

Bergman-Lorentz spaces on tube domains over symmetric cones

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ABSTRACT. We study Bergman-Lorentz spaces on tube domains over symmetric cones, i.e. spaces of holomorphic functions which belong to Lorentz spaces $L(p, q)$. We establish boundedness and surjectivity of Bergman projectors from Lorentz spaces to the corresponding Bergman-Lorentz spaces and real interpolation between Bergman-Lorentz spaces. Finally we ask a question whose positive answer would enlarge the interval of parameters $p \in (1, \infty)$ such that the relevant Bergman projector is bounded on L^p for cones of rank $r \geq 3$.

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1. Introduction

The notations and definitions are those of [11]. We denote by Ω an irreducible symmetric cone in \mathbb{R}^n with rank r and determinant Δ . We denote $T_\Omega = \mathbb{R}^n + i\Omega$ the tube domain in \mathbb{C}^n over Ω . For $\nu \in \mathbb{R}$, we define the weighted measure μ on T_Ω by $d\mu(x + iy) = \Delta^{\nu - \frac{n}{r}}(y) dx dy$. We consider Lebesgue spaces L_ν^p and Lorentz spaces $L_\nu(p, q)$ on the measure space (T_Ω, μ) . The Bergman space A_ν^p (resp. the Bergman-Lorentz space $A_\nu(p, q)$) is the subspace of L_ν^p (resp. of $L_\nu(p, q)$) consisting of holomorphic functions. Our first result is the following.

Theorem 1.1. *Let $1 < p \leq \infty$ and $1 \leq q \leq \infty$.*

- (1) *For $\nu \leq \frac{n}{r} - 1$, the Bergman-Lorentz space $A_\nu(p, q)$ is trivial, i.e. $A_\nu(p, q) = \{0\}$.*
- (2) *Suppose $\nu > \frac{n}{r} - 1$. Equipped with the norm induced by the Lorentz space $L_\nu(p, q)$, the Bergman-Lorentz space $A_\nu(p, q)$ is a Banach space.*

For $p = 2$, the Bergman space A_ν^2 is a closed subspace of the Hilbert space L_ν^2 and the Bergman projector P_ν is the orthogonal projector from L_ν^2 to A_ν^2 . We adopt the notation

$$Q_\nu = 1 + \frac{\nu}{\frac{n}{r} - 1}.$$

Our boundedness theorem for Bergman projectors on Lorentz spaces is the following.

Theorem 1.2. *Let $\nu > \frac{n}{r} - 1$ and $1 \leq q \leq \infty$.*

- (1) *For all $\gamma \geq \nu + \frac{n}{r} - 1$, the weighted Bergman projector P_γ extends to a bounded operator from $L_\nu(p, q)$ to $A_\nu(p, q)$ for all $1 < p < Q_\nu$. In this case, under the restriction $1 < q < \infty$, if $\gamma > \left(\frac{1}{\min(p, q) - 1} - 1\right) \left(\frac{n}{r} - 1\right)$, then P_γ is the identity on $A_\nu(p, q)$.*
- (2) *The weighted Bergman projector P_ν extends to a bounded operator from $L_\nu(p, q)$ to $A_\nu(p, q)$ for all $1 + Q_\nu^{-1} < p < 1 + Q_\nu$. In this case, under the restriction $1 \leq q < \infty$, then P_ν is the identity on $A_\nu(p, q)$.*

For the Bergman projector P_ν , following recent developments, this theorem is extended below to a larger interval of exponents p on tube domains over Lorentz cones ($r = 2$) (see section 7).

Finally our main real interpolation theorem between Bergman-Lorentz spaces is the following.

Theorem 1.3. *Let $\nu > \frac{n}{r} - 1$.*

- (1) *For all $1 < p_1 < Q_\nu$, $1 \leq q_1 \leq \infty$ and $0 < \theta < 1$, the real interpolation space*

$$(A_\nu^1, A_\nu(p_1, q_1))_{\theta, q}$$

identifies with $A_\nu(p, q)$, $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$, $1 < q < \infty$ with equivalence of norms.

