a concrete representation, see Tate [Tat68].

**2.2. In general.** Parshin generalized this method to surfaces by introducing two-dimensional adèles [Par76], [ParF99]. Beilinson's paper [Beĭ80] provides the multidimensional technology. We need to recall this for later use:

We mostly follow the notation in [Beĭ80]. Let $X$ be a Noetherian scheme. For points $\eta_0, \eta_1 \in X$ we write $\eta_0 > \eta_1$ if $\overline{\{\eta_0\}} \ni \eta_1$, $\eta_1 \neq \eta_0$. Denote by $S(X)_n := \{(\eta_0 > \cdots > \eta_n), \eta_i \in X\}$ the set of chains of length $n+1$. The elements of these sets are also known as *flags*. Let $K_n \subseteq S(X)_n$ be an arbitrary subset. For any point $\eta \in X$ define ${}_\eta K := \{(\eta_1 > \cdots > \eta_n)$ s.t. $(\eta > \eta_1 > \cdots > \eta_n) \in K_n\}$, a subset of $S(X)_{n-1}$. Let $\mathcal{F}$ be a *coherent* sheaf on $X$. For $n = 0$ and $n \geq 1$ we define inductively

$$\begin{aligned} A(K_0,\mathcal{F}) :=& \prod\nolimits_{\eta\in K_0} \varprojlim_i \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta^i \\ A(K_n,\mathcal{F}) :=& \prod\nolimits_{\eta\in X} \varprojlim_i A({}_\eta K_n\, , \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta^i). \end{aligned} \tag{2.2}$$

The sheaves $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,\eta}/\mathfrak{m}_\eta^i$ are usually only quasi-coherent, so we complement this partial definition as follows: For a *quasi-coherent* sheaf $\mathcal{F}$ we define $A(K_n,\mathcal{F}) := \varinjlim_{\mathcal{F}_j} A(K_n,\mathcal{F}_j)$, where $\mathcal{F}_j$ runs through all coherent subsheaves of $\mathcal{F}$ (and hereby reducing to Equation 2.2). As it is built successively from ind-limits and Mittag-Leffler pro-limits, $A(K_n,-)$ is a covariant exact functor from quasi-coherent sheaves to abelian groups. Next, we observe that $S(X)_\bullet$ carries a natural structure of a simplicial set (omitting the $i$-th entry in a flag yields faces; duplicating the $i$-th entry in a flag degeneracies). This turns

$$\mathbf{A}^\bullet(U,\mathcal{F}) := A(S(U)_\bullet,\mathcal{F}) \qquad \text{(for } U \text{ Zariski open)}$$

into a sheaf of cosimplicial abelian groups (actually even cosimplicial $\mathcal{O}_X$-module sheaves) and via the unreduced Dold-Kan correspondence into a complex of sheaves, which we may denote by $\mathbf{A}^i_{\mathcal{F}}$.

**Theorem 6** ([Beĭ80, §2])**.** *(Beilinson) For a Noetherian scheme $X$ and a quasi-coherent sheaf $\mathcal{F}$ on $X$, the $\mathbf{A}^i(-,\mathcal{F})$ are flasque sheaves and*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathbf{A}^0_{\mathcal{F}} \longrightarrow \mathbf{A}^1_{\mathcal{F}} \longrightarrow \cdots$$

*is a flasque resolution.*

See Huber [Hub91a], [Hub91b] for a detailed proof.

*Remark* 2. There are also discussions circling around this construction in Hübl-Yekutieli [HY96], Osipov [Osi07], Parshin [Par83]. A very interesting perspective on the relation of the Grothendieck residue complex and adèles can be found in Yekutieli [Yek03]. Beilinson actually defines $S(X)_n$ so that also degenerate flags with $\eta_i = \eta_{i+1}$ are allowed, but one can check that this yields a slightly larger, but quasi-isomorphic complex [Hub91b, §5.1].

*Example* 2. Suppose $X$ is an integral smooth proper curve. We may read the set $X^p$ of codimension $p$ points as length one flags. One computes

$$A(X^0, \mathcal{O}_X) = k(X) \qquad A(X^1, \mathcal{O}_X) = \prod_{x \in X^1} \widehat{\mathcal{O}}_{X,x}$$
$$A(S(X)_1, \mathcal{O}_X) = \prod\nolimits'_{x \in X^1} \widehat{\mathcal{K}}_{X,x}$$

so that Theorem 6 reduces to the Equation 2.1.

It is also instructive to have a detailed look at a computation in dimension two:

*Example* 3 (generic behaviour). For a commutative and unital ring $R$ and a prime $P \subset R$, $\varinjlim_{f \notin P} R[f^{-1}]$ denotes the localization $R_P$. For any such $f$, we define $R[f^{-1}] = \varinjlim_i R\langle f^{-i}\rangle$ for $i \to \infty$, where $\langle f^{-i}\rangle$ denotes the $R$-sub*module* generated by $f^{-i}$ inside $R[f^{-1}]$. Combining both colimits writes $R_P$ as a colimit of finitely generated $R$-modules. We shall abbreviate this colimit by writing $\varinjlim_{f \notin P} R\langle f^{-\infty}\rangle$. Now suppose $X := \operatorname{Spec} k[s,t]$ and $\triangle := \{(0) > (s) > (s,t)\} \in S(X)_2$ is a singleton set. Then

$$\begin{aligned}
A(\triangle, \mathcal{O}_X) = A({}_{(0)}\triangle, k(s,t)) &= \varinjlim_{f \notin (0)} A({}_{(0)}\triangle, k[s,t]\langle f^{-\infty}\rangle) \\
&= \varinjlim_{f \notin (0)} \varprojlim_i A({}_{(s)(0)}\triangle, k[s,t]\langle f^{-\infty}\rangle \otimes k[s,t]_{(s)}/(s^i)) \\
&= \varinjlim_{f \notin (0)} \varprojlim_i \varinjlim_{g \notin (s)} A({}_{(s)(0)}\triangle, k[s,t]\langle f^{-\infty}\rangle\langle g^{-\infty}\rangle/(s^i)) \\
&= \varinjlim_{f \notin (0)} \varprojlim_i \varinjlim_{g \notin (s)} \varprojlim_j k[s,t]_{(s,t)}\langle f^{-\infty}\rangle\langle g^{-\infty}\rangle/(s^i)/(s,t)^j.
\end{aligned}$$

and this yields

$$\begin{aligned}
&= \varinjlim_{f \notin (0)} \varprojlim_i \varinjlim_{g \notin (s)} k[[s,t]]\langle f^{-\infty}\rangle\langle g^{-\infty}\rangle/(s^i) \\
&= \varinjlim_{f \notin (0)} \varprojlim_i k[[s,t]][(k[s,t]-(s))^{-1}]\langle f^{-\infty}\rangle/(s^i) \\
&= \varinjlim_{f \notin (0)} k((t))[[s]]\langle f^{-\infty}\rangle = k((t))((s)).
\end{aligned}$$

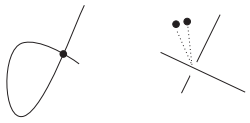
Note that this computation has not provided us with a canonical isomorphism to $k((t))((s))$. Already writing $\mathbf{A}^2_k$ as $\operatorname{Spec} k[s,t]$ involved the choice of coordinates $s, t$.

We hope that the structural similarities to the discussion in §1 have become clear. Again, we get an isomorphism $\simeq k((t))((s))$ making it tempting to define a two-dimensional residue as

$$\operatorname{res}_t \operatorname{res}_s f\mathrm{d}s \wedge \mathrm{d}t = a_{-1,-1} \qquad \text{where} \qquad f = \textstyle\sum_{s,t} a_{k,l} s^k t^l.$$

While this would work (cf. [Par76], [ParF99], but beware of the topological pitfalls explained by Yekutieli [Yek92]) it is a priori again entirely unclear whether this construction is independent of the choice of the isomorphism.

*Example* 4 (exceptional behaviour). An example where $A(\triangle, \mathcal{O}_X)$ has two summands arises at singularities. Note that for $\operatorname{char} k \neq 2$ the prime ideal $(s^3 + s^2 - t^2)$ in $k[s,t]$ does not remain prime under the adelic completion because the new element $\sqrt{1+s} = \sum_{k\geq 0} \binom{1/2}{k} s^k$ enables a factorization. Instead, we get two irreducible components.



For the flag $\triangle := \{(0) > (s,t)\}$ we obtain

$$\begin{aligned}
A(\triangle, \mathcal{O}_X) &= A({}_{(0)}\triangle\,, k(s)[t]/(s^3+s^2-t^2)) \\
&= \varinjlim_{f\notin(0)} A({}_{(0)}\triangle\,, k[s,t]/(s^3+s^2-t^2)\left\langle f^{-\infty}\right\rangle) \\
&= \varinjlim_{f\notin(0)} \varprojlim_{i} \varinjlim_{g\notin(s,t)} k[s,t]/(t^i, s^i, s^3+s^2-t^2)\left\langle f^{-\infty}\right\rangle\left\langle g^{-\infty}\right\rangle \\
&= \varinjlim_{f\notin(0)} k[[s,t]]/(s\sqrt{1+s}+t)(s\sqrt{1+s}-t)\left\langle f^{-\infty}\right\rangle \\
&= k((s)) \oplus k((s)),
\end{aligned}$$

so that the image of $t$ is $(-s\sqrt{1+s}, +s\sqrt{1+s})$. For the last step in the computation note that the colimit is an Artinian ring, so it is isomorphic to the product over the localizations at its maximal ideals.

A detailed description of the behaviour of adèles especially for flags along singular subvarieties can be found in [Par83], [ParF99]. One can give a precise dictionary between direct summand decompositions in adèles and fibers of singularities under normalization. We recommend Yekutieli [Yek92, §3.3] for a thorough discussion.

**Definition 7** (see [FK00])**.** *For $n \geq 1$ an* $n$-local field with last residue field $k$ *is a complete discrete valuation field whose residue field is an* $(n-1)$*-local field with last residue field* $k$*. Moreover, we call* $k$ *itself the only* 0*-local field with last residue field* $k$*.*

In the formulation of the following proposition, we write $A_Y(-,-)$ to refer to the adèles belonging to the scheme $Y$.

**Proposition 8** (Structure theorem, [Beĭ80, p. 2, 2nd paragr.])**.** *Suppose $X$ is a finite type reduced scheme of pure dimension $n$ over a field $k$ and $\triangle$ a finite subset of*

$$\{(\eta_0 > \cdots > \eta_n) \text{ such that } \operatorname{codim}_X \eta_i = i\} \qquad \subseteq S(X)_n.$$

*Define*

$$\triangle' := \{(\eta_1 > \cdots > \eta_n) \text{ such that } (\eta_0 > \cdots > \eta_n) \in \triangle \text{ for some } \eta_0\}.$$

(1) *Then $A(\triangle, \mathcal{O}_X)$ is a finite direct product of $n$-local fields $\prod K_i$ such that each last residue field is a finite field extension of $k$. Moreover,*

$$A(\triangle', \mathcal{O}_X) \overset{(*)}{\subseteq} \prod \mathcal{O}_i \subseteq \prod K_i = A(\triangle, \mathcal{O}_X), \tag{2.3}$$

*where $\mathcal{O}_i$ denotes the ring of integers of $K_i$ and $(*)$ is a finite ring extension. Each $K_i$ is non-canonically isomorphic to $k'((t_1))\cdots((t_n))$ for $k'/k$ finite.*

(2) *If we instead regard $\triangle'$ as a flag in the closed subscheme $\overline{\{\eta_1\}}$ the decomposition as in Equation 2.3 also exists for $A_{\overline{\{\eta_1\}}}(\triangle', \mathcal{O}_{\overline{\{\eta_1\}}})$. Its field factors equal the residue fields of the $\mathcal{O}_i$ in Equation 2.3. In particular, up to the finite extensions $(*)$, the $n$-local field structure of the $K_i$ in $A_X(\triangle, \mathcal{O}_X)$ is induced from*

$$\begin{array}{ccccc}
A_{\overline{\{\eta_0\}}}(\triangle, \mathcal{O}_X) & & & & \\
\uparrow & & & & \\
A_{\overline{\{\eta_0\}}}(\triangle', \mathcal{O}_X) & \longrightarrow & A_{\overline{\{\eta_1\}}}(\triangle', \mathcal{O}_X) & & \\
 & & \uparrow & & \\
 & & A_{\overline{\{\eta_1\}}}(\triangle'', \mathcal{O}_X) & \longrightarrow & A_{\overline{\{\eta_2\}}}(\triangle'', \mathcal{O}_X) \\
 & & & & \uparrow \\
 & & & & \vdots
\end{array}$$

(3) *For a coherent sheaf $\mathcal{F}$, $A(\triangle, \mathcal{F}) \cong \mathcal{F} \otimes_{\mathcal{O}_X} A(\triangle, \mathcal{O}_X)$.*

*Beware:* Even if $\triangle$ consists only of one flag, the products in Equation 2.3 may have several factors. See Example 4.

The first published proof (of a mild variation) of the above result was given by Yekutieli [Yek92, Theorem 3.3.2]. A different proof can be found in [BGW16a]. We now have described the multidimensional generalization of the infinite-dimensional $k$-vector space $\widehat{\mathcal{K}}_{X,x}$ appearing in §1.

Next, we need to describe the higher analogues of the operator ideals $I^+, I^-$. Since these might seem quite involved, let us axiomatize the precise input datum which the following constructions require:

**Definition 9** ([Beĭ80])**.** *Let $k$ be a field. An* ($n$-fold) cubically decomposed algebra[2] *over $k$ is the datum $(A, (I_i^{\pm}), \tau)$:*

- *an associative $k$-algebra $A$;*
- *two-sided ideals $I_i^+, I_i^-$ such that $I_i^+ + I_i^- = A$ for $i = 1, \dots, n$;*
- *writing $I_i^0 := I_i^+ \cap I_i^-$ and $I_{tr} := I_1^0 \cap \cdots \cap I_n^0$, a $k$-linear map (called* trace*)*

$$\tau : I_{tr}/[I_{tr}, A] \to k.$$

The essence of Beilinson's residue construction uses nothing but the above datum. The reader should therefore not be discouraged by the involved actual construction of it:

Below $\mathrm{Hom}_k(-,-)$ refers to plain $k$-vector space homomorphisms without any further conditions.

**Definition 10** ([Beĭ80])**.** *Suppose $X/k$ is a finite type reduced scheme of pure dimension $n$.*

(1) *Let $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ be given and $M$ a finitely generated $\mathcal{O}_{\eta_0}$-module. Then a* lattice *in $M$ is a finitely generated $\mathcal{O}_{\eta_1}$-module $L \subseteq M$ such that $\mathcal{O}_{\eta_0} \cdot L = M$.*

(2) *For any quasi-coherent sheaf $M$ on $X$ define $M_\triangle := A(\triangle, M)$.*

(3) *Write $\triangle' := {}_{\eta_0}\triangle = \{(\eta_1 > \cdots > \eta_n)\}$. Suppose $M_1, M_2$ are finitely generated $\mathcal{O}_{\eta_0}$-modules. Let $\mathrm{Hom}_\triangle(M_1, M_2)$ be the $k$-submodule of those $f \in \mathrm{Hom}_k(M_{1\triangle}, M_{2\triangle})$ such that for all lattices $L_1 \subset M_1, L_2 \subset M_2$, there exist lattices $L_1' \subset M_1, L_2' \subset M_2$ such that*

$$L_1' \subseteq L_1, \qquad L_2 \subseteq L_2', \qquad f(L_{1\triangle'}') \subseteq L_{2\triangle'}, \qquad f(L_{1\triangle'}) \subseteq L_{2\triangle'}' \tag{2.4}$$

*and for all such $L_1, L_1', L_2, L_2'$ the induced $k$-linear map*

$$\overline{f} : (L_1/L_1')_{\triangle'} \to (L_2'/L_2)_{\triangle'} \tag{2.5}$$

*lies in $\mathrm{Hom}_{\triangle'}(L_1/L_1', L_2'/L_2)$. Define $\mathrm{Hom}_\varnothing(-,-)$ as $\mathrm{Hom}_k(-,-)$.*

(4) *Define $I_{1\triangle}^+(M_1, M_2)$ to consist of those $f \in \mathrm{Hom}_\triangle(M_1, M_2)$ such that there exists a lattice $L \subset M_2$ with $f(M_{1\triangle}) \subseteq L_{\triangle'}$. Respectively, $I_{1\triangle}^-(M_1, M_2)$ consists of those such that there exists a lattice $L \subset M_1$ with $f(L_{\triangle'}) = 0$. Next, for $i = 2, \dots, n$ and both $+/-$ define $I_{i\triangle}^{\pm}(M_1, M_2)$ as those $f \in \mathrm{Hom}_\triangle(M_1, M_2)$ such that for all lattices $L_1, L_1', L_2, L_2'$ as in Equation 2.4 we have*

$$\overline{f} \in I_{(i-1)\triangle'}^{\pm}(L_1/L_1', L_2'/L_2). \tag{2.6}$$

[2]This definition is slightly more general than in [Bra14, Definition 6] because we do not demand that $A$ is unital.

A discussion around this type of structure can be found in Osipov [Osi07]. It can be related to topologizations of $n$-local fields [Cám13], [Yek92]. We refer the reader especially to Yekutieli's work in the context of topological higher local fields [Yek15]. Note the similarity to Definitions 1 and 2. The above definition leads us to the central object of study:

**Definition 11** ([Beĭ80])**.** *In the context of the previous definition, let*

$$E_\triangle := \mathrm{Hom}_\triangle(\mathcal{O}_{\eta_0}, \mathcal{O}_{\eta_0}) \subseteq \mathrm{End}_k(\mathcal{O}_{X\triangle}, \mathcal{O}_{X\triangle}).$$

*Write* $I^\pm_{i\triangle} \subseteq E_\triangle$ *for* $I^\pm_{i\triangle}(\mathcal{O}_{\eta_0}, \mathcal{O}_{\eta_0})$ *and* $i = 1, \dots, n$.