- (2) *For all $1 < p_0 < p_1 < Q_\nu$ (resp. $1 + Q_\nu^{-1} < p_0 < p_1 < 1 + Q_\nu$), $1 \leq q_0, q_1 \leq \infty$ and $0 < \theta < 1$, the real interpolation space*

$$(A_\nu(p_0, q_0), A_\nu(p_1, q_1))_{\theta, q}$$

identifies with $A_\nu(p, q)$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $1 < q < \infty$ (resp. $1 \leq q < \infty$) with equivalence of norms.

- (3) *For all $Q_\nu \leq p_1 < 1 + Q_\nu$, $1 \leq q_1 \leq \infty$, the Bergman-Lorentz spaces $A_\nu(p, q)$, $1 + Q_\nu^{-1} < p < Q_\nu$, $1 \leq q < \infty$ are real interpolation spaces between A_ν^1 and $A_\nu(p_1, q_1)$ with equivalence of norms.*
- (4) *For all $1 < p_0 \leq 1 + Q_\nu^{-1}$, $Q_\nu \leq p_1 < 1 + Q_\nu$, $1 \leq q_0, q_1 \leq \infty$, the Bergman-Lorentz spaces $A_\nu(p, q)$, $1 + Q_\nu^{-1} < p < Q_\nu$, $1 \leq q < \infty$ are real interpolation spaces between $A_\nu(p_0, q_0)$ and $A_\nu(p_1, q_1)$ with equivalence of norms.*

The plan of the paper is as follows. In section 2, we overview definitions and properties of Lorentz spaces on a non-atomic σ -finite measure space. This section encloses results on real interpolation between Lorentz spaces. In section 3, we define Bergman-Lorentz spaces on a tube domain T_Ω over a symmetric cone Ω . We produce examples and we establish Theorem 1.1. In section 4, we study the density of the subspace $A_\nu(p, q) \cap A_\gamma^t$ in the Banach space $A_\nu(p, q)$ for $\nu, \gamma > \frac{n}{r} - 1$, $1 < p, t \leq \infty$, $1 \leq q < \infty$. Relying on boundedness results for the Bergman projectors P_γ , $\gamma \geq \nu + \frac{n}{r} - 1$ and P_ν on Lebesgue spaces L_ν^p [2, 4, 18], we then provide a proof of Theorem 1.2. In section 5, we prove Theorem 1.3. We next deduce a result of dependence of the Bergman space $A_\nu(p, q)$ on the parameters p, q . In section 6, we come back to the density of the subspace $A_\nu(p, q) \cap A_\gamma^t$ in $A_\nu(p, q)$ and we prove a stronger result than the ones in section 4. Section 7 consists of four questions. A positive answer to the first question would enlarge the interval of parameters $p \in (1, \infty)$ such that the Bergman projector P_ν is bounded on L_ν^p for upper rank cones ($r \geq 3$). The second question addresses the density of the subspace $A_\nu(p, q) \cap A_\gamma^t$ in the Banach space $A_\nu(p, q)$. The third question concerns a possible extension of Theorem 1.3. The fourth question concerns the dependence on the parameters p, q of the Bergman-Lorentz space $A_\nu(p, q)$.

These results were first presented in the PhD dissertation of the second author [12]. Similar results, with the real interpolation method replaced by the complex interpolation method, were proved in [5].

2. Lorentz spaces on measure spaces

2.1. Definitions and preliminary topological properties. Throughout this section, the notation (E, μ) is fixed for a non-atomic σ -finite measure space. We refer to [20], [14], [13], [7] and [8]. Also cf. [12].

Definition 2.1. Let f be a measurable function on (E, μ) and finite μ -a.e.. The distribution function μ_f of f is defined on $[0, \infty)$ by

$$\mu_f(\lambda) = \mu(\{x \in E : |f(x)| > \lambda\}).$$

The non-increasing rearrangement function f^* of f is defined on $[0, \infty)$ by

$$f^*(t) = \inf\{\lambda \geq 0 : \mu_f(\lambda) \leq t\}.$$

Theorem 2.2. (Hardy-Littlewood) [7, Theorem 2.2.2] *Let f and g be two measurable functions on (E, μ) . Then*

$$\int_E |f(x)g(x)|d\mu(x) \leq \int_0^\infty f^*(s)g^*(s)ds.$$

In particular, let g be a positive measurable function on (E, μ) and let F be a measurable subset of E of bounded measure $\mu(F)$. Then

$$\int_F g(x)d\mu(x) \leq \int_0^{\mu(F)} g^*(s)ds.$$

Definition 2.3. Let $1 \leq p, q \leq \infty$. The Lorentz space $L(p, q)$ is the space of measurable functions on (E, μ) such that

$$\|f\|_{p,q} = \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \quad \text{if } 1 \leq p < \infty \quad \text{and} \quad 1 \leq q < \infty$$

(resp.

$$\|f\|_{p,\infty} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty \quad \text{if } 1 \leq p \leq \infty).$$

We first recall that (cf. e.g. [13, Proposition 1.4.5, assertion (16)]):

$$(2.1) \quad \|f\|_{p,\infty} = \sup_{\lambda>0} \lambda \mu_f(\lambda)^{\frac{1}{p}}.$$

We next recall that for $p = \infty$ and $q < \infty$, this definition gives way to the space of vanishing almost everywhere functions on (E, μ) . It is also well known that for $p = q$, the Lorentz space $L(p, p)$ coincides with the Lebesgue space $L^p(E, d\mu)$. More precisely, we have the equality

$$(2.2) \quad \|f\|_p = \left(\int_0^\infty f^*(t)^p dt \right)^{\frac{1}{p}}.$$

In the sequel we shall adopt the following notation:

$$L^p = L^p(E, d\mu).$$

We shall need the following two results.

Proposition 2.4. [7, Lemma 4.4.5] *The functional $\|\cdot\|_{p,q}$ is a quasi-norm (it satisfies all properties of a norm except the triangle inequality) on $L(p, q)$. If $1 < p < \infty$ and $1 \leq q \leq \infty$ (and if $p = q = 1$), the Lorentz space $L(p, q)$ is equipped with a norm equivalent to the quasi-norm $\|\cdot\|_{p,q}^*$.*

Theorem 2.5. [7, Theorem 4.4.6], [14, (2.3)] *Let $1 \leq p < \infty$, $1 \leq q \leq \infty$. Equipped with the quasi-norm $\|\cdot\|_{p,q}$, the Lorentz space $L(p, q)$ is a quasi-Banach space (a Banach space with the norm referred to in the previous proposition if $1 < p < \infty$ and $1 \leq q \leq \infty$).*

We next record the following results stated in [7, 13, 14].

Proposition 2.6. [13, Proof of Theorem 1.4.11] *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Every Cauchy sequence in the Lorentz space $(L(p, q), \|\cdot\|_{p,q})$ contains a subsequence which converges a.e. to its limit in $L(p, q)$.*

Theorem 2.7. [7, Corollary 4.4.8], [13, Theorem 1.4.17] and [14, (2.7)] *Let $1 < p < \infty$ and $1 \leq q < \infty$. The topological dual space $(L(p, q))'$ of the Lorentz space $L(p, q)$ identifies with the Lorentz space $L(p', q')$ with respect to the duality pairing*

$$(\star) \quad (f, g) = \int_{T_\Omega} f(z) \overline{g(z)} d\mu(z).$$

We finally record the nested property of Lorentz spaces. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two quasi-normed vector spaces. We say that X continuously embeds in Y and we write $X \hookrightarrow Y$ if $X \subset Y$ and there exists a positive constant C such that

$$\|x\|_Y \leq C\|x\|_X \quad \forall x \in X.$$

It is easy to check that X identifies with Y if and only if $X \hookrightarrow Y$ and $Y \hookrightarrow X$.

Proposition 2.8. [13, Proposition 1.4.10 and Exercise 1.4.8] *For all $1 \leq p < \infty$ and $1 \leq q < r \leq \infty$ we have the continuous embedding*

$$L(p, q) \hookrightarrow L(p, r).$$

This embedding is strict.

2.2. Interpolation via the real method between Lorentz spaces.

We begin with an overview of the theory of real interpolation between quasi-Banach spaces (cf. e.g. [7, Chapters 4 and 5]) and [8, Chapters 3 and 5] for Banach spaces, and [8, Sections 3.10 and 3.11] for quasi-Banach spaces).

Definition 2.9. A pair (X_0, X_1) of quasi-Banach spaces is called a compatible couple if there is some Hausdorff topological vector space in which X_0 and X_1 are continuously embedded.