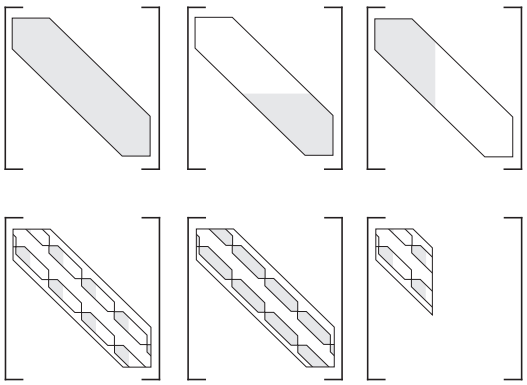
*Example* 5 (toy example [Bra14])*.* The above definition can easily be confusing. It is helpful to look at the structurally simpler, but essentially equivalent case of infinite matrix algebras first: For any associative algebra $R$ define

$$E(R) := \{\phi = (\phi_{ij})_{i,j\in\mathbf{Z}}, \phi_{ij} \in R \mid \exists K_\phi : |i-j| > K_\phi \Rightarrow \phi_{ij} = 0\} \tag{2.7}$$

and equip it with the usual matrix multiplication. Then

$$I^+(R) := \{\phi \in E(R) \mid \exists B_\phi : i < B_\phi \Rightarrow \phi_{ij} = 0\}$$
$$I^-(R) := \{\phi \in E(R) \mid \exists B_\phi : j > B_\phi \Rightarrow \phi_{ij} = 0\}$$

define two-sided ideals in $E(R)$ with $I^+(R) + I^-(R) = E(R)$. We may iterate this construction so that $I^\pm_i := (EE\cdots I^\pm \cdots E)(R)$ (with $I^\pm$ in the $i$-th place) defines a two-sided ideal of $E^n(R) = E\cdots E(R)$. One checks that $(E^nR, \{I^\pm_i\}, \mathrm{tr})$ is an $n$-fold cubically decomposed algebra [Bra14, §1.1].



The top row displays typical matrices from $E(R)$, $I^+(R)$, $I^-(R)$ respectively. The lower row illustrates double infinite matrix constructions, namely $E(I^-(R))$, $E(E(R))$ and $I^-(I^-(R))$ respectively. Although defined in a more complicated way, the ideals of Definition 10 have the same structural properties as these infinite matrix ideals. Note that $E^n(R)$ has a natural $R$-linear action on the Laurent polynomial ring $R[t_1^\pm, \dots, t_n^\pm]$, see [Bra14, §1.1].

**Proposition 12** ([Beĭ80, Theorem (a)])**.** *Suppose $X/k$ is a finite type reduced scheme of pure dimension $n$. Suppose $\triangle = \{(\eta_0 > \cdots > \eta_n)\}$ is a single-element set such that* $\mathrm{codim}_X \overline{\{\eta_i\}} = i$.

(1) *Then* $(E_\triangle, (I^\pm_{i\triangle}), \mathrm{tr}_{I_{tr}})$ *is a unital cubically decomposed algebra over* $k$*, where* $\mathrm{tr}_{I_{tr}}$ *refers to Tate's operator trace (cf. Proposition 3).*
(2) *For every* $f \in I_{tr}$ *there exists a finite-dimensional* $f$*-stable* $k$*-vector subspace* $W \subseteq E_\triangle$ *such that* $\mathrm{tr}_{I_{tr}} f = \mathrm{tr}_W f$*.*

**Proof.** One easily sees that the $I_i^\pm$ are two-sided ideals. For $I_i^+ + I_i^- = E_\triangle$ pick any lattice on the suitable level of the inductive definition and any vector space idempotent projecting on it, call it $P^+$. Then $P^- := \mathbf{1} - P^+$ contains the lattice in the kernel. Clearly, $\mathbf{1} = P^+ + P^-$ and $P^\pm \in I_i^\pm$. It remains to check that Tate's trace is defined on $I_{tr} = I_1^0 \cap \cdots \cap I_n^0$, i.e. that all operators in this ideal are finite-potent, one can argue by induction: Suppose $f \in I_{tr}(V,V)$ for some $V$. In particular $f \in I_n^0(V,V)$, i.e. there exists a lattice $L \subset V$ such that $fL = 0$ and a lattice $L' \subset V$ such that $fV \subseteq L'$. We observe that $f^{\circ 3^n} : V \to V$ factors as

$$f^{\circ 3^n} : V \xrightarrow{f^{\circ 3^{n-1}}} L' \xrightarrow{\text{quot}} \frac{L'}{L \cap L'} \xrightarrow{\overline{f}^{\circ 3^{n-1}}} \frac{L'}{L \cap L'} \xrightarrow{f^{\circ 3^{n-1}}} L' \xrightarrow{\text{incl}} V. \tag{2.8}$$

As $L, L'$ are lattices, $L \cap L'$ is a lattice, so we may take $L_1' = L_2 := L \cap L'$ and $L_1 = L_2' := L'$ as choices in Equation 2.4. As we also have $f \in I_{n-1}^0$, this yields that $\overline{f} \in I_{n-1}^0(L'/(L \cap L'), L'/(L \cap L'))$. Thus, using $V := L'/(L \cap L')$ the middle term $\overline{f}^{\circ 3^{n-1}}$ in Equation 2.8 again satisfies the assumptions for the induction step, just replace $n$ with $n-1$. Proceed down to $n = 1$, where the middle term $\overline{f}^{\circ 1}$ is a morphism of finite-dimensional $k$-vector spaces. Combining all induction steps, this shows that for every $f \in I_{tr}$, $f^{\circ 3^n}$ factors through a finite-dimensional $k$-vector space $W$, so a power of $f$ indeed has finite-dimensional image over $k$, i.e. $f$ is finite-potent. Similarly, the computation of the trace can be reduced to a classical trace: Again, we use induction. Assume $f \in I_n^0$. As the lattices $L, L'$ (chosen as above) are $f$-stable, using axiom **T2** twice yields

$$\mathrm{tr}_V f = \mathrm{tr}_{L'} f + \mathrm{tr}_{V/L'} f = (\mathrm{tr}_{L \cap L'} f + \mathrm{tr}_{L'/L \cap L'} f) + \mathrm{tr}_{V/L'} f.$$

As $\overline{f} \equiv 0$ in the quotient $V/L'$ as well as $f \mid_L = 0$ when restricted to $L$ (and thus $L \cap L'$), axiom **T3** reduces the above to $\mathrm{tr}_{L'/L \cap L'} \overline{f}$. Hence, we have reduced to $\overline{f} : L'/(L \cap L') \to L'/(L \cap L')$. As before it follows that if we also have $f \in I_{n-1}^0(V,V)$, then $\overline{f} \in I_{n-1}^0(L'/(L \cap L'), L'/(L \cap L'))$ and using $V := L'/(L \cap L')$ we again satisfy our initial assumptions for the induction step. If $f \in I_{tr}$, this inductively yields

$$\mathrm{tr}_V f = \cdots = \mathrm{tr}_W \overline{f},$$

where $W$ is a finite-dimensional $k$-vector space. Hence, by **T1** the last trace $\mathrm{tr}_W \overline{f}$ is the ordinary trace of an endomorphism. For $f \in [I_{tr}, A]$ use **T5** to see that $\mathrm{tr}_V f = 0$. □

## 3. Beilinson's construction

In this section we try to be brief. A motivated explanation can be found in §4.

**3.1. Beilinson's functional.** Let us recall Beilinson's construction of the cocycle [Beĭ80]. We begin with some general considerations:

**Definition 13.** *For $V$ a vector space and $V' \subseteq V$ a subspace, we define*

$$V' \wedge \bigwedge^{r-1} V = \left\{ \begin{array}{l} \text{subspace of } \bigwedge^r V \text{ generated by} \\ v' \wedge v_1 \wedge \cdots \wedge v_{r-1} \text{ with } v' \in V',\ v_i \in V \end{array} \right\}$$

*Beware:* Note that $V' \wedge (-)$ is by no means an exact functor in any possible sense. It behaves quite differently from $V' \otimes (-)$.

Let $\mathfrak{g} := A_{Lie}$ be the Lie algebra of an associative algebra $A$ and $M$ a $\mathfrak{g}$-module. Then one has the Chevalley-Eilenberg complex $C_i^{\mathrm{Lie}}(\mathfrak{g}, M) := M \otimes \bigwedge^i \mathfrak{g}$, see [Lod92, §10.1.3] for details. Its homology is ordinary Lie homology. We abbreviate $C_i^{\mathrm{Lie}}(\mathfrak{g}) := C_i^{\mathrm{Lie}}(\mathfrak{g}, k)$ for trivial coefficients. Let $\mathfrak{j} \subseteq \mathfrak{g}$ be a Lie ideal. Then the vector spaces

$$CE(\mathfrak{j})_r := \mathfrak{j} \wedge \bigwedge^{r-1} \mathfrak{g} \tag{3.1}$$

for $r \geq 1$ and $CE(\mathfrak{j})_0 := k$ define a subcomplex of $C_r^{\mathrm{Lie}}(\mathfrak{g}, k)$ via the identification

$$j \wedge f_1 \wedge \cdots \wedge f_{r-1} \approx 1 \otimes j \wedge f_1 \wedge \cdots \wedge f_{r-1}.$$

The differential turns into the nice expression (cf. [Beĭ80, first equation])

$$\delta(f_0 \wedge f_1 \wedge \ldots \wedge f_r) := \textstyle\sum_{0 \leq i < j \leq r} (-1)^{i+j} [f_i, f_j] \wedge f_0 \wedge \ldots \widehat{f_i} \ldots \widehat{f_j} \ldots \wedge f_r. \tag{3.2}$$

*Beware:* Due to the difference between $\mathfrak{j} \wedge (-)$ and $\mathfrak{j} \otimes (-)$ the homology of $CE(\mathfrak{j})_\bullet$ does *not* agree with the Lie homology $H_n(\mathfrak{g}, \mathfrak{j})$ with $\mathfrak{j}$ seen as a $\mathfrak{g}$-module. It is better viewed as relative Lie homology, as explained in §4.

Now suppose $A$ is given the extra structure of a cubically decomposed algebra (cf. Definition 9), i.e.

- two-sided ideals $I_i^+, I_i^-$ such that $I_i^+ + I_i^- = A$ for $i = 1, \ldots, n$;
- writing $I_i^0 := I_i^+ \cap I_i^-$ and $I_{tr} := I_1^0 \cap \cdots \cap I_n^0$, a $k$-linear map

$$\tau : I_{tr}/[I_{tr}, A] \to k.$$

For any elements $s_1, \ldots, s_n \in \{+, -, 0\}$ we define the *degree* $\deg(s_1, \ldots, s_n) := 1 + \#\{i \mid s_i = 0\}$.

*Notation.* Below, we shall frequently encounter indices $s_1, \ldots, s_n$ and when they appear in a subscript or superscript, we often shorten this to writing "$s_1 \ldots s_n$" instead. Moreover, for exponents $s \in \{+, -\}$ we write

$$(-1)^+ := +1 \qquad \text{and} \qquad (-1)^- := -1$$

and $-s$ denotes the opposite sign. Read $(-1)^{s_1 + \cdots + s_n}$ as $(-1)^{s_1} \cdots (-1)^{s_n}$.

Given the above datum, Beilinson constructs a very interesting family of complexes:

**Definition 14** ([Beĭ80])**.** *Define*

$$(3.3)\qquad {}^{\wedge}T^p_\bullet := \bigoplus_{\substack{s_1\dots s_n\in\{\pm,0\}\\ \deg(s_1\dots s_n)=p}} \bigcap_{i=1}^{n} \begin{cases} CE(I_i^+)_\bullet & \text{for } s_i = +\\ CE(I_i^-)_\bullet & \text{for } s_i = -\\ CE(I_i^+)_\bullet \cap CE(I_i^-)_\bullet & \text{for } s_i = 0 \end{cases}$$

*and* ${}^{\wedge}T^0_\bullet := CE(\mathfrak{g})_\bullet$*. View them as complexes in the subscript index* $(-)_\bullet$*.*

Each $CE(I_i^\pm)_\bullet$ is a complex and all their differentials are defined by the same formula, namely Equation 3.2. Thus, the intersection of these complexes has a well-defined differential and is a complex itself. Next, Beilinson shows that

$$(3.4)\qquad 0 \longrightarrow {}^{\wedge}T^{n+1}_\bullet \longrightarrow \cdots \longrightarrow {}^{\wedge}T^1_\bullet \longrightarrow {}^{\wedge}T^0_\bullet \longrightarrow 0$$

is an exact sequence (now indexed by the superscript) with respect to a suitably defined differential coming from a structure as a cubical object (see [Beĭ80, §1] or [Bra14, Lemma 18]). Thus, we obtain a bicomplex

$$(3.5)\qquad \begin{array}{ccccccccc} & & & & & & \to\cdots\to & {}^{\wedge}T^0_2 & \to 0\\ & & \text{etc.} & & \downarrow & & & \downarrow & \\ 0\to & {}^{\wedge}T^{n+1}_1 & \to & {}^{\wedge}T^n_1 & \to\cdots\to & {}^{\wedge}T^0_1 & \to 0\\ & \downarrow & & \downarrow & & \downarrow & & & \\ 0\to & {}^{\wedge}T^{n+1}_0 & \to & {}^{\wedge}T^n_0 & \to\cdots\to & {}^{\wedge}T^0_0 & \to 0. \end{array}$$

Its support is horizontally bounded in degrees $[n+1, 0]$, vertically $(+\infty, 0]$. As a result, the associated two bicomplex spectral sequences are convergent. Since the rows are exact, the one with $E^0$-page differential in direction '$\to$' vanishes already on the $E^1$-page. Thus, this (and therefore both) spectral sequences converge to zero. Now focus on the second spectral sequence, the one with $E^0$-page differential in direction '$\downarrow$'. Since $E^{n+2}_{\bullet,\bullet} = 0$ by horizontal concentration in $[n+1, 0]$, the differential $d : E^{n+1}_{n+1,1} \to E^{n+1}_{0,n+1}$ on the $(n+1)$-st page must be an isomorphism. Upon composing its inverse with suitable edge maps, Beilinson gets a morphism

(3.6)

$$\phi_{Beil} : H_{n+1}(\mathfrak{g}, k) \xrightarrow{\sim} H_{n+1}(CE(\mathfrak{g})) \xrightarrow{\text{edge}} E^{n+1}_{0,n+1} \xrightarrow[d^{-1}]{\sim} E^{n+1}_{n+1,1} \xrightarrow{\text{edge}} H_1({}^{\wedge}T^{n+1}_\bullet) \xrightarrow{\tau} k.$$

For the left-hand side isomorphism note that $H_{n+1}(\mathfrak{g}, k) \cong H_{n+1}(CE(\mathfrak{g}))$ just by definition of Lie homology (*Beware:* this is true for $CE(\mathfrak{j})$ if and only if $\mathfrak{j} = \mathfrak{g}$), and ${}^{\wedge}T^0_\bullet := CE(\mathfrak{g})_\bullet$ by definition. For the right-hand side map $\tau$ observe that

$$(3.7)\qquad H_1({}^{\wedge}T^{n+1}_\bullet) = H_1(\textstyle\bigcap_{i=1}^n \bigcap_{s=\{+,-\}} CE(I_i^s)_\bullet) = \frac{\mathfrak{j}}{[\mathfrak{j},\mathfrak{g}]}$$

for $\mathfrak{j} := \bigcap_{i=1}^{n} \bigcap_{s=\{+,-\}} I_i^s = I_{tr}$. Using the Universal Coefficient Theorem in Lie algebra homology, this is the same as giving an element in $H^{n+1}(\mathfrak{g}, k) \cong H_{n+1}(\mathfrak{g}, k)^*$. This is the proof for Beilinson's result [Beĭ80, Lemma 1 (a)]. We summarize:

**Proposition 15.** *(Beilinson) For every cubically decomposed algebra $(A, (I_i^{\pm}), \tau)$ and $\mathfrak{g} := A_{Lie}$ there is a canonical morphism*

$$\phi_{Beil} : H_{n+1}(\mathfrak{g}, k) \longrightarrow k,$$

*or equivalently a canonical Lie cohomology class in $H^{n+1}(\mathfrak{g}, k)$. It is functorial in morphisms of cubically decomposed algebras.*

Thus, if a commutative $k$-algebra $K$ embeds as $K \hookrightarrow A$, we get a morphism

$$\begin{gathered} \mathrm{res} : \Omega^n_{K/k} \xrightarrow{(\diamond)} H_{n+1}(\mathfrak{g}, k) \xrightarrow{\phi_{Beil}} k \\ f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n \longmapsto f_0 \wedge f_1 \wedge \cdots \wedge f_n \longmapsto \phi_{Beil}(f_0 \wedge \cdots \wedge f_n) \end{gathered}$$

It turns out to be the residue. This is essentially [Beĭ80, Lemma 1 (b) and Theorem (a)]. For a very explicit proof of this see [Bra14, Theorem 4 and Theorem 5]. Note that $(\diamond)$ is not really a morphism; it does not respect the relation $\mathrm{d}(xy) = x\mathrm{d}y + y\mathrm{d}x$. This washes out after composing with $\phi_{Beil}$.

*Remark* 3 (reduces to Tate's theory). It is a general fact from homological algebra that the connecting morphism coming from the snake lemma agrees with the inverse of the suitable differential in the bicomplex spectral sequence applied to the two-row bicomplex which one feeds into the snake lemma. If we apply this remark to Equation 1.4, we readily see how Equation 3.6 transforms into Equation 1.5. This also justifies why $d^{-1} : E^{n+1}_{0,n+1} \to E^{n+1}_{n+1,1}$ is a natural choice to consider.

## 4. Etiology

I will try to explain how one could read Tate's original article and naturally be led to Beilinson's generalization. Clearly, I am just writing down a possible interpretation here and quite likely it has no connection whatsoever with the actual development of the ideas. Since the original papers [Tat68], [Beĭ80] say very little about the underlying creative process, this might be of some use. Of course, logically, this section is superfluous.

I would have liked to begin by explaining Cartier's idea. Tate writes "I arrived at this treatment of residues by considering the special features of the one-dimensional case, after discussing with Mumford an approach of Cartier to Grothendieck's higher dimensional residue symbol" [Tat68, p. 1]. Pierre Cartier told me that he has never published his approach, it was only disseminated in seminar talks by Adrien Douady, whom we sadly cannot ask anymore. It seems possible that the original formulation of Cartier's method has fallen into oblivion. Similarly, John Tate told me that he does not remember more about the history than what is documented in his article.

So allow me to take Tate's method for granted and proceed to Beilinson's generalization.