Definition 2.10. Let (X_0, X_1) be a compatible couple of quasi-Banach spaces. Denote $\overline{X} = X_0 + X_1$. Let $t > 0$ and $a \in \overline{X}$. We define the functional $K(t, a, \overline{X})$ by

$$K(t, a, \overline{X}) = \inf \{ \|a_0\|_{X_0} + t\|a_1\|_{X_1} : a = a_0 + a_1, a_0 \in X_0, a_1 \in X_1 \}.$$

For $0 < \theta < 1$ and $1 \leq q \leq \infty$, the real interpolation space between X_0 and X_1 is the space

$$(X_0, X_1)_{\theta, q} := \{ a \in \overline{X} : \|a\|_{\theta, q, \overline{X}} < \infty \}$$

with

$$\|a\|_{\theta, q, \overline{X}} := \left(\int_0^\infty \left(t^{-\theta} K(t, a, \overline{X}) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}.$$

The following proposition is proved in [7, Proposition 5.1.8] for Banach spaces; for quasi-Banach Banach spaces, we refer to [8, Sections 3.10 and 3.11].

Proposition 2.11. For $0 < \theta < 1$ and $1 \leq q \leq \infty$, the functional $\|\cdot\|_{\theta, q, \overline{X}}$ is a quasi-norm on the real interpolation space $(X_0, X_1)_{\theta, q}$. Endowed with this quasi-norm, $(X_0, X_1)_{\theta, q}$ is a quasi-Banach space.

Definition 2.12. Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples of quasi-Banach spaces and let T be a linear operator defined on $\overline{X} := X_0 + X_1$ and taking values in $\overline{Y} := Y_0 + Y_1$. Then T is said to be admissible with respect to the couples \overline{X} and \overline{Y} if for $i = 0, 1$, the restriction of T is a bounded operator from X_i to Y_i .

We next state the following fundamental theorem.

Theorem 2.13. [7, Theorem 5.1.12], [8, Theorem 3.11.8] Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples of quasi-Banach spaces and let $0 < \theta < 1$, $1 \leq q \leq \infty$. Let T be an admissible linear operator with respect to the couples \overline{X} and \overline{Y} such that

$$\|Tf_i\|_{Y_i} \leq M_i \|f_i\|_{X_i} \quad (f_i \in X_i, \quad i = 0, 1).$$

Then T is a bounded operator from $(X_0, X_1)_{\theta, q}$ to $(Y_0, Y_1)_{\theta, q}$. More precisely, we have

$$\|Tf\|_{(Y_0, Y_1)_{\theta, q}} \leq M_0^{1-\theta} M_1^\theta \|f\|_{(X_0, X_1)_{\theta, q}}$$

for all $f \in (X_0, X_1)_{\theta, q}$.

The following theorem gives the real interpolation spaces between Lebesgue spaces and Lorentz spaces on the measure space (E, μ) .

Theorem 2.14. [7, Theorem 5.1.9], [8, Theorems 5.2.1 and 5.3.1] Let $0 < \theta < 1$, $1 \leq q \leq \infty$. Let $1 \leq p_0 < p_1 \leq \infty$ and define the exponent p by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We have the identifications with equivalence of norms:

a) $(L^{p_0}, L^{p_1})_{\theta, q} = L(p, q)$;

b) $(L(p_0, q_0), L(p_1, q_1))_{\theta, q} = L(p, q)$ for $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1 \leq \infty$.

Definition 2.15. Let (R, μ) and (S, ν) be two non-atomic σ -finite measure spaces. Suppose $1 \leq p < \infty$, $1 \leq q \leq \infty$. Let T be a linear operator defined on the simple functions on (R, μ) and taking values on the measurable functions on (S, ν) . Then T is said to be of restricted weak type (p, q) if there is a positive constant M such that

$$t^{\frac{1}{q}}(T\chi_F)^*(t) \leq M\mu(F)^{\frac{1}{p}} \quad (t > 0)$$

for all measurable subsets F of R . This estimate can also be written in the form

$$\|T\chi_F\|_{q, \infty} \leq M\|\chi_F\|_{p, 1}$$

or equivalently, in view of equality (2.1),

$$\sup_{\lambda > 0} \lambda \mu_{T\chi_F}(\lambda)^{\frac{1}{q}} \leq M\mu(F)^{\frac{1}{p}}.$$

In the next two statements, $L_R(p, 1)$ and $L_S(q, \infty)$ denote the corresponding Lorentz spaces on the respective measure spaces (R, μ) and (S, ν) .