Firstly, let us reformulate Tate's original construction. As explained in §1, it begins with an exact sequence of Lie modules

$$0 \longrightarrow I^0 \longrightarrow I^+ \oplus I^- \longrightarrow E \longrightarrow 0. \tag{4.1}$$

We may read $I^+ \oplus I^-$ as a Lie algebra itself and hope for $I^0$ being a Lie ideal in there, so that we could view the sequence as an extension of Lie algebras. However, this fails (e.g. $[x \oplus x, a \oplus b] = [x, a] \oplus [x, b]$ has no reason to be diagonal). There is an easy remedy, we quotient out

$$0 \longrightarrow I^0 \longrightarrow I^+ \oplus I^- \longrightarrow \left(I^+ \oplus I^-\right)/I^0 \longrightarrow 0 \tag{4.2}$$

by $I^-$ and push the sequence out along the quotient map, giving

$$0 \longrightarrow I^0 \xrightarrow{i} I^+ \xrightarrow{j} I^+/I^0 \longrightarrow 0. \tag{4.3}$$

Now $I^0$ is indeed a Lie ideal in $I^+$ so that this is an extension of Lie algebras. We may take the homology of Lie algebras with trivial coefficients, i.e. $H_i(-) := H_i(-, k)$. If $C_i^{\mathrm{Lie}}(-)$ denotes the underlying Chevalley-Eilenberg complex, we get an obvious induced morphism $j_* : C_i^{\mathrm{Lie}}(I^+) \to C_i^{\mathrm{Lie}}(I^+/I^0)$, which we would like to fit into a long exact sequence. To this end, define *relative Lie homology* $H_i(I^+ \operatorname{\mathsf{rel}} I^0)$ simply as the co-cone of this morphism $j_*$, so that we get a long exact sequence

$$\cdots \to H_{i+1}(I^+/I^0) \xrightarrow{d} H_i(I^+ \operatorname{\mathsf{rel}} I^0) \to H_i(I^+) \to H_i(I^+/I^0) \xrightarrow{d} \cdots. \tag{4.4}$$

*Remark* 4. This is not to be confused with the long exact sequence in Lie homology $H_i(E, -)$ coming from viewing Equation 4.3 as a short exact sequence of coefficient modules. In Equation 4.4 we change the Lie algebra, not the coefficients.

It would be nice to have a more explicit description of the relative homology groups. Instead of just defining them as an abstract co-cone of complexes, define it (quasi-isomorphically) as the kernel of the map $j_*$ of Chevalley-Eilenberg complexes. Explicitly,

$$0 \to C_i^{\mathrm{Lie}}(I^+ \operatorname{\mathsf{rel}} I^0) \to \bigwedge^i I^+ \to \bigwedge^i (I^+/I^0) \to 0. \tag{4.5}$$

We see that $C_i^{\mathrm{Lie}}(I^+ \operatorname{\mathsf{rel}} I^0) = I^0 \wedge \bigwedge^{i-1} I^+$, the subspace spanned by those exterior tensors with at least one slot lying in $I^0$; see Definition 13. Next, let us address the question to compute the connecting homomorphism $d$ in Equation 4.4. Recall that it is constructed by spelling out the underlying complexes and applying the snake lemma. In the homological degree $H_2 \xrightarrow{d}$

$H_1$, this unravels as the snake map of

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^0 \wedge I^+ & \longrightarrow & I^+ \wedge I^+ & \longrightarrow & (I^+/I^0) \wedge (I^+/I^0) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle [-,-]} & & \downarrow{\scriptstyle [-,-]} & & \downarrow{\scriptstyle [-,-]} & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^+ & \longrightarrow & I^+/I^0 & \longrightarrow & 0 \end{array} \tag{4.6}$$

and by comparison with Diagram 1.4 we find that the connecting homomorphism

$$H_2(I^+/I^0) \longrightarrow H_1(I^+ \text{ rel } I^0) \tag{4.7}$$

agrees (after precomposing with $E \cong (I^+ \oplus I^-)/I^0 \twoheadrightarrow I^+/I^0$) with the snake map used in Tate's construction, see Equation 1.5. We leave it to the reader to spell this out in detail. In summary: Tate's residue can be read as a connecting homomorphism in relative Lie homology.

In the one-dimensional theory we have the notion of a lattice as in Definition 1, e.g. these are the

$$t^i k[[t]] \subset k((t))$$

for any $i \in \mathbf{Z}$ – here we temporarily allow ourselves to use explicit coordinates for the sake of exposition. As we proceed to the two-dimensional theory, the analogue of $k((t))$ will look like $k((s))((t))$ and we get a more complicated pattern of lattices: First of all, there are the "$t$-lattices" like $t^i k((s))[[t]]$ and the quotient of any two such $t$-lattices, say of the pair

$$t^i k((s))[[t]] \subset t^j k((s))[[t]] \qquad \text{with} \qquad j \leq i,$$

is a finite-dimensional $k((s))$-vector space; in this example it is the span

$$\simeq k((s)) \left\langle t^j, t^{j+1}, \ldots, t^{i-1} \right\rangle.$$

In any such space we now get a notion of an "$s$-lattice", namely just in the previous sense, e.g. if $i = j+1$ the quotient is just the span $\simeq k((s)) \left\langle t^j \right\rangle$ and the $s$-lattices would be of the shape $s^i k[[s]] \left\langle t^j \right\rangle \subset k((s)) \left\langle t^j \right\rangle$ for any $i \in \mathbf{Z}$. Two things are important to keep in mind here:
Firstly, for the sake of presentation we have described this in explicit coordinates here. Of course we need to replace the vague notion of "$t$-lattices" and "$s$-lattices" by something which makes no reference to coordinates. See Definition 10 for Beilinson's beautiful solution.
Secondly, there is a true asymmetry between $t$ and $s$. Note that for a field $k((s))((t))$ the roles of $s$ and $t$ are not interchangeable, unlike for $k[[s]][[t]]$. For example, $\sum_{i \geq 0} s^{-i} t^i$ lies in this field, but $\sum_{i \geq 0} t^{-i} s^i$ does not describe an actual element of $k((s))((t))$. This is why we chose to speak of "$s$-lattices" in a *quotient* of $t$-lattices, rather than trying to deal with something like $s^i k[[s]]((t))$. Note for example that $\bigcup_{i \in \mathbf{Z}} s^i k[[s]]((t)) \subsetneq k((s))((t))$. To avoid all pitfalls, it would be best to work in appropriate categories of ind-pro limits right from the start, as in [BGW16c], but this is of course an anachronism.

Based on having two lattice structures instead of just one, in dimension two Beilinson deals with four ideals $I_1^{\pm}, I_2^{\pm}$ instead of just a single pair as in Tate's construction. We may read the exact sequence in Equation 4.1 as a quasi-isomorphism

$$\left[I^0 \longrightarrow I^+ \oplus I^-\right]_{1,0} \xrightarrow{\sim} E$$

with a two-term complex concentrated in homological degrees $[1, 0]$. View these ideals as representing the $t$-lattices of above (e.g. $I_1^+$ would be endomorphisms whose image lies in some $t$-lattice). Then replicating the analogous structure for $s$-lattices leads to the bicomplex

$$\begin{bmatrix} I_1^0 \cap I_2^0 & \longrightarrow & I_1^0 \cap I_2^+ \oplus I_1^0 \cap I_2^- \\ \downarrow & & \downarrow \\ I_1^+ \cap I_2^0 \oplus I_1^- \cap I_2^0 & \longrightarrow & I_1^+ \oplus I_1^- \oplus I_2^+ \oplus I_2^- \end{bmatrix} \xrightarrow{\sim} E.$$

Accordingly, in the theory for $n$ dimensions one gets a structure of $n$ cascading notions of lattices, and correspondingly $2^n$ ideals $I_i^{\pm}$. The above gets replaced by a quasi-isomorphism with an $n$-hypercube. It is a matter of taste whether one prefers to work with multi-complexes or with the ordinary total complex. We prefer the latter, giving a complex concentrated in homological degrees $[n + 1, 0]$, see Equation 3.4 and Equation 5.7.

In order to construct the residue map in dimension two, it seems natural to perform the mechanism of dimension one twice, once for each layer of lattices. Hence, one should study the connecting homomorphism analogous to the one in Equation 4.7. However, things get a bit more complicated, because if we try to compose two such connecting homomorphisms, we find that the input of the second step should be the relative Lie homology group which is the output of the first step. This leads to *bi-relative Lie homology*, defined just as the kernel on the left-hand side in

$$0 \to C_i^{\mathrm{Lie}}(I_1^+ \;\mathsf{rel}\; I_1^0 \;\mathsf{rel}\; I_2^0) \to C_i^{\mathrm{Lie}}(I_1^+ \;\mathsf{rel}\; I_1^0) \to C_i^{\mathrm{Lie}}(I_1^+/I_2^0 \;\mathsf{rel}\; I_1^0/I_2^0) \to 0.$$

Here we allow ourselves to write $I_1^+/I_2^0$ as a shorthand for $\frac{I_1^+}{I_2^0 \cap I_1^+}$ to improve legibility. Now we are able to compose the associated connecting homomorphism with the one of Equation 4.7, giving something like

$$H_3(I^+/I_1^0 I_2^0) \xrightarrow{d} H_2(I^+/I_2^0 \;\mathsf{rel}\; I_1^0) \xrightarrow{d} H_1(I^+ \;\mathsf{rel}\; I_1^0 \;\mathsf{rel}\; I_2^0).$$

We should make the bi-relative Lie homology more explicit: Unwinding complexes as in Equation 4.5, we see that

$$0 \to C_i^{\mathrm{Lie}}(I_1^+ \;\mathsf{rel}\; I_1^0 \;\mathsf{rel}\; I_2^0) \to I_1^0 \wedge \bigwedge^{i-1} I_1^+ \to I_1^0/I_2^0 \wedge \bigwedge^{i-1}(I_1^+/I_2^0) \to 0$$

and therefore

$$C_i^{\mathrm{Lie}}(I_1^+ \;\mathsf{rel}\; I_1^0 \;\mathsf{rel}\; I_2^0) = \bigcap_{i=1,2} \left(I_i^0 \wedge \bigwedge^{i-1} I_1^+\right). \tag{4.8}$$

The reader will have no difficulty in checking that $i$-fold multi-relative Lie homology can be defined accordingly, and leads to further intersections of

subcomplexes as in Equation 4.8. This explains the underlying structure of Beilinson's complex ${}^{\wedge}T^p_\bullet$, see Equation 3.3 (or, this is my interpretation. There is no mention of relative homology in [Beĭ80]). In fact, ${}^{\wedge}T^p_\bullet$ is a tiny bit more complicated because it works with all $2^n$ ideals $I_i^{\pm}$ and $E$ instead of quotienting out the $I^-$-ideals and working with $I^+$ only, i.e. without the simplification coming from switching from Equation 4.2 to Equation 4.3.

Let us pause for a second. What happens if we ignore Remark 4 and phrase Tate's construction in terms of a long exact sequence, this time with varying coefficients? The diagram 4.6 turns into

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I^0 \otimes E & \longrightarrow & I^+ \otimes E & \longrightarrow & (I^+/I^0) \otimes E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle [-,-]} & & \downarrow{\scriptstyle [-,-]} & & \downarrow{\scriptstyle [-,-]} & & \\ 0 & \longrightarrow & I^0 & \longrightarrow & I^+ & \longrightarrow & I^+/I^0 & \longrightarrow & 0 \end{array}$$

and Equation 4.7 gets replaced by

$$H_1(E, I^+/I^0) \longrightarrow H_0(E, I^0).$$

Besides the index shift, this map also gives Tate's residue[3]. Hence, it is actually possible to set up the entire theory using Lie homology with coefficients instead of relative Lie homology. This is the path taken in the previous paper [Bra14]; the corresponding variant of Beilinson's complex ${}^{\wedge}T^p_\bullet$ is called ${}^{\otimes}T^p_\bullet$ in *loc. cit.* Both variants in general give different maps (and begin and end in different homology groups), but still they are largely compatible [Bra14, Lemma 23] and both give the multi-dimensional residue [Bra14, Theorem 4 and 5].

The coefficient variant is more manageable for explicit computations: The problem with complexes like $I^0 \wedge \bigwedge^{i-1} I^+$ is that it is difficult to write down explicit bases for these spaces because the only natural candidate are pure tensors

$$f_0 \otimes f_1 \otimes \cdots \otimes f_{i-1}$$

with $f_0, \ldots, f_{i-1}$ ascendingly taken from an ordered basis of $I^+$ so that $f_0 \in I^0$. Performing calculations, it quickly becomes very tedious to maintain elements in this standard ordered shape.

In the next section §5 we propose yet another point of view. First of all, motivated by the strong relation between the Hodge $n$-part of Hochschild homology and Lie homology, we replace Lie homology by (the full) Hochschild homology. This poses no problem since all the Lie algebras/ideals we have encountered above are actually coming from associative algebras and ordinary ideals. For example, the sequence in Equation 4.4 will be replaced

[3]I find it noteworthy that essentially the same computation admits at least two (quite different) homological interpretations.

by

$$\cdots \to HH_{i+1}(I^+/I^0) \xrightarrow{d} HH_i(I^+ \text{ rel } I^0) \to HH_i(I^+) \to HH_i(I^+/I^0) \xrightarrow{d} \cdots.$$

However, now a substantial simplification occurs: In certain circumstances relative Hochschild homology agrees with absolute Hochschild homology, in the sense that the natural morphism

$$HH_i(I^0) \longrightarrow HH_i(I^+ \text{ rel } I^0)$$

sometimes happens to be an isomorphism. This is known as *excision*; it is easily seen to be wrong for arbitrary ideals but it turns out that the ideals $I_i^0$ have the necessary property. This spares us from having to work with multi-relative homology at all. Instead, we can just compose the corresponding $n$ connecting maps, one by one, and we will prove that this again gives the same map, but now its construction necessitates much less effort, §6. We will also see that it is much easier to compute this map explicitly, saving us from a lot of trouble we had to go through in [Bra14].

## 5. Hochschild and cyclic picture

In this section we will formulate an analogue of Beilinson's construction in the context of Hochschild (and later also cyclic) homology. We follow the natural steps:

(1) We replace Lie homology with Hochschild homology. This is harmless since cubically decomposed algebras come with an associative product structure anyway. There is a natural map

$$\varepsilon : H_\bullet(A_{Lie}, M_{Lie}) \longrightarrow H_\bullet(A, M),$$

ultimately explaining numerous similarities.

(2) The Hochschild complex is modelled on chain groups $A \otimes \cdots \otimes A$ instead of exterior powers. Thus, the only reasonable replacement of the mixed exterior powers/relative homology groups

$$CE(\mathfrak{j})_r := \mathfrak{j} \wedge \textstyle\bigwedge^{r-1} \mathfrak{g}$$

in the original construction are the groups $J \otimes A \otimes \cdots \otimes A$ for $J$ an ideal. This is very convenient, as this just gives Hochschild homology with coefficients $H_r(A, J)$. Alternatively, one could work with relative Hochschild groups. We will return to a relative perspective in §6.

To set up notation, let us very briefly recall the necessary structures in Hochschild homology. See [Lod92, Ch. I] for a detailed treatment. Suppose $A$ is an arbitrary (not necessarily unital) associative $k$-algebra. Let $M$ be an $A$-bimodule over $k$, or equivalently a left-$A \otimes_k A^{\mathrm{op}}$-module. Define chain groups $C_i(A, M) := M \otimes_k A^{\otimes i}$ and a differential $b : C_i(A, M) \to C_{i-1}(A, M)$,

given by

$$\begin{aligned} m \otimes a_1 \otimes \cdots \otimes a_i &\mapsto ma_1 \otimes a_2 \otimes \cdots \otimes a_i \\ &\quad + \textstyle\sum_{j=1}^{i-1} (-1)^j\, m \otimes a_1 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_i \\ &\quad + (-1)^i\, a_i m \otimes a_1 \otimes \cdots \otimes a_{i-1}. \end{aligned} \tag{5.1}$$

We call the homology of the complex $(C_\bullet(A,M), b)$ its *Hochschild homology*, denoted by $H_i(A,M)$. Write $A_{Lie}$ for the Lie algebra associated to $A$ via $[x,y] := x \cdot y - y \cdot x$. There is a canonical morphism

$$\begin{aligned} \varepsilon : C_i^{\mathrm{Lie}}(A_{Lie}, M_{Lie}) &\to C_i(A,M) \\ m \otimes a_1 \wedge \cdots \wedge a_i &\mapsto m \otimes \sum_{\pi \in \mathfrak{S}_i} \mathrm{sgn}(\pi) a_{\pi^{-1}(1)} \otimes \cdots \otimes a_{\pi^{-1}(i)}, \end{aligned} \tag{5.2}$$

where $\mathfrak{S}_i$ is the symmetric group on $i$ letters. This is a morphism of complexes, in particular it induces a morphism $H_i(A_{Lie}, M_{Lie}) \to H_i(A,M)$.

For the rest of this section assume $A$ is *unital*. Clearly $A$ is a bimodule over itself and we write $HH_i(A) := H_i(A,A)$ as an abbreviation (see §6.2 for the correct definition when $A$ is not unital). A $k$-algebra morphism $f : A \to A'$ induces a map $f_* : HH_i(A) \to HH_i(A')$. The motivation for using Hochschild homology in the context of residue theory stems from the following famous isomorphism:

**Proposition 16.** *(Hochschild-Kostant-Rosenberg) Suppose $A/k$ is a commutative smooth $k$-algebra. Then the morphism*

$$\begin{aligned} \Omega^n_{A/k} &\longrightarrow HH_n(A) \\ f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n &\longmapsto \sum_{\pi \in \mathfrak{S}_n} \mathrm{sgn}(\pi) f_0 \otimes f_{\pi^{-1}(1)} \otimes \cdots \otimes f_{\pi^{-1}(n)} \end{aligned} \tag{5.3}$$

*is an isomorphism of graded commutative algebras.*

See [Lod92, Theorem 3.4.4]. Let us now assume that $\mathbf{Q} \subseteq k$: On $A^{\otimes(i+1)}$ recall that there is an action by Connes' cyclic permutation operator

$$t : a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto (-1)^i\, a_i \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_{i-1}.$$

Define the cyclic chain groups by $CC_i(A) := A^{\otimes(i+1)}/(1-t)$; this is the quotient by the action of $t$ on pure tensors. As was discovered by Connes, it turns out that the differential $b$ remains well-defined on these quotients. Its homology is known as *cyclic homology* and denoted by $HC_i(A)$. We shall also need *Connes' periodicity sequence* [Lod92, Theorem 2.2.1]: There is a long exact sequence

$$\cdots \longrightarrow HH_i(A) \xrightarrow{I} HC_i(A) \xrightarrow{S} HC_{i-2}(A) \xrightarrow{B} HH_{i-1}(A) \longrightarrow \cdots \tag{5.4}$$

where $I$ is induced from the obvious inclusion/quotient map on the level of complexes.