Proposition 2.16. [7, Theorem 5.5.3] *Let (R, μ) and (S, ν) be two non-atomic σ -finite measure spaces. Suppose $1 \leq p < \infty$, $1 \leq q \leq \infty$. Let T be a linear operator defined on the simple functions on (R, μ) and taking values on the measurable functions on (S, ν) . We suppose that T is of restricted weak type (p, q) . Then T uniquely extends to a bounded operator from $L_R(p, 1)$ to $L_S(q, \infty)$.*

Theorem 2.17 (Stein-Weiss). [7, Theorem 4.5.5] *Let (R, μ) and (S, ν) be two measure spaces. Suppose $1 \leq p_0 < p_1 < \infty$ and $1 \leq q_0, q_1 \leq \infty$ with $q_0 \neq q_1$. Suppose further that T is a linear operator defined on the simple functions on (R, μ) and taking values on the measurable functions on (S, ν) and suppose that T is of restricted weak types (p_0, q_0) and (p_1, q_1) . If $1 \leq r \leq \infty$, then T has a unique extension to a linear operator, again denoted by T , which is bounded from $L_R(p, r)$ into $L_S(q, r)$ where*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad 0 < \theta < 1.$$

If in addition, the inequalities $p_j \leq q_j$ ($j = 0, 1$) hold, then T is of strong type (p, q) , i.e. there exists a positive constant C such that

$$\|Tf\|_{L^q(S, \nu)} \leq C\|f\|_{L^p(R, \mu)} \quad (f \in L^p(R, \mu)).$$

We finish this section with the Wolff reiteration theorem.

Definition 2.18. If (X_0, X_1) is a compatible couple of quasi-Banach spaces, then a quasi-Banach space X is said to be an intermediate space between X_0 and X_1 if X is continuously embedded between $X_0 \cap X_1$ and $X_0 + X_1$, i.e.

$$X_0 \cap X_1 \hookrightarrow X \hookrightarrow X_0 + X_1.$$

We remind the reader that the real interpolation space $(X_0, X_1)_{\theta, q}$, $0 < \theta < 1$, $1 \leq q \leq \infty$ is an intermediate space between X_0 and X_1 . In this direction, we recall the following density theorem. The given reference is for Banach spaces; for quasi-Banach spaces, we refer to [8, Section 3.11].

Theorem 2.19. [7, Theorem 2.9] *Let (X_0, X_1) be a compatible couple of quasi-Banach spaces and suppose $0 < \theta < 1$, $1 \leq q < \infty$. Then the subspace $X_0 \cap X_1$ is dense in $(X_0, X_1)_{\theta, q}$.*

We next state the Wolff reiteration theorem.

Theorem 2.20. [21, 16] *Let X_2 and X_3 be intermediate quasi-Banach spaces of a compatible couple (X_1, X_4) of quasi-Banach spaces. Let $0 < \varphi, \psi < 1$ and $1 \leq q, r \leq \infty$ and suppose that*

$$X_2 = (X_1, X_3)_{\varphi, q}, \quad X_3 = (X_2, X_4)_{\psi, r}.$$

Then (up to equivalence of norms)

$$X_2 = (X_1, X_4)_{\rho, q}, \quad X_3 = (X_1, X_4)_{\theta, r}$$

where

$$\rho = \frac{\varphi\psi}{1 - \varphi + \varphi\psi}, \quad \theta = \frac{\psi}{1 - \varphi + \varphi\psi}.$$

3. The Bergman-Lorentz spaces on tube domains over symmetric cones

3.1. Symmetric cones: definitions and preliminary notions. Materials of this section are essentially from [11]. We give some definitions and useful results.

Let Ω be an irreducible open cone of rank r inside a vector space V of dimension n , endowed with an inner product (\cdot, \cdot) for which Ω is self-dual. Such a cone is called a symmetric cone in V . Let $G(\Omega)$ be the group of transformations of Ω , and G its identity component. It is well-known that there exists a subgroup H of G acting simply transitively on Ω , that is, every $y \in \Omega$ can be written uniquely as $y = ge$ for some $g \in H$ and a fixed $e \in \Omega$. The notation Δ is for the determinant of Ω .

We first recall the following lemma.

Lemma 3.1. [2, Corollary 3.4 (i)] *The following inequality is valid.*

$$\Delta(y) \leq \Delta(y + v) \quad (y, v \in \Omega).$$

We denote by d_Ω the H -invariant distance on Ω . The following lemma will be useful.

Lemma 3.2. [2, Theorem 2.38] *Let $\delta > 0$. There exists a positive constant $\gamma = \gamma(\delta, \Omega)$ such that for $\xi, \xi' \in \Omega$ satisfying $d_\Omega(\xi, \xi') \leq \delta$, we have*

$$\frac{1}{\gamma} \leq \frac{\Delta(\xi)}{\Delta(\xi')} \leq \gamma.$$

In the sequel, we write as usual $V = \mathbb{R}^n$.

3.2. Bergman-Lorentz spaces on tube domains over symmetric cones. Proof of Theorem 1.1. Let Ω be an irreducible symmetric cone in \mathbb{R}^n with rank r , determinant Δ and fixed point \mathbf{e} . We denote $T_\Omega = \mathbb{R}^n + i\Omega$ the tube domain in \mathbb{C}^n over Ω . For $\nu \in \mathbb{R}$, we define the weighted measure μ on T_Ω by $d\mu(x + iy) = \Delta^{\nu - \frac{n}{r}}(y) dx dy$. For a measurable subset A of T_Ω , we denote by $|A|$ the (unweighted) Lebesgue measure of A , i.e. $|A| = \int_A dx dy$.

Definition 3.3. Since the determinant Δ is a polynomial in \mathbb{R}^n , it can be extended in a natural way to \mathbb{C}^n as a holomorphic polynomial we shall denote $\Delta\left(\frac{x+iy}{i}\right)$. It is known that this extension is zero free on the simply connected region T_Ω in \mathbb{C}^n . So for each real number α , the power function Δ^α can also be extended as a holomorphic function $\Delta^\alpha\left(\frac{x+iy}{i}\right)$ on T_Ω .

The following lemma will be useful.

Lemma 3.4. [2, Remark 3.3 and Lemma 3.20] *Let $\alpha > 0$.*

(1) *We have*

$$\left| \Delta^{-\alpha}\left(\frac{z}{i}\right) \right| \leq \Delta^{-\alpha}(\Im z) \quad (z \in T_\Omega).$$

(2) *We suppose $\nu > \frac{n}{r} - 1$ and $p > 0$. The following estimate*

$$\int_\Omega \left(\int_{\mathbb{R}^n} \left| \Delta^{-\alpha}\left(\frac{x + i(y + \mathbf{e})}{i}\right) \right|^p dx \right) \Delta^{\nu - \frac{n}{r}}(y) dy < \infty$$

$$\text{holds if and only if } \alpha > \frac{\nu + \frac{2n}{r} - 1}{p}.$$

We denote by d the Bergman distance on T_Ω . We remind the reader that the group $\mathbb{R}^n \times H$ acts simply transitively on T_Ω . The following lemma will also be useful.

Lemma 3.5. [2, Proposition 2.42] *The measure $\Delta^{-\frac{2n}{r}}(y) dx dy$ is $\mathbb{R}^n \times H$ -invariant on T_Ω .*

The following corollary is an easy consequence of Lemma 3.2.

Corollary 3.6. *Let $\delta > 0$. There exists a positive constant $C = C(\delta)$ such that for $z, z' \in \Omega$ satisfying $d(z, z') \leq \delta$, we have*

$$\frac{1}{C} \leq \frac{\Delta(\Im z)}{\Delta(\Im z')} \leq C.$$

The next proposition will lead us to the definition of Bergman-Lorentz spaces.

Proposition 3.7. *The measure space (T_Ω, μ) is a non-atomic σ -finite measure space.*

In view of this proposition, all the results of the previous section are valid on the measure space (T_Ω, μ) . We shall denote by $L_\nu(p, q)$ the corresponding Lorentz space on (T_Ω, μ) and we denote by $\|\cdot\|_{L_\nu(p, q)}$ its associated (quasi-)

norm. Moreover, we write L_ν^p for the weighted Lebesgue space $L^p(T_\Omega, d\mu)$ on (T_Ω, μ) .