*Remark* 5. At the expense of a more complicated definition of the cyclic chain groups, all of these facts remain available without the simplifying assumption $\mathbf{Q} \subset k$; see [Lod92, Theorem 2.1.5, we work with $H^\lambda$ of *loc. cit.*]. We leave the necessary modifications to the reader.

We shall moreover employ the map (recall that $\mathfrak{g} := A_{Lie}$)

$$\begin{aligned} I' : H_n(\mathfrak{g}, \mathfrak{g}) &\longrightarrow H_{n+1}(\mathfrak{g}, k) \\ f_0 \otimes f_1 \wedge \cdots \wedge f_n &\longmapsto (-1)^n \otimes f_0 \wedge \cdots \wedge f_n \end{aligned} \tag{5.5}$$

in Lie homology. The $(-1)^n$ is needed to make the differentials compatible.

**Proposition 17.** *(Connes, Loday-Quillen) Suppose $A/k$ is a commutative smooth $k$-algebra and* $\operatorname{char} k = 0$. *Then there is a canonical isomorphism*

$$HC_n(A) \to \Omega^n_{A/k}/\mathrm{d}\Omega^{n-1}_{A/k} \oplus \textstyle\bigoplus_{i\geq 1} H^{n-2i}_{\mathrm{dR}}(A)$$

*and $I : HH_n(A) \to HC_n(A)$ identifies with the quotient map $\Omega^n_{A/k} \to \Omega^n_{A/k}/\mathrm{d}\Omega^{n-1}_{A/k}$ and zero on the lower deRham summands.*

See [Lod92, Theorem 3.4.12 and remark]. The direct summand decomposition on the right-hand side can be identified with the Hodge decomposition of cyclic homology due to Gerstenhaber and Schack [GS87].

**5.1. Hochschild setup.** Let $A$ be a cubically decomposed algebra over $k$. We define $A$-bimodules $N^0 := A$ and for $p \geq 1$

$$N^p := \bigoplus\nolimits_{\substack{s_1 \ldots s_n \in \{+,-,0\} \\ \deg(s_1 \ldots s_n) = p}} I_1^{s_1} \cap I_2^{s_2} \cap \cdots \cap I_n^{s_n} \tag{5.6}$$

with degree $\deg(s_1, \ldots, s_n) := 1 + \#\{i \mid s_i = 0\}$ as before. Each $I_i^{\pm}$ is a two-sided ideal and thus an $A$-bimodule.

We shall denote the components $f = (f_{s_1 \ldots s_n})$ of elements in $N^p$ with indices in terms of $s_1, \ldots, s_n \in \{+, -, 0\}$. Clearly $N^p = 0$ for $p > n+1$. We get an exact sequence of $A$-bimodules

$$0 \longrightarrow N^{n+1} \xrightarrow{\partial} N^n \xrightarrow{\partial} \cdots \xrightarrow{\partial} N^0 \longrightarrow 0 \tag{5.7}$$

by using the following differential

$$\begin{aligned} (\partial f)_{s_1 \ldots s_n} &:= \sum_{\{i \mid s_i = +,-\}} (-1)^{\#\{j \mid j > i \text{ and } s_j = 0\}} f_{s_1 \ldots 0 \ldots s_n} \text{ (for } N^i \to N^{i-1},\ i \geq 2) \\ \partial f &:= \sum_{s_1 \ldots s_n \in \{+,-\}} (-1)^{s_1 + \cdots + s_n} f_{s_1 \ldots s_n} \text{ (for } N^1 \to N^0) \end{aligned}$$

It is straight-forward to check that $\partial^2 = 0$ holds, but more details are found in [Bra14, §4] nonetheless. As tensoring with $(-) \otimes A^{\otimes(r-1)}$ is exact, we can functorially take the Hochschild complex and obtain a bicomplex with

exact rows, fairly similar to the bicomplex that we have encountered before in Equation 3.5,

$$\begin{array}{ccccccccc} & & & & & & \to\cdots\to & C_2(A,N^0) & \to 0 \\ & & \text{etc.} & & \downarrow & & & \downarrow & \\ (5.8) \quad 0\to & C_1(A,N^{n+1}) & \to & C_1(A,N^n) & \to\cdots\to & C_1(A,N^0) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & & & \\ 0\to & C_0(A,N^{n+1}) & \to & C_0(A,N^n) & \to\cdots\to & C_0(A,N^0) & \to 0 \end{array}$$

As before its support is horizontally bounded in degrees $[n+1,0]$, vertically $(+\infty,0]$; we get an analogous differential on the $E^{n+1}$-page, which is an isomorphism. Proceeding as before, *but this time considering degree $n$ instead of $n+1$*, we obtain

(5.9)

$$\phi_{HH} : HH_n(A) \xrightarrow{\sim} H_n(A,N^0) \xrightarrow{\text{edge}} E^{n+1}_{0,n} \xrightarrow[d^{-1}]{\sim} E^{n+1}_{n+1,0} \xrightarrow{\text{edge}} H_0(A,N^{n+1}) \xrightarrow{\tau} k.$$

The consideration with the trace $\tau$ of the cubically decomposed algebra is exactly the same as before since

$$H_0(A,N^{n+1}) = \frac{N^{n+1}}{[N^{n+1},A]},$$

but $N^{n+1} = I_1^{s_1} \cap I_2^{s_2} \cap \cdots \cap I_n^{s_n} = I_{tr}$, so we obtain exactly the same object as in the Lie counterpart, see Equation 3.7. In particular, the trace $\tau$ is applicable for the same reasons as before. This leads to the following new construction:

**Proposition 18.** *For every cubically decomposed algebra $(A,(I_i^{\pm}),\tau)$ over $k$, there is a canonical morphism*

$$\phi_{HH} : HH_n(A) \longrightarrow k.$$

*It is functorial in morphisms of cubically decomposed algebras.*

Let us explain how to obtain an explicit formula for the fairly abstract construction of $\phi_{HH}$. To this end, we employ the following tool from the theory of spectral sequences:

**Lemma 19** ([Bra14, Lemma 19])**.** *Suppose we are given a bounded exact sequence*

$$S^\bullet = [S^{n+1} \to S^n \to \cdots \to S^0]_{n+1,0}$$

*of bounded below complexes of $k$-vector spaces; or equivalently a correspondingly bounded bicomplex.*

(1) *There is a second quadrant homological spectral sequence $(E^r_{p,q}, d_r)$ converging to zero such that*

$$E^1_{p,q} = H_q(S^p_\bullet). \qquad (d_r : E^r_{p,q} \to E^r_{p-r,q+r-1})$$

(2) *The following differentials are isomorphisms:*

$$d_{n+1} : E^{n+1}_{n+1,0} \to E^{n+1}_{0,n}\,.$$

(3) *If $H_p : S^p \to S^{p+1}$ is a contracting homotopy for $S^\bullet$, then*

$$(d_{n+1})^{-1} = H_n\delta_1 H_{n-1}\cdots\delta_{n-1}H_1\delta_n H_0 = H_n\textstyle\prod_{i=1,\dots,n}(\delta_i H_{n-i}). \tag{5.10}$$

This result can be applied to the bicomplex of Equation 5.8. The required contracting homotopy can be constructed from a suitable family of commuting idempotents in the cubically decomposed algebra as in Definition 20:

**Definition 20** ([Bra14, Def. 14])**.** *Suppose $A$ is an $n$-fold unital cubically decomposed algebra. $A$* system of good idempotents *are pairwise commuting elements $P_i^+ \in A$ (with $i = 1,\dots,n$) such that the following conditions are met:*

- $(P_i^+)^2 = P_i^+$.
- $P_i^+ A \subseteq I_i^+$.
- $P_i^- A \subseteq I_i^-$ *(where $P_i^- := \mathbf{1}_A - P_i^+$).*

The elements $P_i^-$ then are pairwise commuting idempotents as well. We can use the contracting homotopy developed in an earlier paper:

**Lemma 21** ([Bra14, Lemma 16])**.** *Let $A$ be unital and $\{P_i^+\}$ a system of good idempotents. An explicit contracting homotopy $H : N^i \to N^{i+1}$ for the complex $N^\bullet$ of Equation 5.7 is given by*

$$\begin{aligned}(Hf)_{s_1\dots s_n} = (-1)^{\deg(s_1\dots s_n)}\,(-1)^{s_1+\dots+s_b}\,P_1^{s_1}\cdots P_b^{s_b}\\ \sum_{\gamma_1\dots\gamma_{b+1}\in\{\pm\}} (-1)^{\gamma_1+\dots+\gamma_b}\,P_{b+1}^{-\gamma_{b+1}} f_{\gamma_1\dots\gamma_{b+1}s_{b+2}\dots s_n}\end{aligned} \tag{5.11}$$

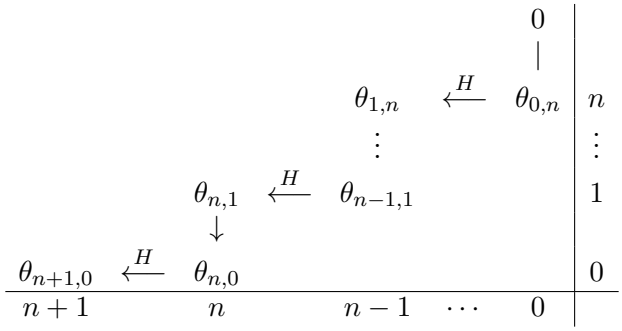
*for $N^i \to N^{i+1}$ with $i \geq 1$, where $b$ is the largest index such that $s_1,\dots,s_b \in \{\pm\}$ or $b = 0$ if none. It is given by*

$$(Hf)_{s_1\dots s_n} = (-1)^{s_1+\dots+s_n} P_1^{s_1}\cdots P_n^{s_n} f \tag{5.12}$$

*for $N^0 \to N^1$.*

By tensoring $(-)\otimes A^{\otimes(r-1)}$ this induces a contracting homotopy for the rows in the bicomplex of Equation 5.8. The evaluation of the formula in Equation 5.10 corresponds to following a zig-zag in the bicomplex which can be depicted graphically as:

(5.13)



If $\theta_{0,n} = f_0 \otimes \cdots \otimes f_n$ represents an element in $E^{n+1}_{0,n}$ arising from the first part of the definition of $\phi_{HH}$ (cf. Equation 5.9)

$$HH_n(A) \xrightarrow{\sim} H_n(A, N^0) \xrightarrow{\text{edge}} E^{n+1}_{0,n} \ni \theta_{0,n},$$

we can compute $d^{-1} : E^{n+1}_{0,n} \xrightarrow{\sim} E^{n+1}_{n+1,0}$ by Equation 5.10. We claim:

**Lemma 22.** *Let $A$ be unital and $\{P_i^+\}$ a system of good idempotents. Starting with $\theta_{0,n} = f_0 \otimes \cdots \otimes f_n$, we get for $s_1, \ldots, s_{n-p} \in \{+,-\}$ the formula*

$$\begin{aligned}\theta_{p+1,n-p|s_1\ldots s_{n-p}0\ldots 0} &= (-1)^{n+(n-1)+\ldots+(n-p+1)} \\ &(-1)^{2+3+\cdots+(p+1)}\,(-1)^{s_1+\cdots+s_{n-p}}\,P_1^{s_1}\cdots P_{n-p}^{s_{n-p}} \\ &\left(\prod_{i=n-p+1}^{n}\left(\sum_{\gamma_i\in\{\pm\}}(-1)^{\gamma_i}\,P_i^{-\gamma_i} f_i P_i^{\gamma_i}\right)\right) f_0 \otimes f_1 \otimes \cdots \otimes f_{n-p}\end{aligned}$$

*for the terms in Fig. 5.13. Here the product (whose factors need not commute) is to be expanded left to right as the index $i$ increases, so $i = n-p+1$ contributes the leftmost factor, $i = n$ the rightmost. The product is to be read as the identity map for $p = 0$.*

This is the Hochschild counterpart of [Bra14, Proposition 24]. The proof will be very similar to the one given for the Lie homology counterpart in [Bra14], but actually quite a bit *less* involved.

**Proof.** We prove this by induction on $p$, starting from $p = 0$. In this case, the claim reads

$$\theta_{1,n|s_1\ldots s_n} = (-1)^{s_1+\cdots+s_n}\,P_1^{s_1}\cdots P_n^{s_n} f_0 \otimes f_1 \otimes \cdots \otimes f_n,$$

which is clearly true in view of Equation 5.12. Next, assume the claim is known for a given $p$ and we want to treat the case $p+1$, i.e. we need to evaluate a Hochschild differential $b$ and pick a preimage as in the step

$$\begin{array}{ccc} & & \theta_{p+1,n-p} \\ & & \downarrow b \\ \theta_{p+2,n-p-1} & \xleftarrow{H} & \theta_{p+1,n-p-1} \end{array}$$

of Fig. 5.13. According to our induction hypothesis, we get $\theta_{p+1,n-p|s_1\ldots s_{n-p}0\ldots 0} = Mf_0 \otimes f_1 \otimes \cdots \otimes f_{n-p}$ with the auxiliary expression

$$\begin{aligned}M = (-1)^{n+(n-1)+\ldots+(n-p+1)}&(-1)^{2+3+\cdots+(p+1)}\,(-1)^{s_1+\cdots+s_{n-p}}\,P_1^{s_1}\cdots P_{n-p}^{s_{n-p}} \\ &\prod_{i=n-p+1}^{n}\left(\sum_{\gamma_i\in\{\pm\}}(-1)^{\gamma_i}\,P_i^{-\gamma_i} f_i P_i^{\gamma_i}\right).\end{aligned}$$

The Hochschild differential $b$ naturally decomposes into three parts (cf. Equation 5.1)

$$\begin{aligned}
\theta^{(A)}_{p+1,n-p-1} &= Mf_0f_1 \otimes f_2 \otimes \cdots \otimes f_{n-p},\\
\theta^{(B)}_{p+1,n-p-1} &= \textstyle\sum_{j=1}^{n-p-1} (-1)^j\, Mf_0 \otimes f_1 \otimes \cdots \otimes f_jf_{j+1} \otimes \cdots \otimes f_{n-p},\\
\theta^{(C)}_{p+1,n-p-1} &= (-1)^{n-p} f_{n-p}Mf_0 \otimes f_1 \otimes \cdots \otimes f_{n-p-1}
\end{aligned}$$

(here we have suppressed the subscript $(-)_{s_1\dots s_{n-p}0\dots0}$ for the sake of readability). Next, we need to evaluate $\theta^{(-)}_{p+2,n-p-1} := H\theta^{(-)}_{p+1,n-p-1}$ for the cases $A, B, C$. Let us consider case $C$: In this case, we just use Equation 5.11 and plugging in $M$, we get $\theta^{(C)}_{p+2,n-p-1|s_1\dots s_{n-p-1}0\dots0} =$

$$\begin{aligned}
&(-1)^{n-p}\,(-1)^{\deg(s_1\dots s_{n-p-1}0\dots0)}\\
&\quad (-1)^{s_1+\cdots+s_{n-p-1}}\, P_1^{s_1}\cdots P_{n-p-1}^{s_{n-p-1}}\\
&\qquad \sum_{\gamma_1\dots\gamma_{n-p}\in\{\pm\}} (-1)^{\gamma_1+\cdots+\gamma_{n-p-1}}\, P_{n-p}^{-\gamma_{n-p}} f_{n-p}\\
&\quad (-1)^{n+(n-1)+\dots+(n-p+1)}(-1)^{2+3+\cdots+(p+1)}\,(-1)^{\gamma_1+\cdots+\gamma_{n-p}}\, P_1^{\gamma_1}\cdots P_{n-p}^{\gamma_{n-p}}\\
&\qquad \left(\sum_{\gamma_{n-p+1}\in\{\pm\}} (-1)^{\gamma_{n-p+1}}\, P_{n-p+1}^{-\gamma_{n-p+1}} f_{n-p+1} P_{n-p+1}^{\gamma_{n-p+1}}\right)\cdot(\cdots)\cdot\\
&\qquad \left(\sum_{\gamma_n\in\{\pm\}} (-1)^{\gamma_n} P_n^{-\gamma_n} f_n P_n^{\gamma_n}\right) f_0 \otimes f_1 \otimes \cdots \otimes f_{n-p-1}
\end{aligned}$$

This fairly complicated expression unwinds into something much simpler by several observations: (1) There is a large cancellation in the sign terms $(-1)^{(\dots)}$, (2) we have $\deg(s_1,\dots,s_{n-p-1},0,\dots,0) = p+2$, (3) the pairwise commutativity of the idempotents allows us to reorder terms so that we obtain the expression $\sum_{\gamma_1\dots\gamma_{n-p-1}\in\{\pm\}} P_1^{\gamma_1}\cdots P_{n-p-1}^{\gamma_{n-p-1}}$, but this is just the identity operator by using the fact $P_i^+ + P_i^- = \mathbf{1}$. Finally, we arrive at

$$\begin{aligned}
\theta^{(C)}_{p+2,n-p-1|s_1\dots s_{n-p-1}0\dots0} = (-1)^{n+(n-1)+\dots+(n-p+1)+(n-p)}&(-1)^{2+3+\cdots+(p+2)}\\
&(-1)^{s_1+\cdots+s_{n-p-1}}\, P_1^{s_1}\cdots P_{n-p-1}^{s_{n-p-1}}\\
&\left(\sum_{\gamma_{n-p}\in\{\pm\}} (-1)^{\gamma_{n-p}}\, P_{n-p}^{-\gamma_{n-p}} f_{n-p} P_{n-p}^{\gamma_{n-p}}\right)\cdots\\
&\cdots\left(\sum_{\gamma_n\in\{\pm\}} (-1)^{\gamma_n} P_n^{-\gamma_n} f_n P_n^{\gamma_n}\right) f_0 \otimes f_1 \otimes \cdots \otimes f_{n-p-1}.
\end{aligned}$$