The following corollary is an immediate consequence of Theorem 2.19 and assertion a) of Theorem 2.14.

Corollary 3.8. *The subspace $C_c^\infty(T_\Omega)$ consisting of C^∞ functions with compact support on T_Ω is dense in the Lorentz space $L_\nu(p, q)$ for all $1 < p < \infty$, $1 \leq q < \infty$.*

Definition 3.9. The Bergman-Lorentz space $A_\nu(p, q)$, $1 \leq p, q \leq \infty$ is the subspace of the Lorentz space $L_\nu(p, q)$ consisting of holomorphic functions. In particular $A_\nu(p, p) = A_\nu^p$, where $A_\nu^p = Hol(T_\Omega) \cap L_\nu^p$ is the usual weighted Bergman space on T_Ω . In fact, A_ν^p , $1 \leq p \leq \infty$ is a closed subspace of the Banach space L_ν^p . The Bergman projector P_ν is the orthogonal projector from the Hilbert space L_ν^2 to its closed subspace A_ν^2 .

For each $F \in A_\nu(p, q)$, we shall adopt the notation:

$$\|F\|_{A_\nu(p,q)} = \|F\|_{L_\nu(p,q)}$$

Example 3.10. Let $\nu > \frac{n}{r} - 1$, $1 < p < \infty$, $1 \leq q \leq \infty$. The function $F(z) = \Delta^{-\alpha}(\frac{z+ie}{i})$ belongs to the Bergman-Lorentz $A_\nu(p, q)$ if $\alpha > \frac{\nu+2n-1}{p}$. Indeed we can find positive numbers p_0 and p_1 such that $1 \leq p_0 < p < p_1 \leq \infty$ and $\alpha > \frac{\nu+2n-1}{p_i}$ ($i = 0, 1$). By assertion 2) of Lemma 3.4, the holomorphic function F belongs to $L_\nu^{p_i}$ ($i = 0, 1$). The conclusion follows by assertion a) of Theorem 2.14 and Theorem 2.19.

Remark 3.11. (1) We could not provide examples showing that

$$A_\nu(p, q_0) \neq A_\nu(p, q_1)$$

if $q_0 \neq q_1$. However, in the one-dimensional case $n = r = 1$, $\Omega = (0, \infty)$ (T_Ω is the upper half-plane), it is easy to prove that for $\nu > 0$, $0 < p < \infty$ the function $(z+i)^{-\frac{\nu+1}{p}}$ belongs to $A_\nu(p, \infty)$, but does not belong to $A_\nu(p, p) = A_\nu^p$. In fact, by assertion (2) of Lemma 3.4, the function $(z+i)^{-\beta}$, $\beta \in \mathbb{R}$, belongs to A_ν^p if and only if $\beta > \frac{\nu+1}{p}$.

(2) In section 5 below, we shall show that $A_\nu(p_0, q_0) \neq A_\nu(p_1, q_1)$ unless $p_0 = p_1$, $q_0 = q_1$, in the following two cases:

- (a) $1 < p_0, p_1 < Q_\nu$ and $1 < q_0, q_1 < \infty$;
- (b) $1 + Q_\nu^{-1} < p_0, p_1 < 1 + Q_\nu$ and $1 \leq q_0, q_1 < \infty$.

Lemma 3.12. *Let $\nu \in \mathbb{R}$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ and let $f \in A_\nu(p, q)$. For every compact set K of \mathbb{C}^n contained in T_Ω , there is a positive constant C_K such that*

$$|f(z)| \leq C_K \|f\|_{A_\nu(p,q)} \quad (z \in K).$$

Proof. Suppose first $p = \infty$. The interesting case is $q = \infty$. In this case, $L_\nu(p, q) = L^\infty$ and $A_\nu(p, q) = A^\infty$ is the space of bounded holomorphic functions on (T_Ω, μ) . The relevant result is straightforward.