In a similar fashion we can deal with the cases $A, B$, however in both these cases we obtain a term $P_i^{\gamma_i}P_i^{-\gamma_i} = P_i^{\gamma_i}(\mathbf{1} - P_i^{\gamma_i}) = 0$, so that these terms vanish. We leave the details to the reader (a similar cancellation occurs in the proof of [Bra14, Proposition 24], the cancellation is explained by the

very beautiful identity[4] $H^2 = 0$, which holds for this particular contracting homotopy). Hence, $\theta_{p+2,n-p-1} = \theta^{(C)}_{p+2,n-p-1}$, giving the claim. □

**Theorem 23.** *Let $(A, (I_i^{\pm}), \tau)$ be a unital cubically decomposed algebra over $k$ and $\{P_i^+\}$ a system of good idempotents. Then the explicit formula*

$$\phi_{HH}(f_0 \otimes \cdots \otimes f_n) = (-1)^n \tau \left( \prod_{i=1}^{n} \sum_{\gamma_i \in \{\pm\}} (-1)^{\gamma_i} P_i^{-\gamma_i} f_i P_i^{\gamma_i} \right) f_0$$

*holds, where the product over $i$ is to be expanded from left to right, i.e. $i = 1$ corresponds to the leftmost factor.*

**Proof.** Use the lemma with $p = n$ and compose with the trace $\tau$ as in the definition of $\phi_{HH}$ in Equation 5.9. □

**Corollary 24.** *Let $(A, (I_i^{\pm}), \tau)$ be a unital cubically decomposed algebra over $k$, and let $\mathfrak{g} := A_{Lie}$ be the associated Lie algebra. Then the diagram*

$$\begin{array}{ccc} H_n(\mathfrak{g}, \mathfrak{g}) & \xrightarrow{\varepsilon} & HH_n(A) \\ {\scriptstyle I'}\downarrow & & \downarrow{\scriptstyle \phi_{HH}} \\ H_{n+1}(\mathfrak{g}, k) & \xrightarrow[\phi_{Beil}]{} & k \end{array}$$

*commutes up to sign. Here $\varepsilon$ refers to the comparison map from Equation 5.2. The composition $\phi_{HH} \circ \varepsilon$ is given by the commutator formula*

(5.14)
$$f_0 \otimes f_1 \wedge \cdots \wedge f_n \mapsto (-1)^n \tau \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \sum_{\gamma_1 \ldots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_n} (P_1^{-\gamma_1} \operatorname{ad}(f_{\sigma^{-1}(1)}) P_1^{\gamma_1}) \cdots (P_n^{-\gamma_n} \operatorname{ad}(f_{\sigma^{-1}(n)}) P_n^{\gamma_n}) f_0.$$

*If $n = 1$ and $[f_0, f_1] = 0$, then this specializes to*

$$f_0 \otimes f_1 \mapsto \tau[P_1^+ f_0, f_1]. \tag{5.15}$$

The last equation links these formulae with the classical one-dimensional case as found in Equation 1.6.

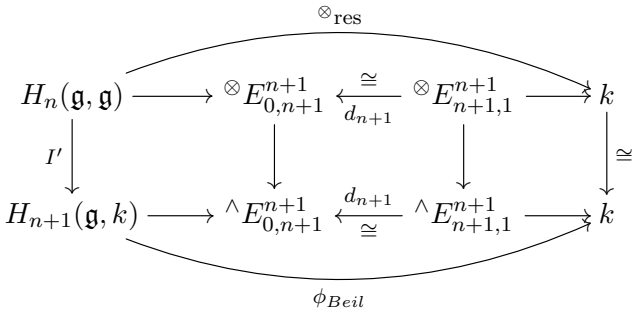
**Proof.** Let $\{P_i^+\}$ be any system of good idempotents. A direct computation of $\phi_{HH} \circ \varepsilon$ yields the explicit formula

$$f_0 \otimes f_1 \wedge \cdots \wedge f_n \mapsto (-1)^n \tau \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma) \sum_{\gamma_1 \ldots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_n} (P_1^{-\gamma_1} f_{\sigma^{-1}(1)} P_1^{\gamma_1}) \cdots (P_n^{-\gamma_n} f_{\sigma^{-1}(n)} P_n^{\gamma_n}) f_0,$$

which agrees (up to sign) with the morphism $^{\otimes}\operatorname{res}_*$ described in [Bra14, Theorem 25, and following discussion]. The commutativity then follows from

[4] pointed out to me by the anonymous referee of [Bra14]

[Bra14, Lemma 23]: Extended on the right with the trace, this reads




in the notation of the reference. The formula $P^{-\gamma}\,\mathrm{ad}(f)P^{\gamma}g = P^{-\gamma}[f,P^{\gamma}g] = P^{-\gamma}fP^{\gamma}g - P^{-\gamma}P^{\gamma}gf = P^{-\gamma}fP^{\gamma}g$ (since $P^{-\gamma}P^{\gamma}=0$) implies Equation 5.14. For $n=1$, this specializes to

$$\begin{aligned} f_0\otimes f_1 &\mapsto -\tau\sum_{\gamma\in\{\pm\}}(-1)^{\gamma}P_1^{-\gamma}[P_1^{\gamma}f_0,f_1] \\ &= \tau(-P_1^{+}([f_0,f_1]-[P_1^{+}f_0,f_1])+P_1^{-}[P_1^{+}f_0,f_1]) \end{aligned}$$

and if $[f_0,f_1]=0$ (as would be the case if $f_0,f_1$ are functions on a variety) this simplifies to Equation 5.15 by using $P_1^{+}+P_1^{-}=\mathbf{1}$. □

After these general considerations regarding cubically decomposed algebras, let us turn to geometry.

**Proposition 25.** *Let $k$ be a field and $k'/k$ a finite field extension. For the equicharacteristic $n$-local field*

$$K := k'((t_1))\cdots((t_n)),$$

*consider the $\phi_{HH}$ associated to its standard cubically decomposed algebra $E_K$ (we refer to* [Yek15], [BGW16a], [BGW16b]*, or see the proof for an explanation).*

(1) *Then for all $\beta\in k'$, we have*

$$\phi_{HH}(\beta\cdot t_1^{c_{0,1}}\dots t_n^{c_{0,n}}\otimes\cdots\otimes t_1^{c_{n,1}}\dots t_n^{c_{n,n}}) = \mathrm{Tr}_{k'/k}(\beta)\textstyle\prod_{i=1}^{n}c_{i,i}$$

*whenever $\forall i:\sum_{p=0}^{n}c_{p,i}=0$ and zero otherwise.*

(2) *Precomposed with the HKR map (cf. Equation 5.3), this yields*

$$\begin{aligned} \Omega^n_{K/k} &\longrightarrow HH_n(K)\longrightarrow k \\ \beta\cdot f_0\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_n &\longmapsto \mathrm{Tr}_{k'/k}(\beta)\det\begin{pmatrix} c_{1,1} & \cdots & c_{n,1}\\ \vdots & \ddots & \vdots\\ c_{1,n} & \cdots & c_{n,n}\end{pmatrix} \end{aligned} \tag{5.16}$$

*for $f_p = t_1^{c_{p,1}}\cdots t_n^{c_{p,n}}$ $(0\le p\le n)$ whenever $\forall i:\sum_{p=0}^{n}c_{p,i}=0$, and zero otherwise.*

(3) *For $f \in K$ given by $f = \sum f_{\alpha_1 \dots \alpha_n} t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ (with coefficients $f_{\alpha_1 \dots \alpha_n} \in k'$), we have*

$$\begin{aligned} \Omega^n_{K/k} &\longrightarrow HH_n(K) \longrightarrow k \\ f \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n &\longmapsto \mathrm{Tr}_{k'/k}(f_{-1,\dots,-1}). \end{aligned}$$

We supplement this result with two statements which elucidate the behaviour when plugging in infinite series.

**Supplement 1.** *Under the same assumptions as in the proposition, we further have the following properties:*

(1) (Series variant) *Assume $f_0, \dots, f_n \in K$ are arbitrary elements. Concretely, say*

$$f_m := \sum_{c_1 \dots c_n \in \mathbf{Z}} f^m_{c_1 \dots c_n} t_1^{c_1} \cdots t_n^{c_n} \qquad \text{for} \qquad 0 \leq m \leq n \tag{5.17}$$

*and $f^m_{c_1 \dots c_n} \in k'$. Then Equation 5.16 extends to the following: The $n$-form $f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n$ gets sent to*

(5.18)

$$\sum_{c_{0,1} \dots c_{0,n}} \cdots \sum_{\substack{c_{n,1} \dots c_{n,n} \\ \text{such that } \forall 1 \leq i \leq n\colon c_{0,i} + \cdots + c_{n,i} = 0}} \mathrm{Tr}_{k'}(f^0_{c_{0,1} \dots c_{0,n}} \cdots f^n_{c_{n,1} \dots c_{n,n}}) \det \begin{pmatrix} c_{1,1} & \cdots & c_{n,1} \\ \vdots & \ddots & \vdots \\ c_{1,n} & \cdots & c_{n,n} \end{pmatrix},$$

*which is always a finite sum.*

(2) (Approximation) *Moreover, there exists some natural number $N$ (depending on $f_0, \dots, f_n$) such that for the Laurent polynomial truncations*

$$\tilde{f}_m := \sum_{-N \leq c_1 \dots c_n \leq N} f^m_{c_1 \dots c_n} t_1^{c_1} \cdots t_n^{c_n}$$

*we have*

$$\phi_{HH}(f_0 \otimes \cdots \otimes f_n) = \phi_{HH}(\tilde{f}_0 \otimes \cdots \otimes \tilde{f}_n).$$

*Furthermore, under the map in Equation 5.16 both $f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n$ and $\tilde{f}_0 \mathrm{d}\tilde{f}_1 \wedge \cdots \wedge \mathrm{d}\tilde{f}_n$ are being sent to the same value.*

The following proves both the Proposition as well as the Supplement.

**Proof. (1)** Yekutieli gives a construction of the cubically decomposed algebra $E_K$ [Yek15]. Alternatively, write the underlying vector space of the $n$-local field as

$$k'((t_1))((t_2)) \dots ((t_n)) = \varinjlim_{i_n} \overleftarrow{\lim_{j_n}} \cdots \varinjlim_{i_1} \overleftarrow{\lim_{j_1}} \frac{1}{t_1^{i_1} \cdots t_n^{i_n}} k'[t_1, \dots, t_n]/(t_1^{j_1}, \dots, t_n^{j_n}).$$

Following [BGW16b, Example 10], this defines an $n$-Tate object in the category of finite-dimensional $k'$-vector spaces and the main results of [BGW16b] imply that its endomorphism algebra in the category of $n$-Tate objects carries a cubically decomposed structure, which we may also take to be $E_K$.

The equivalence of both approaches was shown in [BGW16a, Theorem 3.8]. Moreover, loc. cit. shows that viewing this $n$-Tate object as a $k'$-vector space is a faithful functor, i.e. any such endomorphism can be thought of as a $k'$-linear map. For $f \in I_{tr}$, the trace is evaluated as follows: First, pick $i_n$ big enough such that the image lies in

$$L_1 := \varprojlim_{j_n} \cdots \varinjlim_{i_1} \varprojlim_{j_1} \frac{1}{t_1^{i_1} \cdots t_n^{i_n}} k'[t_1, \ldots, t_n]/(t_1^{j_1}, \ldots, t_n^{j_n}),$$

and then $i'_n$ small enough such that $f$ sends

$$L'_1 := \varprojlim_{j_n} \cdots \varinjlim_{i_1} \varprojlim_{j_1} \frac{1}{t_1^{i_1} \cdots t_n^{i'_n}} k'[t_1, \ldots, t_n]/(t_1^{j_1}, \ldots, t_n^{j_n})$$

to zero. Such values for $i_n$ and $i'_n$ exist since $f$ lies (in particular) in $I_1^0$. Using axiom **T2** of Tate's trace, Proposition 3, the trace of $f$ agrees with the trace of $f\,|_{L_1/L'_1}$. We see that this step has reduced computing the trace of an endomorphism of $n$ limit-colimit pairs (of finite-dimensional vector spaces), to computing the trace for just $(n-1)$ limit-colimit pairs. This holds since the limit over $j_n$ in the quotient $L_1/L'_1$ becomes eventually stationary, so we can drop the limit. Repeating this reduction, it suffices to evaluate the trace on a finite-dimensional vector space, where by axiom **T1** it agrees with the ordinary trace. Moreover, as these reduction steps just restrict the ranges of exponents of the $t_1^? \cdots t_n^?$ appearing, some finite system of such monomials forms a $k'$-basis.

**(2)** Henceforth, in order to distinguish clearly between $t_i$ as the multiplication operator $x \mapsto t_i \cdot x$, or as the monomials in the formal series expansion (Equation 5.17), we write the latter in bold letters $\mathbf{t}_i$ for the duration of this proof. Define idempotents $P_i^+$ by

$$P_i^+ \sum f_{\lambda_1 \ldots \lambda_n} \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n} := \sum \delta_{\lambda_i \geq 0} f_{\lambda_1 \ldots \lambda_n} \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n}. \tag{5.19}$$

Define $P_i^- = \mathbf{1} - P_i^+$. We know that $\operatorname{im} P_i^+ \subseteq I_i^+$ is a lattice and $P_i^-(\operatorname{im} P_i^+) = 0$, so we have a system of good idempotents in the sense of Definition 20. Thus, by Theorem 23 we have

$$\phi_{HH}(f_0 \otimes \cdots \otimes f_n) = (-1)^n \operatorname{tr}_k M = (-1)^n \operatorname{Tr}_{k'/k}(\operatorname{tr}_{k'} M) \tag{5.20}$$

for the operator $M$ defined by

$$M := \sum_{\gamma_1 \ldots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_n} P_1^{-\gamma_1} f_1 P_1^{\gamma_1} \cdots P_n^{-\gamma_n} f_n P_n^{\gamma_n} f_0. \tag{5.21}$$

The remaining computation is essentially the same as in the proof of [Bra14, Theorem 26], so we just sketch the key steps:

*(a)* We first handle monomials. Suppose

$$f_m := \beta_m \mathbf{t}_1^{c_{m,1}} \cdots \mathbf{t}_n^{c_{m,n}}$$

for $c_{m,i} \in \mathbf{Z}$ and $0 \leq m \leq n$; $1 \leq i \leq n$. We compute

$$P_m^- f_m P_m^+ \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n} = \delta_{0 \leq \lambda_m < -c_{m,m}} \beta_m \mathbf{t}_1^{\lambda_1 + c_{m,1}} \cdots \mathbf{t}_n^{\lambda_n + c_{m,n}} \qquad \text{for } 1 \leq m \leq n.$$

This formula closely mimics the one-dimensional computation in Lemma 4. With this we can explicitly compute the action of $M$ on a monomial. We get

$$\begin{aligned} M \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n} = (\beta_0 \cdots \beta_m) \\ \prod_{i=1}^{n} (\delta_{0 \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{p,i} < -c_{i,i}} - \delta_{-c_{i,i} \leq \lambda_i + c_{0,i} + \sum_{p=i+1}^n c_{p,i} < 0}) \\ \mathbf{t}_1^{\lambda_1 + c_{0,1} + \sum_{p=1}^n c_{p,1}} \cdots \mathbf{t}_n^{\lambda_n + c_{0,n} + \sum_{p=1}^n c_{p,n}}. \end{aligned}$$

It is clear that this operator can have a non-zero trace only if $\forall i : \sum_{p=0}^n c_{p,i} = 0$ holds, because otherwise it is visibly nilpotent and we can invoke axiom **T3** of Tate's trace. This proves the vanishing part of the claim. Now assume this condition holds and simplify the formula for $M$ accordingly. An eigenvalue count reveals

$$\operatorname{tr}_{k'} M = \beta_0 \cdots \beta_n \textstyle\prod_{i=1}^n (-c_{i,i}) = (-1)^n \beta_0 \cdots \beta_n \prod_{i=1}^n c_{i,i}, \tag{5.22}$$

where $M$ is still viewed as an endomorphism of a $k'$-vector space. See the proof of [Bra14, Theorem 26] for the full details. Finally, $\operatorname{tr}_k M = \operatorname{Tr}_{k'/k}(\operatorname{tr}_{k'} M)$ computes the value in question; the signs $(-1)^n$ from Equation 5.20 and Equation 5.22 cancel each other out.

*(b)* Now we handle infinite series in order to obtain the statements of the Supplement, Equation 5.18. We return to the operator $M$ of Equation 5.21, but this time suppose

$$f_m := \sum_{c_1 \ldots c_n \in \mathbf{Z}} f_{c_1 \ldots c_n}^m \mathbf{t}_1^{c_1} \cdots \mathbf{t}_n^{c_n} \qquad \text{for} \qquad 0 \leq m \leq n, \tag{5.23}$$

i.e., we allow arbitrary elements $f_0, \ldots, f_n \in K$. Since under *(a)* we had established Equation 5.22 for monomials, it extends linearly to all monomials in the above formal series expansion (there is a subtlety, we refer to Elaboration 1 below for a discussion). Thus, we obtain

$$\operatorname{tr}_{k'} M = (-1)^n \sum_{\substack{c_{0,1} \ldots c_{0,n} \in \mathbf{Z} \\ \text{such that for all } 1 \leq i \leq n\text{: } c_{0,i} + \cdots + c_{n,i} = 0}} \cdots \sum_{c_{n,1} \ldots c_{n,n} \in \mathbf{Z}} (f_{c_{0,1} \ldots c_{0,n}}^0 \cdots f_{c_{n,1} \ldots c_{n,n}}^n) \textstyle\prod_{i=1}^n c_{i,i},$$

giving the first claim of the Supplement.

**(3)** We return to Formula 5.22. Plugging in the antisymmetrizer coming from the HKR map, we get

$$= \beta \textstyle\sum_{\pi \in \mathfrak{S}_n} \operatorname{sgn}(\pi) \prod_{i=1}^n c_{\pi(i),i},$$

which up to the factor $\beta$ is exactly the Leibniz formula for the determinant.

**(4)** In this special case, let $f_0 := f$ and $f_m = t_m$ for $1 \leq m \leq n$ and proceed

basically as before. We compute

$$P_m^- f_m P_m^+ \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n} = \delta_{\lambda_m=0} \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_m^{\lambda_m+1} \cdots \mathbf{t}_n^{\lambda_n} \qquad \text{for } 1 \le m \le n$$

on monomials. As before, we use this to compute the trace of the operator

$$M := \sum_{\gamma_1 \ldots \gamma_n \in \{\pm\}} (-1)^{\gamma_1 + \cdots + \gamma_n} P_1^{-\gamma_1} f_1 P_1^{\gamma_1} \cdots P_n^{-\gamma_n} f_n P_n^{\gamma_n} f_0,$$

which this time unwinds as

$$M \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n} = \sum_{c_{0,1} \ldots c_{0,n}} f_{c_{0,1} \ldots c_{0,n}} \prod_{i=1}^{n} (-\delta_{\lambda_i + c_{0,i} = -1}) \mathbf{t}_1^{\lambda_1 + c_{0,1} + 1} \cdots \mathbf{t}_n^{\lambda_n + c_{0,n} + 1}$$

and we see that only the summand with $c_{0,i} = -1 - \lambda_i$ remains, giving

$$= (-1)^n f_{(-1-\lambda_1) \ldots (-1-\lambda_n)} \mathbf{t}_1^{\lambda_1 + (-1-\lambda_1) + 1} \cdots \mathbf{t}_n^{\lambda_n + (-1-\lambda_n) + 1}.$$

This is nilpotent unless $\lambda_1 = \cdots = \lambda_n = 0$ and in this case indeed has the trace $\mathrm{Tr}_{k'/k}(f_{-1,\ldots,-1})$, proving the claim.

**(4)** It remains to prove the Approximation statement of the Supplement. Let us explain why Equation 5.18 is a finite sum. This will also demonstrate how one could make the choice of $N$ in the Approximation statement of the Supplement effective. For $0 \le p \le n$ write

$$f_p := \sum_{c_{p,1} \ldots c_{p,n}} f^p_{c_{p,1} \ldots c_{p,n}} t_1^{c_{p,1}} \cdots t_n^{c_{p,n}}$$

and since these elements lie in $k'((\ldots))((t_n))$, they have a finite lowest $t_n$-degree monomial, so we can find some integer $N_0$ such that we have $c_{p,n} \ge -N_0$ for all non-zero terms in the series. This is true for all $0 \le p \le n$, so we can pick $N_0$ to be a joint bound among all $p$. Hence, the constraint

$$\underset{\ge -N_0}{c_{0,n}} + \cdots + \underset{\ge -N_0}{c_{n,n}} = 0$$

in Equation 5.18 forces $c_{p,n} \le nN_0$ for all $p$, because otherwise the sum cannot be zero anyway. It follows that it suffices if the indices $c_{0,n}, \ldots, c_{n,n}$ run through a finite set and this truncation does not change the value of Equation 5.18. This being achieved, replace each $f_p$ by

$$f_p = \sum_{\substack{c_{p,n} \\ \text{from finite set}}} \left( \sum_{c_{p,1} \ldots c_{p,n-1}} f^p_{c_{p,1} \ldots c_{p,n}} t_1^{c_{p,1}} \cdots t_{n-1}^{c_{p,n-1}} \right) t_n^{c_{p,n}},$$

where each term inside the bracket is an element of $k'((\cdots))((t_{n-1}))$. We may repeat the same argument inductively, e.g., the next step shows that it also suffices to pick the indices $c_{0,n-1}, \ldots, c_{n,n-1}$ from a finite set. Concluding this induction, we have shown that the a priori infinite sum in Equation 5.18 is just a sum over a finite subset of choices for $\{c_{p,i}\}_{p,i}$, and this finite subset lies in some sufficiently big box with edges $[-N, N]$. This $N$ can be taken in the Approximation statement. □

This finishes the proof. However, the reader might find the quick reduction from infinite series to checking something on monomials a little daring, so let us add some details.

*Elaboration* 1. Let us provide some more details regarding the handling of infinite expressions around Equation 5.23 in the above proof. When we speak of "extending linearly" right below Equation 5.23, we do *not* mean the following: We have checked the identity in part *(a)* of the proof for monomials $f_m := \mathbf{t}_1^{c_{m,1}} \cdots \mathbf{t}_n^{c_{m,n}}$, then these monomials span the subspace of Laurent polynomials

$$k'[t_1^{\pm}, \dots, t_n^{\pm}] \subset k'((t_1)) \cdots ((t_n)), \tag{5.24}$$

which is much smaller than all of $K$, and thus does *not* give the claim we make in *(b)*. Instead, we mean the following: Let us call a map $\varphi : K \to K$ *monomial* if it has the following property,

$$\begin{aligned} \varphi : K &\longrightarrow K \\ \sum_{\lambda_1 \dots \lambda_n} f_{\lambda_1 \dots \lambda_n} \mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n} &\longmapsto \sum_{\lambda_1 \dots \lambda_n} f_{\lambda_1 \dots \lambda_n} \varphi(\mathbf{t}_1^{\lambda_1} \cdots \mathbf{t}_n^{\lambda_n}), \end{aligned} \tag{5.25}$$

and it is understood that we demand the sum on the right-hand side to make sense, i.e. for every monomial $\mathbf{t}_1^{d_1} \cdots \mathbf{t}_n^{d_n}$ the computation of its coefficient in $\varphi(f) \in K$ reduces to a finite summation. Let us illustrate this concept with a well-known example. If $f \in K$, then the multiplication map

$$\begin{aligned} \varphi_f : K &\longrightarrow K \\ x &\longmapsto f \cdot x \end{aligned}$$

is monomial. Indeed, while it is customary to write a product in a Laurent series ring $R((t))$ as $\left(\sum a_i t^i\right)\left(\sum b_j t^j\right)$, it is the first exercise for defining the ring structure that expanding this purely formal expression reduces, for each coefficient, to a finite summation. The same verification shows that $\varphi_f$ is indeed monomial. Secondly, we note that the operators $P_i^{\pm}$ of Equation 5.19 are monomial; they are in fact defined exactly in the shape of Equation 5.25. Thirdly, both finite sums and the composition of monomial operators are again monomial. Combining these three facts, we see that the operator $M$ in Equation 5.21 is monomial. Thus, when we speak of "extending linearly" in order to settle part *(b)* in the above proof, we mean that $M$ is uniquely determined by its action on monomials via the Formula 5.25, and unlike the situation in Equation 5.24, this indeed pins down $M$ on all of $K$.

*Elaboration* 2. If the reader wants a more philosophical comparison between these two different notions of linear extension, one may observe that for any ring $R$, we have $R((t)) \subseteq \prod_{\lambda \in \mathbf{Z}} R$, which is nothing but identifying a Laurent series by its sequence of coefficients, and by induction $K \subseteq \prod_{\lambda_1 \dots \lambda_n \in \mathbf{Z}^n} k'$. Since in the proof of the Supplement we only want to prove the equality of two elements, we can just as well do this in the bigger space $\prod_{\lambda_1 \dots \lambda_n \in \mathbf{Z}^n} k'$.

Here the concept of $\varphi$ being monomial translates into a condition of a linear extension along a product, as opposed to a coproduct, which would be the extension à la Equation 5.24.

Next, we shall relate various $\phi_{HH}$ for different cubically decomposed algebras. To clarify the distinction, let us agree to write $\phi^A_{HH} : HH_n(A) \to k$ instead of $\phi_{HH}$ plain.

**Theorem 26** (Local formula)**.** *Suppose $X/k$ is a finite type reduced scheme of pure dimension $n$ over a perfect field $k$. Suppose $\triangle = (\eta_0 > \cdots > \eta_n) \in S(X)_n$ with $\operatorname{codim}_X \overline{\{\eta_i\}} = i$. Then there is a canonical finite decomposition*

$$A(\triangle, \mathcal{O}_X) \cong \prod K_j$$

*with each $K_j$ an $n$-local field. Let $E_\triangle$ be the cubically decomposed algebra of Proposition 12.*

(1) *Each $E_j := \{f \in E_\triangle \mid fK_j \subseteq K_j,\ fK_r = 0 \text{ for } r \neq j\}$ with ideals $J_i^\pm := I_i^\pm \cap E_j$ is a cubically decomposed algebra over $k$ and for $f \in HH_n(\mathcal{O}_{\eta_0})$ we have*

$$\phi^{E_\triangle}_{HH}(f) = \textstyle\sum_j \phi^{E_j}_{HH}(f). \tag{5.26}$$

(2) *There exists (non-canonically) an isomorphism $K_j \simeq k_j((t_1))\cdots((t_n))$ with $k_j/k$ a finite field extension such that for all $\beta \in k_j$*

$$\phi^{E_j}_{HH}(\beta \cdot t_1^{c_{0,1}} \dots t_n^{c_{0,n}} \otimes \cdots \otimes t_1^{c_{n,1}} \dots t_n^{c_{n,n}}) = \operatorname{Tr}_{k_j/k}(\beta) \textstyle\prod_{i=1}^n c_{i,i}$$

*whenever $\forall i : \sum_{p=0}^n c_{p,i} = 0$ and zero otherwise.*

(3) *Precomposed with the HKR map (cf. Equation 5.3), this yields*

$$\begin{aligned} \Omega^n_{K_j/k} &\longrightarrow HH_n(K_j) \longrightarrow k \\ \beta \cdot f_0 \mathrm{d}f_1 \wedge \cdots \wedge \mathrm{d}f_n &\longmapsto \operatorname{Tr}_{k_j/k}(\beta) \det \begin{pmatrix} c_{1,1} & \cdots & c_{n,1} \\ \vdots & \ddots & \vdots \\ c_{1,n} & \cdots & c_{n,n} \end{pmatrix} \end{aligned}$$

*for $f_p = t_1^{c_{p,1}} \cdots t_n^{c_{p,n}}$ $(0 \leq p \leq n)$ whenever $\forall i : \sum_{p=0}^n c_{p,i} = 0$ and zero otherwise.*

(4) *For $f \in K_j$ given by $f = \sum f_{\alpha_1 \dots \alpha_n} t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ (with coefficients $f_{\alpha_1 \dots \alpha_n} \in k_j$) we have*

$$\begin{aligned} \Omega^n_{K_j/k} &\longrightarrow HH_n(K_j) \longrightarrow k \\ f \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n &\longmapsto \operatorname{Tr}_{k_j/k}(f_{-1,\dots,-1}). \end{aligned} \tag{5.27}$$

A word of warning: In (2), while there always exists an isomorphism $K_j \simeq k_j((t_1))\cdots((t_n))$ such that the above claims hold, it is by no means true that any isomorphism between these fields has these properties. This very subtle behaviour is discussed extensively in [Yek15] and [BGW16a].

**Proof.** Almost all of the first claim follows directly from Proposition 8. **(1)** Observe that the $E_j$ are associative algebras. Define $J_i^\pm := I_i^\pm \cap E_j$ with $I_i^\pm$ the ideals of the cubically decomposed algebra structure of $E_\triangle$, see Proposition 12. It is clear that the $J_i^\pm$ are two-sided ideals in $E_j$ and we claim that $J_i^+ + J_i^- = E_j$. To see this, let $x \in E_j$ be given. We have $A(\triangle, \mathcal{O}_X) = \prod K_j$, so let $e_j$ be the idempotent of the $j$-th factor. It is clear that $e_j \in E_\triangle$, because it is a multiplication operator. Write $x = x^+ + x^-$ with $x^\pm \in I_i^\pm$. Now $e_j x e_j = e_j x^+ e_j + e_j x^- e_j$. Since the $I_i^\pm$ are ideals, $e_j x^\pm e_j \in I_i^\pm$, but also $e_j x^\pm e_j \in E_j$. It follows that $e_j x^\pm e_j \in I_i^\pm \cap E_j = J_i^\pm$. On the other hand, $e_j x e_j = x$. The converse inclusion is obvious, so we have $J_i^+ + J_i^- = E_\triangle \cap E_j = E_j$. Since $J_{tr} = \bigcap_{i=1,\dots,n} \bigcap_{s\{\pm\}} J_i^s \subseteq I_{tr}$ we can use the trace map of $E_\triangle$. This proves that $(E_j, \{J_i^\pm\}, \mathrm{tr}_{I_{tr}})$ is a cubically decomposed algebra. In particular, the maps $\phi_{HH}^{E_j}$ exist. The embedding $\mathcal{O}_{\eta_0} \hookrightarrow A(\triangle, \mathcal{O}_X) \cong \prod K_j$ is diagonal, i.e. $f \mapsto (f, \dots, f)$. As a result, the associated multiplication operator in $E_\triangle$ is diagonal in the $K_j$, therefore Equation 5.26 holds. **(2)** For the evaluation of $\phi_{HH}^{E_j}$, we want to pick an isomorphism of fields

$$\rho : K_j \xrightarrow{\sim} k_j((t_1)) \cdots ((t_n))$$

with the following properties: (1) $\rho$ is an isomorphism of fields, (2) $\rho$ is an isomorphism of $k$-vector spaces, and (3) $\rho$ induces an isomorphism of cubically decomposed algebras $(E_j, \{J_i^\pm\}, \mathrm{tr}_{I_{tr}})$ to the cubically decomposed algebra structure of $k_j((t_1)) \cdots ((t_n))$, as in Proposition 25:

$$(E_j, \{J_i^\pm\}, \mathrm{tr}_{I_{tr}}) \longrightarrow E_{k_j((t_1))\cdots((t_n))}, \qquad f \longmapsto \rho \circ f \circ \rho^{-1}.$$

The existence of such a $\rho$ follows from [BGW16a, Theorem 0.2, (3)], and it hinges on $k$ being perfect. Since the construction of $\phi_{HH}$ is intrinsic to the cubically decomposed algebra structure, this isomorphism implies that we may perform our computation on the level of $k_j((t_1)) \cdots ((t_n))$, so the entire claim reduces to Proposition 25. □

As we have just seen how the entire result rests solely on Proposition 25 in the end, we may also state its Supplement here, for the sake of completeness.

**Supplement 2.** *Under the same assumptions as in the theorem, we further have the following properties:*

(1) (Series variant) *Assume $f_0, \dots, f_n \in K_j$ are arbitrary elements. Concretely, say*

$$f_m := \sum_{c_1 \dots c_n \in \mathbf{Z}} f_{c_1 \dots c_n}^m t_1^{c_1} \cdots t_n^{c_n} \qquad \text{for} \qquad 0 \leq m \leq n$$

*with $f^m_{c_1\dots c_n} \in k_j$ under the isomorphism given in claim (2) of the theorem. Then the $n$-form $f_0\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_n$ gets sent to*

$$\sum_{c_{0,1}\dots c_{0,n}}\cdots\sum_{\substack{c_{n,1}\dots c_{n,n}\\ \text{such that } \forall 1\le i\le n:\ c_{0,i}+\cdots+c_{n,i}=0}} \mathrm{Tr}_{k_j}(f^0_{c_{0,1}\dots c_{0,n}}\cdots f^n_{c_{n,1}\dots c_{n,n}})\det\begin{pmatrix} c_{1,1} & \cdots & c_{n,1}\\ \vdots & \ddots & \vdots\\ c_{1,n} & \cdots & c_{n,n}\end{pmatrix}.$$

(2) (Approximation) *Moreover, there exists some natural number $N$ (depending on $f_0,\dots,f_n$) such that for the Laurent polynomial truncations*

$$\tilde{f}_m := \sum_{-N\le c_1\dots c_n\le N} f^m_{c_1\dots c_n}t_1^{c_1}\cdots t_n^{c_n}$$

*we have*

$$\phi^{E_j}_{HH}(f_0\otimes\cdots\otimes f_n) = \phi^{E_j}_{HH}(\tilde{f}_0\otimes\cdots\otimes\tilde{f}_n).$$

*Furthermore, under the map in Equation 5.27, both $f_0\mathrm{d}f_1\wedge\cdots\wedge\mathrm{d}f_n$ and $\tilde{f}_0\mathrm{d}\tilde{f}_1\wedge\cdots\wedge\mathrm{d}\tilde{f}_n$ are being sent to the same value.*

## 6. A new approach

**6.1. Introduction.** We want to change our perspective. Let $(A,(I_i^{\pm}),\tau)$ be a cubically decomposed algebra. So far we have always worked in the category of $A$-bimodules and considered exact sequences of $A$-bimodules like

$$0\longrightarrow I_n^0\xrightarrow{\text{diag}} I_n^+\oplus I_n^-\xrightarrow{\text{diff}} A\longrightarrow 0 \tag{6.1}$$

or their higher-dimensional counterparts as in Equation 5.7. This approach corresponds to viewing Hochschild homology as a functor

$$A\text{-bimodules}\to k\text{-vector spaces},\qquad M\mapsto H_i(A,M).$$

However, Hochschild homology can also be regarded as a functor

$$\text{associative } k\text{-algebras}\to k\text{-vector spaces},\qquad A\mapsto HH_i(A).$$

In this section we want to transform the mechanisms of §3, §5 from the former to the latter perspective.

**6.2. Recollections.** We shall need to work with *non-unital* algebras, so let us briefly recall the necessary material (see [Wod89] for details). Hochschild homology was defined and described in §5 for an arbitrary associative algebra $A$. We may read $A$ as a bimodule over itself and if $A$ is unital we write $HH_i(A) := H_i(A,A)$. If $A$ is not unital, all definitions still make sense and we write $HH_i^{naiv}(A) := H_i(A,A)$ for these groups, following [Lod92, §1.4.3]. However, this is not a good definition in general, so usually one proceeds

differently: There is a unitalization $A^+$ along with a canonical map $k \hookrightarrow A^+$ of unital associative algebras, and one defines[5]

$$HH_i(A) := \operatorname{coker}\left(HH_i(k) \to HH_i(A^+)\right), \tag{6.2}$$

see [Lod92, §1.4] for details; this parallels a similar construction in algebraic $K$-theory. If $A$ happens to be unital, this agrees with the previous definition as in §5, i.e. it agrees with $HH_i^{naiv}$. In general, there is only the obvious morphism $\kappa : HH_i^{naiv}(A) \to HH_i(A)$ (sending a pure tensor to itself in $A^+$) which need neither be injective nor surjective.

If $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of $A$-bimodules, the sequence $0 \to C_\bullet(A, M') \to C_\bullet(A, M) \to C_\bullet(A, M'') \to 0$ is obviously an exact sequence of complexes, so there is a long exact sequence in Hochschild homology

$$\cdots \to H_i(A, M') \to H_i(A, M) \to H_i(A, M'') \xrightarrow{\partial} H_{i-1}(A, M') \to \cdots. \tag{6.3}$$

We denote the connecting homomorphism by $\partial$. If $I$ is a two-sided ideal in $A$, this yields the sequence

(6.4)

$$\cdots \to H_i(A, I) \to H_i(A, A) \xrightarrow{\mu} H_i(A, A/I) \xrightarrow{\partial} H_{i-1}(A, I) \to H_{i-1}(A, A) \to \cdots$$

Moreover, if $M$ is an $A/I$-bimodule, it is also an $A$-bimodule via $A \twoheadrightarrow A/I$. Then there is an obvious change-of-algebra map $\nu : C_i(A, M) \to C_i(A/I, M)$. Clearly $A/I$ is an $A/I$-bimodule and thus there are canonical maps

$$j : C_i(A, A) \xrightarrow{\mu} C_i(A, A/I) \xrightarrow{\nu} C_i(A/I, A/I),$$

where $\mu$ is the morphism inducing the respective arrow in Equation 6.4. One also defines the *relative Hochschild homology* complex $\mathcal{K}_\bullet(A \to A/I)$, the precise definition is somewhat involved, see [Wod89, beginning of §3, where instead of $\mathcal{C}$ one uses the Hochschild version $\mathcal{K}$, defined on the same page 598 in line 5]. We write $HH_i(A \text{ rel } I) := H_i\mathcal{K}_\bullet(A \to A/I)$ for its homology (*Beware:* The notation $HH_i(A, I)$ is customary. However, it is easily confused with $H_i(A, I)$, which also plays a role here, so we have opted for the present clearer distinction). We may regard $I$ as an associative algebra itself, but unless $A = I$ it will not be unital.

**Proposition 27** ([Wod88], [Wod89, Theorem 3.1])**.** *Suppose $A$ is an associative algebra and $I$ a two-sided ideal. Suppose both have at least one-sided local units. Then the canonical morphisms*

$$HH_i^{naiv}(I) \xrightarrow{\kappa} HH_i(I) \xrightarrow{\Diamond} HH_i(A \text{ rel } I) \tag{6.5}$$

*are both isomorphisms. There is a quasi-isomorphism*

$$\mathcal{K}_\bullet(A \to A/I) \simeq_{\text{qis}} \ker(C_\bullet(A, A) \xrightarrow{j} C_\bullet(A/I, A/I)). \tag{6.6}$$

[5]This is not the definition given in our main reference [Wod89]; here $HH_i(A)$ is the homology of $\mathcal{K}$, cf. p. 598, l. 5 in *loc. cit.*, defined in terms of the bar complex. The equivalence of definitions follows from the paragraph before Theorem 3.1 in *loc. cit.*

It is noteworthy that only the right-most term in Equation 6.5 actually depends on $A$.

**Proof.** For the proof, combine [Wod89, Theorem 3.1 and Cor. 4.5] for the first claim: The existence of local units implies $H$-unitality. For the second claim, $A$ is $H$-unital, so the bar complex in the definiton of $\mathcal{K}$ in *loc. cit.* p. 598 in line 5 is zero up to quasi-isomorphism. Applying this to the definition of $\mathcal{K}_\bullet(A \to A/I)$ in §3 in *loc. cit.* gives the second claim. For an alternative presentation, combine the treatment [Lod92, §1.4.9] with the generality of [Lod92, E.1.4.6]. The $H$-unitality of $A/I$ follows from [Wod89, Cor. 3.4]. □

Basically by construction, we get a long exact sequence in homology

(6.7)
$$\cdots \to HH_i(A \text{ rel } I) \to HH_i(A) \to HH_i(A/I) \xrightarrow{\delta} HH_{i-1}(A \text{ rel } I) \to \cdots.$$

Although different, it is not unrelated to the sequence in Equation 6.4:

**Lemma 28.** *Suppose $A$ is an associative algebra and $I$ a two-sided ideal with at least one-sided local units. Then the diagram*

(6.8)
$$\begin{array}{ccccccccc} \cdots & \longrightarrow & H_i(A,I) & \longrightarrow & H_i(A,A) & \longrightarrow & H_i(A,A/I) & \xrightarrow{\partial} & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \kappa} & & \downarrow{\scriptstyle \lambda} & & \\ \cdots & \longrightarrow & HH_i(A \,\mathsf{rel}\, I) & \longrightarrow & HH_i(A) & \longrightarrow & HH_i(A/I) & \xrightarrow{\delta} & \cdots \end{array}$$

*is commutative.*

**Proof.** Trivial if $A$ is unital. In general: We construct this on the level of complexes $C_\bullet(-,-)$. The middle downward arrow maps pure tensors to themselves, $A \to A^+$ in $HH_i(A^+)$ and then to the cokernel as given by Equation 6.2. Similarly, the right-hand side downward arrow is induced by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_i \mapsto a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_i},$$

where $a_0 \in A/I$, $a_1, \ldots, a_n \in A$ and $\bar{\cdot} : A \twoheadrightarrow A/I$ is the quotient map, again sent to $(A/I)^+$ and then to the respective cokernel. For the left-hand side we can wlog. use the presentation on the right-hand side of Equation 6.6 for $HH_i(A \text{ rel } I)$. The downward arrow is then given by the analogous formula, but $a_0 \in I$ and so $\overline{a_0} = 0$ in $A/I$, so that it is clear that the image lies in the kernel of $j : C_i(A,A) \to C_i(A/I, A/I)$. □

**6.3. The construction.** Let $(A^n, (I_i^\pm), \tau)$ be an $n$-fold cubically decomposed algebra over $k$. Define

(6.9)
$$A^{n-1} := I_n^0 \qquad J_i^\pm := I_i^\pm \cap A^{n-1} \qquad (\text{for } i = 0, \ldots, n-1).$$

Then $(A^{n-1}, (J_i^\pm), \tau)$ is an $(n-1)$-fold cubically decomposed algebra over $k$.

**Definition 29.** *We say that an $n$-fold cubically decomposed algebra $(A, (I_i^{\pm}), \tau)$* has local units on all levels *(or is 'good') if $A^s$ has local left units (or local right units) for $s = 1, \dots, n$.*

Evaluating Equation 6.9 inductively, we find $A^s = (I_{s+1}^0 \cap \dots \cap I_n^0) \cap A$. Define

$$\Lambda : A^n \longrightarrow A^n/A^{n-1}, \qquad x \longmapsto x^+, \tag{6.10}$$

where $x = x^+ + x^-$ is any decomposition with $x^{\pm} \in I_n^{\pm}$ (always exists and gives well-defined map). This map does *not* equal the natural quotient map! Using the relative Hochschild homology sequence, Equation 6.7, coming from the exact sequence of associative algebras

$$0 \longrightarrow A^{n-1} \longrightarrow A^n \xrightarrow{\text{quot}} A^n/A^{n-1} \longrightarrow 0, \tag{6.11}$$

the connecting homomorphism induces a map $\delta$ and we employ it to define a map

$$d : HH_{i+1}(A^n) \xrightarrow{\Lambda} HH_{i+1}(A^n/A^{n-1}) \xrightarrow{\delta} HH_i(A^{n-1}). \tag{6.12}$$

We can repeat this construction and obtain a morphism:

**Definition 30.** *Suppose $(A, (I_i^{\pm}), \tau)$ is an $n$-fold cubically decomposed algebra over $k$ which has local units on all levels. Then there is a canonical map*

$$\phi_C : HH_n(A) \longrightarrow HH_0(I_{tr}) \longrightarrow k, \qquad \alpha \mapsto \tau \underbrace{d \circ \dots \circ d}_{n \text{ times}} \alpha.$$

*Analogously, for cyclic homology $\phi_C : HC_n(A) \to k$ (see the lemma below why we call this $\phi_C$ as well).*

**Lemma 31.** *The map $\phi_C$ factors over $HH_n(A) \xrightarrow{I} HC_n(A) \longrightarrow k$.*

**Proof.** Let $d'$ be the analogue of the map in Equation 6.12 with cyclic homology. Both $\Lambda$ and the connecting map are compatible with $I$ so that

$$\begin{array}{ccc} HH_n(A) & \xrightarrow{d \circ \dots \circ d} & HH_0(I_{tr}) \\ \Big\downarrow{\scriptstyle I} & & \Big\downarrow{\scriptstyle I} \\ HC_n(A) & \xrightarrow[d' \circ \dots \circ d']{} & HC_0(I_{tr}) \end{array}$$

commutes, but the right-hand side downward arrow is an isomorphism, giving the claim. □

**Theorem 32.** *Suppose $(A, (I_i^{\pm}), \tau)$ is a unital $n$-fold cubically decomposed algebra over $k$ which has local units on all levels. Then $\phi_C : HH_n(A) \to k$ agrees up to sign with $\phi_{HH}$, namely*

$$\phi_C = (-1)^{\frac{n(n-1)}{2}} \phi_{HH}.$$

**Proof. (1)** We proceed by induction. Firstly, we construct a commutative diagram and a map $\Psi$:
(6.13)

$$\begin{array}{ccccc}
H_s(A,A^s) & \longleftarrow & H_s(A^s,A^s) & \xrightarrow{\kappa} & HH_s(A^s) \\
\downarrow{\scriptstyle\Lambda} & & \downarrow{\scriptstyle\Lambda} & & \downarrow{\scriptstyle\Lambda} \\
H_s(A,\frac{A^s}{A^{s-1}}) & \longleftarrow & H_s(A^s,\frac{A^s}{A^{s-1}}) & \xrightarrow{\lambda} & HH_s(\frac{A^s}{A^{s-1}}) \\
\downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\delta} \\
H_{s-1}(A,A^{s-1}) & \longleftarrow & H_{s-1}(A^s,A^{s-1}) & \longrightarrow & HH_{s-1}(A^{s-1})
\end{array}$$

with $\Psi$ running down the left side and $d$ down the right side.

The leftward arrows are the change-of-algebra maps along $A^s \hookrightarrow A$.

The commutativity of the upper left square is immediate, the one on the right agrees with the rightmost square in Lemma 28 (rotated). The downward arrows in the middle row come from the connecting homomorphism in the long exact sequences arising from Equation (6.11) (as in Equation (6.3) and Equation (6.7), combined with Proposition 27). The commutativity of the lower squares then follows from Lemma 28. **(2)** Next, we patch the outer columns of the diagram as in Equation 6.13 for $s = n, n-1, \ldots, 1$ under each other, giving

$$\begin{array}{ccccc}
H_n(A,A^n) & \xleftarrow{\cong} & H_n(A^n,A^n) & \xrightarrow[\kappa]{\cong} & HH_n(A^n) \\
\downarrow{\scriptstyle\Psi} & & & & \downarrow{\scriptstyle d} \\
H_{n-1}(A,A^{n-1}) & & & & HH_{n-1}(A^{n-1}) \\
\vdots{\scriptstyle\Psi}\downarrow & & & & \vdots{\scriptstyle d}\downarrow \\
H_0(A,A^0) & \longleftarrow & H_0(A^1,A^0) & \longrightarrow & HH_0(A^0)
\end{array}$$

The middle column of the previous diagram does not fit to be glued into this pattern, so we omit it, except for the top and bottom row. The morphisms in the top row are isomorphisms since $A$ (unlike the $A^s$ for $s < n$) is unital. We evaluate the terms in the lowest row and compose with the trace $\tau$, giving the diagram

$$\begin{array}{ccccc}
\frac{A^0}{[A,A^0]} & \longleftarrow & \frac{A^0}{[A^1,A^0]} & \longrightarrow & HH_0(A^0) \\
\downarrow & & \downarrow & & \downarrow \\
k & \xleftarrow[\cong]{} & k & \xrightarrow[\cong]{} & k.
\end{array}$$

Since the trace $\tau$ factors through $[A, A^0]$ (note that $A^0 = I_{tr}$), it is clear that the arrows in the bottom row must be isomorphisms. Thus, $\phi_C = \tau d^{\circ n} = \tau \Psi^{\circ n}$. Note that this comparison only works because in the top and bottom row all terms are isomorphic, whereas on the intermediate rows it is not

clear whether there should exist arrows from the left to the right column (or reversely). It remains to compute $\tau\Psi^{\circ n}$:
**(3)** Consider the diagram with exact rows

$$\begin{array}{ccccc} I_s^0 & \xrightarrow{\text{diag}} & I_s^+ \oplus I_s^- & \longrightarrow & A^s \\ \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle pr_{I_s^+}} & & \Big\downarrow{\scriptstyle (1)} \\ A^{s-1} & \longrightarrow & I_s^+ & \longrightarrow & A^s/I_s^- \\ \Big\downarrow{\scriptstyle =} & & \Big\downarrow{\scriptstyle \text{incl}} & & \Big\downarrow{\scriptstyle (2)} \\ A^{s-1} & \longrightarrow & A^s & \longrightarrow & A^s/A^{s-1} \end{array} \qquad \Lambda \tag{6.14}$$

(here for readability we have omitted intersecting all the ideals with $A^s$; everything is understood to be subobjects of $A^s$). The map $pr_{I_s^+}$ is the projection $(x^+, x^-) \mapsto x^+$. Pick the arrows (1) and (2) such that the diagram becomes commutative. We find both are given by $x \mapsto x^+$ where $x = x^+ + x^-$ with $x^\pm \in I_s^\pm$ is any decomposition of $x$. Moreover, the composition on the right is indeed $\Lambda$. Taking the long exact sequences in Hochschild homology of the top and bottom row yields

$$\begin{array}{ccccc} H_s(A, I_s^+ \oplus I_s^-) & \longrightarrow & H_s(A, A^s) & \xrightarrow{\partial} & H_{s-1}(A, I_s^0) \\ \Big\downarrow{\scriptstyle \text{incl}\circ pr_{I_s^+}} & & \Big\downarrow{\scriptstyle \Lambda} & & \Big\downarrow{\scriptstyle \cong} \\ H_s(A, A^s) & \longrightarrow & H_s(A, A^s/A^{s-1}) & \xrightarrow[\partial]{} & H_{s-1}(A, A^{s-1}) \end{array}$$

Now by the commutativity of the above diagram $\Psi = \partial \circ \Lambda : H_s(A, A^s) \to H_{s-1}(A, A^{s-1})$ (as on the left in Equation 6.13) can be computed just by unwinding the connecting map in the top row. It stems from the bimodule exact sequence in the top row of Equation 6.14: Evaluating this is an easy chase of the snake map, compare with the proof of Lemma 4: Pick some system of good idempotents. We need to pick a lift of $a_0 \otimes a_1 \otimes \cdots \otimes a_s \in C_s(A, A^s)$ to $C_s(A, I_s^+ \oplus I_s^-)$. We may take $f_\gamma := (-1)^\gamma P_s^\gamma a_0 \otimes a_1 \otimes \cdots \otimes a_s$ for $\gamma \in \{\pm\}$ respectively. We need to apply the differential $b$, resulting in

$$\begin{aligned} bf_\gamma = (-1)^\gamma \big(&P_s^\gamma a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_s + \sum_{j=1}^{s-1} (-1)^j P_s^\gamma a_0 \otimes \cdots \otimes a_j a_{j+1} \otimes \cdots \otimes a_s \\ &+ (-1)^s a_s P_s^\gamma a_0 \otimes a_1 \otimes \cdots \otimes a_{s-1}\big) \in C_{s-1}(A, I_1^\gamma) \end{aligned}$$

Next, we need to determine the preimage in $C_{s-1}(A, I_s^0) = C_{s-1}(A, A^{s-1})$, which is

$$\begin{aligned} \Psi(a_0 \otimes \cdots \otimes a_s) &= \textstyle\sum_{\gamma\in\{\pm\}} (-1)^\gamma P_s^{-\gamma}(bf_\gamma) \\ &= (-1)^s \left(\textstyle\sum_{\gamma\in\{\pm\}} (-1)^\gamma P_s^{-\gamma} a_s P_s^\gamma\right) a_0 \otimes a_1 \otimes \cdots \otimes a_{s-1}. \end{aligned}$$

Hence, by applying this formula inductively, we get

$$\tau\Psi^{\circ n}(a_0 \otimes \cdots \otimes a_n) = (-1)^{1+2+\cdots+n}\tau \prod_{s=1\ldots n} \left(\textstyle\sum_{\gamma\in\{\pm\}}(-1)^{\gamma} P_s^{-\gamma} a_s P_s^{\gamma}\right) a_0.$$

This expression clearly coincides (up to sign) with the one of Theorem 23 so that the previously proven identity $\phi_C = \tau d^{\circ n} = \tau\Psi^{\circ n}$ implies the claim. □

**Corollary 33** (Comparison diagram)**.** *Under the assumptions of the theorem and* $\mathfrak{g} := A_{Lie}$*,*

(1) *the diagram*

$$\begin{array}{ccccc} H_n(\mathfrak{g},\mathfrak{g}) & \xrightarrow{\varepsilon} & HH_n(A) & \xrightarrow{\phi_{HH}} & k \\ {\scriptstyle I'}\downarrow & & {\scriptstyle I}\downarrow & & \downarrow{=} \\ H_{n+1}(\mathfrak{g},k) & \xrightarrow[(-1)^n\varepsilon]{} & HC_n(A) & \xrightarrow[\phi_C]{} & k \end{array}$$

*commutes, where for* $f_0,\ldots,f_n \in \mathfrak{g}$*, the map* $\varepsilon$ *in the bottom row is given by*

$$\varepsilon(f_0 \wedge \cdots \wedge f_n) := \sum_{\pi\in\mathfrak{S}_n} \operatorname{sgn}(\pi)\, f_0 \otimes f_{\pi^{-1}(1)} \otimes \cdots \otimes f_{\pi^{-1}(n)}.$$

(2) *The composed map* $H_n(\mathfrak{g},\mathfrak{g}) \to k$ *agrees with* $H_n(\mathfrak{g},\mathfrak{g}) \xrightarrow{I'} H_n(\mathfrak{g},k) \xrightarrow{\phi_{Beil}} k$*.*

**Proof.** The left-hand side square commutes by direct inspection. Then combine Corollary 24 and Corollary 31. □

## 7. Tate's abstract reciprocity revisited

A prominent feature of Tate's article [Tat68] is his slick proof of the residue theorem for curves. In this section, we want to propose a formulation of such vanishing statements on the level of cubically decomposed algebras. In particular, we want to interpret the "abstract residue formula" of [ADCK89, Lemma 2.4] in the Hochschild picture.

**Theorem 34** (Tautological Reciprocity Law)**.** *Suppose* $(A,(I_i^{\pm}),\tau)$ *is an* $n$*-fold cubically decomposed algebra over* $k$ *with local units on all levels. Then*

$$\phi_C(x) = 0$$

*for any element* $x$ *in the Hochschild homology of any of the ideals* $I_i^+$*,* $I_i^-$ *for any* $i$*.*

**Proof.** *(Case A)* Suppose the ideal is $I := I_1^+$. Since for $\Lambda$ we may take *any* decomposition $x = x^+ + x^-$ with $x^{\pm} \in I_1^{\pm}$, we may just as well take

$x^+ := x$. But that means that $\Lambda$ acts on $x$ just like the quotient map, and we get the dotted arrow in

$$\begin{array}{ccccccc} & & & & HH_m(I) & & \\ & & & \swarrow & \downarrow{\scriptstyle \Lambda} & & \\ \cdots \longrightarrow HH_m(A^n) & \xrightarrow{\text{quot}} & HH_m(A^n/A^{n-1}) & \xrightarrow{\delta} & HH_{m-1}(A^{n-1}) & \longrightarrow & \cdots \end{array}$$

and the exactness of the bottom row implies $d(x) = 0$. And therefore, $\phi_C(x) = 0$.
*(Case B)* Suppose the ideal is $I := I_1^-$. Since for $\Lambda$ we may take *any* decomposition $x = x^+ + x^-$ with $x^\pm \in I_1^\pm$, we may just pick $x^+ := 0$. Thus, $\phi_C(x) = 0$.
*(Case C)* Suppose the ideal is $I := I_i^s$ for $i \geq 2$ and $s \in \{+, -\}$. Then apply the first $i-1$ maps "$d$" in Definition 30, and observe that its value lies in $HH_{n-(i-1)}(A^{n-(i-1)} \cap I_i^s)$, but by the inductive nature of the definition this means that the value lies in the ideal $I_1^s$ for the $(n-i+1)$-fold cubically decomposed algebra $A^{n-(i-1)}$, and thus the above Cases A or B apply to this element. Again, we obtain zero. □

Note that this proof is so simple because of the inductive nature of Definition 30. The next vanishing statement is a little more refined.

**Theorem 35** (Cube Reciprocity Law)**.** *Let $(A, (I_i^\pm), \tau)$ be a unital $n$-fold cubically decomposed algebra with local units on all levels. Let $P^\pm \in A$ be idempotents such that*

$$P^+ + P^- = 1 \qquad \text{and} \qquad P^\pm A \in I_1^\pm.$$

*If $R \subseteq A$ is a subalgebra such that $P^+A$ (or $P^-A$) is a left $R$-submodule of $A$, then*

$$\phi_C(r) = 0$$

*for all $r \in HH_n(R)$.*

**Proof.** *(Case A)* Suppose $P^+A$ is a left $R$-submodule. We define a $k$-linear map of Hochschild groups $\psi : C_i(R) \to C_i(A)$, $R^{\otimes i+1} \to A^{\otimes i+1}$ by

$$r_0 \otimes \cdots \otimes r_i \longmapsto r_0 P^+ \otimes \cdots \otimes r_i P^+.$$

We note that the map $r \mapsto rP^+$ would have no reason to be an algebra homomorphism from $R$ to $A$, so we cannot just induce the above map from a morphism of algebras. Instead, we need to check that the above describes a morphism of complexes by hand. We compute

$$\begin{aligned} b(\psi(r_0 \otimes \cdots \otimes r_i)) = &\sum_{j=0}^{i-1} (-1)^j r_0 P^+ \otimes \cdots \otimes r_j P^+ r_{j+1} P^+ \otimes \cdots \otimes r_i P^+ \\ &+ (-1)^i r_i P^+ r_0 P^+ \otimes r_1 P^+ \otimes \cdots \otimes r_{i-1} P^+. \end{aligned}$$

Since the image of $P^+$ is a left $R$-module, $r_{j+1}P^+ \in \operatorname{im} P^+$, and thus $P^+r_{j+1}P^+ = r_{j+1}P^+$, and then $r_jP^+r_{j+1}P^+ = r_jr_{j+1}P^+$. Thus, we get

$$\begin{aligned} b(\psi(r_0 \otimes \cdots \otimes r_i)) &= \sum_{j=0}^{i-1} (-1)^j r_0P^+ \otimes \cdots \otimes r_jr_{j+1}P^+ \otimes \cdots \otimes r_iP^+ \\ &\quad + (-1)^i r_ir_0P^+ \otimes r_1P^+ \otimes \cdots \otimes r_{i-1}P^+ \\ &= \psi b(r_0 \otimes \cdots \otimes r_i). \end{aligned}$$

Thus, $\psi \circ b = b \circ \psi$ and we conclude that $\psi$ is a morphism of complexes. Next, note that for any $a \in A$, we have $a = aP^+ + aP^-$ with $aP^{\pm} \in I_1^{\pm}$. It follows that our map $\psi$ is a lift of $\Lambda$, i.e. the diagram

(7.1)

$$\begin{array}{ccccccccc} & & & & & & HH_m(R) & & \\ & & & & & \overset{\psi}{\swarrow} & \downarrow{\scriptstyle \Lambda} & & \\ \cdots & \longrightarrow & HH_m(A^n) & \xrightarrow{\text{quot}} & HH_m(A^n/A^{n-1}) & \xrightarrow{\delta} & HH_{m-1}(A^{n-1}) & \longrightarrow & \cdots \end{array}$$

commutes. As in the previous proof, the exactness of the row implies that $d(r) = 0$.

*(Case B)* Now assume $P^-A$ is a left $R$-submodule of $A$ instead. We define $\psi$ as before, just replacing each $P^+$ by $P^-$. Everything goes through, with the exception that $\psi$ now lifts $x \mapsto x^-$ instead of $x \mapsto x^+$. However, since $P^+ + P^- = 1$, we can replace Diagram 7.1 by

$$\begin{array}{ccccccccc} & & & & & & HH_m(R) & & \\ & & & & & \overset{\iota-\psi}{\swarrow} & \downarrow{\scriptstyle \Lambda} & & \\ \cdots & \longrightarrow & HH_m(A^n) & \xrightarrow{\text{quot}} & HH_m(A^n/A^{n-1}) & \xrightarrow{\delta} & HH_{m-1}(A^{n-1}) & \longrightarrow & \cdots \end{array}$$

where $\iota$ is the inclusion of algebras $\iota : R \hookrightarrow A$ (this is an algebra homomorphism). Thus, again $\Lambda$ lifts and we obtain $d(r) = 0$. □

### 7.1. Applications of the Cube Reciprocity Law.

*Example* 6 (Curves, Local Theory). Let $k$ be a field and $X/k$ an integral curve. Write $\eta$ for its generic point. Suppose $x \in X$ is a closed point. Then the adèle object

$$A := A(\eta > x) = \prod_i \widehat{\mathcal{K}}_i$$

is a finite product of 1-local fields with residue fields finite over $k$. The number of factors in the product agrees with the number of preimages of the point $x$ in the normalization of the curve $X' \to X$. If $X/k$ is regular, there is always just one factor, as in §2.1. Example 4 demonstrates the effect of a singular point. Following our formalism, we get an abstract residue symbol

$$\operatorname{res}_{\widehat{\mathcal{K}}_i} : \Omega^1_{\widehat{\mathcal{K}}_i/k} \longrightarrow HH_1(A) \xrightarrow{\phi_C} k. \tag{7.2}$$

Now write

$$\widehat{\mathcal{K}}_i = \widehat{\mathcal{O}}_i \oplus B, \qquad \text{(as Tate vector spaces)} \tag{7.3}$$

where $\widehat{\mathcal{O}}_i$ is the ring of integers of $\widehat{\mathcal{K}}_i$ (this need not agree with $\widehat{\mathcal{O}}_{X,x}$ if $x$ does not lie in the smooth locus; rather it would be a finite ring extension; it always agrees with $\widehat{\mathcal{O}}_{X',x'}$, where $x'$ is the chosen preimage of $x$ in the normalization $X'$), and $B$ is any $k$-vector space complement. As $\widehat{\mathcal{O}}_i$ is a lattice of the Tate vector space, let $P^{\pm}$ be the idempotents underlying the direct sum decomposition of Equation 7.3. Then $\widehat{\mathcal{O}}_i \hookrightarrow \widehat{\mathcal{K}}_i$ is a subalgebra such that $P^+A$ is a left-$\widehat{\mathcal{O}}_i$-module (this is true because $P^+$ maps everything to $\widehat{\mathcal{O}}_i \subseteq \widehat{\mathcal{K}}_i$, and if we act on $\widehat{\mathcal{O}}_i$ by multiplication with an element $f \in \widehat{\mathcal{O}}_i$, this still lies in $\widehat{\mathcal{O}}_i$, and therefore applying $P^+$ again acts as the identity). Hence, by the Cube Reciprocity Law $HH_1(\widehat{\mathcal{O}}_i) \to HH_1(A) \xrightarrow{\phi_C} k$ is the zero map. As a result, we learn that our residue map in Equation 7.2 is trivial on 1-forms without poles and factors as $\Omega^1_{\widehat{\mathcal{K}}_i/k}/\Omega^1_{\widehat{\mathcal{O}}_i/k} \to k$. Of course, this is one of the most obvious properties the residue map should have. We see here that it is encoded in Theorem 35.

*Example* 7 (Curves, Global Theory). We continue the previous example. By Beilinson's resolution, Theorem 6, we have the flasque adèle resolution of the sheaf $\Omega^1_{X/k}$, namely

$$0 \longrightarrow \Omega^1_{X/k} \longrightarrow \mathbf{A}^{(0)}_{\Omega^1} \oplus \mathbf{A}^{(1)}_{\Omega^1} \longrightarrow \mathbf{A}^{(01)}_{\Omega^1} \longrightarrow 0. \tag{7.4}$$

Here $\mathbf{A}^i_{\mathcal{F}} = \mathbf{A}^i \otimes \mathcal{F}$ denotes the adèles for the coherent sheaf $\mathcal{F}$, while the decoration $\mathbf{A}^{(1)}$ denotes the adèles running through all singleton flags $\triangle$ consisting only of closed points $\{(x)\}_{x \in X_0}$, while $\mathbf{A}^{(0)}$ denotes the remaining summand, which agrees with the rational function field $k(X)$ of the curve. Finally, $\mathbf{A}^{(01)}$ are the adèles of all length 2 flags, i.e. those of the shape $(\eta > x)$ for $\eta$ the generic point and $x$ running through the closed points. The adèles also carry the structure of a cubically decomposed algebra [BGW16b]. One way to see this is by using that they are a 1-Tate object, as explained in [BGW16b], Theorem 5 (1), and therefore the endomorphism algebra in the category of Tate vector spaces has a natural structure of a cubically decomposed algebra, see loc. cit., Theorem 5 (2). Feeding this into our abstract machine, we get a residue symbol on the level of adèles,

$$\mathrm{res}_{\mathbf{A}} : HH_1(\mathbf{A}^{(01)}) \xrightarrow{\phi_C} k.$$

Due to the nature of the adèles, there is a projection map of 1-Tate objects (and rings, simultaneously) $\mathbf{A}^{(01)} \longrightarrow \widehat{\mathcal{K}}$, where $\widehat{\mathcal{K}}$ is a local field factor as in the local theory, Example 6. As a result, the residue on the adèles is just the finite sum of the local residues

$$\mathrm{res}_{\mathbf{A}}((\alpha_x)_x) = \sum \mathrm{res}_{\widehat{\mathcal{K}}}(\alpha_x), \tag{7.5}$$

where $x$ runs through the set of closed points. Thus, we can reduce the computation of residues to local fields (this is the analogue of [Tat68, Theorem 3]). We get two reciprocity laws now: Firstly, $\mathbf{A}^{(1)} = \prod_{x \in X} \widehat{\mathcal{O}}_x$ is an $\mathbf{A}^{(1)}$-submodule of $\mathbf{A}^{(01)}$. We get a direct sum splitting $\mathbf{A}^{(01)} = \mathbf{A}^{(1)} \oplus B$ and Theorem 35 implies that residues of 1-forms from $\mathbf{A}^{(1)}$ are zero. This is no real news of course, since this already follows from the local study of Example 6. However, we also get a direct sum splitting $\mathbf{A}^{(01)} = \mathbf{A}^{(0)} \oplus B'$, where $\mathbf{A}^{(0)} = k(X)$ is just the rational function field and this is a $k(X)$-submodule of $\mathbf{A}^{(01)}$. If $X/k$ is proper (and only then!), the finite-dimensionality of the cohomology implies that the assumptions of Theorem 35 are met: Concretely, we could write $\mathbf{A}^{(01)} = k(X) \oplus L$ for a suitably chosen lattice $L$ of the Tate vector space such that, on the level of $k$-vector spaces, this splitting can be identified with

$$\mathbf{A}^{(01)} = k(X) \oplus \underbrace{\frac{\mathbf{A}^{(1)}}{H^0(X, \mathcal{O}_X)} \oplus H^1(X, \mathcal{O}_X)}_{\simeq L}.$$

This is possible since Theorem 6 (applied to $\mathcal{O}_X$) implies that $H^0(X, \mathcal{O}_X) = \mathbf{A}^{(0)} \cap \mathbf{A}^{(1)}$ and $H^1(X, \mathcal{O}_X)$ is isomorphic to the cokernel of $\mathbf{A}^{(0)} + \mathbf{A}^{(1)}$ inside $\mathbf{A}^{(01)}$. Since both cohomology groups are finite-dimensional $k$-vector spaces, $L$ is indeed a lattice. Thus, Theorem 35 tells us that global rational 1-forms have vanishing global residue $\mathrm{res}_{\mathbf{A}}$. By the global-local formula, Equation (7.5), we conclude the following famous fact: For any global rational 1-form $\omega \in \Omega^1_{X/k}(X) \otimes k(X)$, the sum of residues is zero, i.e.

$$\sum \mathrm{res}_{\widehat{\mathcal{K}}_x}(\omega) = 0.$$

This is the analogue of Tate's [Tat68, §3, Corollary], and of course properness enters our argument in exactly the same rôle as in his paper. If $f$ is a non-zero global rational function, $\mathrm{d}\log(f) = \mathrm{d}f/f$ is such a rational 1-form and we learn that the total sum of orders of zeros and poles is zero (when being added up in $k$; so if $\mathrm{char}(k) > 0$ this statement is not as strong as it could be).

*Example* 8 (Less standard fact). Suppose we are in the situation of Example 6. Instead of Equation 7.3, we also have a direct sum splitting $\widehat{\mathcal{K}}_i = \kappa[t^{-1}] \oplus t\kappa[[t]]$, where we have chosen, for the sake of exposition, an isomorphism $\widehat{\mathcal{K}}_i \simeq \kappa((t))$. Note that $\kappa[t^{-1}]$ is also a subalgebra such that the Cube Reciprocity Law applies. It tells us that $\mathrm{res}(t^{-n}\mathrm{d}t^{-m}) = 0$ for all $n, m \geq 0$. While one finds this fact rarely articulated, it is of course also easy to show using the usual calculus of differentials: $t^{-n}(\mathrm{d}t^{-m}) = t^{-n}(-mt^{-m-1}\mathrm{d}t) = -mt^{-n-m-1}\mathrm{d}t$. For $n, m \geq 0$ this visibly (from the usual perspective) can only have non-zero residue if $n = m = 0$, but then this expression is zero thanks to the leading coefficient $m$.

So far, we have only used the Cube Reciprocity Law to establish statements in dimension one. We shall address the higher-dimensional story in a sequel.

## 8. The bigger picture

In this paper, we have first tried to explain the construction of the residue map in [Beĭ80]. Loc. cit., Beilinson does this using Lie homology, and specifically relative Lie homology. This word never appears in [Beĭ80], but we hope to have elucidated why and how this shows up in §4. The essence of the construction lies in

$$\phi_{Beil} : H_{n+1}(\mathfrak{g}, k) \xrightarrow{\sim} H_{n+1}(CE(\mathfrak{g})) \xrightarrow{\text{edge}} E^{n+1}_{0,n+1} \xrightarrow[d^{-1}]{\sim} E^{n+1}_{n+1,1} \xrightarrow{\text{edge}} H_1(\hat{}T^{n+1}_\bullet) \xrightarrow{\tau} k$$

of §3. In the present paper, we have explained how to remove the presence of any relative Lie homology groups by (a) reformulating the theory in Hochschild homology, and (b) showing that the above map can (essentially) also be realized by an iterated use of a *modified* boundary map $d$,

$$\phi_C : HH_n(A) \longrightarrow HH_0(I_{tr}) \longrightarrow k, \qquad \alpha \mapsto \tau d \circ \cdots \circ d\,\alpha. \tag{8.1}$$

This is based on writing the cubically decomposed algebra as an iterated extension, $A^{n-1} \to A^n \to A^n/A^{n-1}$.

As gets developed in joint work with M. Groechenig and J. Wolfson, [BGW16a], one can conveniently package the definition of the adèles of a scheme as an object of the category $\mathcal{T} := n\text{-}\mathsf{Tate}(\mathsf{Vect}_f)$, and then the complicated definition of the cubically decomposed algebra structure, Definition 10, simplifies to the plain $\mathrm{End}_{\mathcal{T}}$ in this category. Now, for any exact sequence of exact categories $\mathcal{C}' \hookrightarrow \mathcal{C} \twoheadrightarrow \mathcal{C}''$, one has an induced long exact sequence in the Hochschild homology of exact categories [Kel99]. Joint work with J. Wolfson in the companion paper [BW] then shows that $\phi_C$ agrees with the iterated use of the boundary map of this long exact sequence.

Thus, unlike the $d$ in Equation 8.1, which comes from non-commutative algebra extensions and the Toeplitz-like twist by $\Lambda$ in Equation 6.12, the localization sequence boundary map gives the right map on the nose. Combined with this paper, we thus can follow the entire journey from Tate's original approach using commutators in [Tat68], to Beilinson's use of relative Lie homology [Beĭ80], to Hochschild homology of non-unital algebras in the present paper, to the Hochschild homology of categories in [BW]. The latter paper has a new version of a Hochschild–Kostant–Rosenberg theorem *with supports*, which also makes a connection to the local cohomology approach of Grothendieck in [Har66].

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