We next suppose $1 < p < \infty$, $1 \leq q \leq \infty$ and $f \in A_\nu(p, q)$. Since $L_\nu(p, q)$ continuously embeds in $L_\nu(p, \infty)$ (Proposition 2.8) it suffices to show that for each $f \in A_\nu(p, \infty)$, we have

$$|f(z)| \leq C_K \|f\|_{A_\nu(p, \infty)} \quad (z \in K).$$

For a compact set K in \mathbb{C}^n contained in T_Ω , we call ρ the Euclidean distance from K to the boundary of T_Ω . We denote $B(z, \frac{\rho}{2})$ the Euclidean ball centered at z , with radius $\frac{\rho}{2}$. We apply successively

- the mean-value property,
- the fact that the function

$$u + iv \in T_\Omega \mapsto \Delta^{\nu - \frac{n}{r}}(v)$$

is uniformly bounded below on every Euclidean ball $B(z, \frac{\rho}{2})$ when z lies on K and

- the second part of Theorem 2.2,

to obtain that

$$\begin{aligned} |f(z)| &= \frac{1}{|B(z, \frac{\rho}{2})|} \left| \int_{B(z, \frac{\rho}{2})} f(u + iv) dudv \right| \\ &\leq \frac{C}{|B(z, \frac{\rho}{2})|} \int_{B(z, \frac{\rho}{2})} |f(u + iv)| \Delta^{\nu - \frac{n}{r}}(v) dudv \\ &\leq \frac{C}{|B(z, \frac{\rho}{2})|} \int_0^{\mu(B(z, \frac{\rho}{2}))} f^*(t) dt \\ &\leq \frac{C}{|B(z, \frac{\rho}{2})|} \int_0^{\mu(K_\rho)} t^{\frac{1}{p}} f^*(t) t^{\frac{1}{p'}} \frac{dt}{t} \\ &\leq \frac{C \|f\|_{A_\nu(p, \infty)}}{|B(z, \frac{\rho}{2})|} \int_0^{\mu(K_\rho)} t^{\frac{1}{p'}} \frac{dt}{t} \leq C_K \|f\|_{A_\nu(p, \infty)} \end{aligned}$$

for each $z \in K$, with $K_\rho = \bigcup_{z \in K} \overline{B}(z, \frac{\rho}{2})$ and $C_K = Cp' \frac{1}{|B(z, \frac{\rho}{2})|} (\mu(K_\rho))^{\frac{1}{p'}}$. We recall that there is a positive constant C_n such that for all $z \in \mathbb{C}^n$ and $\rho > 0$, we have $|B(z, \rho)| = C_n \rho^{2n}$ and we check easily that $\mu(K_\rho) < \infty$. \square

Proof of Theorem 1.1. (1) We suppose that $\nu \leq \frac{n}{r} - 1$. It suffices to show that $A_\nu(p, \infty) = \{0\}$ for all $1 < p < \infty$. Given $F \in A_\nu(p, \infty)$, we first prove the following lemma.

Lemma 3.13. *Let F be a holomorphic function in T_Ω . Then for general $\nu \in \mathbb{R}$, the following estimate holds.*

$$(3.1) \quad |F(x + iy)| \Delta^{\frac{\nu + \frac{n}{r}}{p}}(y) \leq C_p \|F\|_{A_\nu(p, \infty)} \quad (x + iy \in T_\Omega).$$

Proof of the Lemma. We recall the following inequality [2, Proposition 5.5]:

$$\begin{aligned} |F(x + iy)| &\leq C \int_{d(x+iy, u+iv) < 1} |F(u + iv)| \frac{dudv}{\Delta^{\frac{2n}{r}}(v)} \\ (3.2) \quad &\leq C' \Delta^{-\nu - \frac{n}{r}}(y) \int_{d(x+iy, u+iv) < 1} |F(u + iv)| d\mu(u + iv). \end{aligned}$$

The latter inequality follows by Corollary 3.6. Now by Theorem 2.2, we have

$$\begin{aligned} \int_{d(x+iy, u+iv) < 1} |F(u+iv)| d\mu(u+iv) &\leq \int_0^{\mu(B_{berg}(x+iy, 1))} t^{\frac{1}{p}} F^*(t) t^{\frac{1}{p'}} \frac{dt}{t} \\ (3.3) \qquad \qquad \qquad &\leq p' \|F\|_{A_\nu(p, \infty)} (\mu(B_{berg}(x+iy, 1)))^{\frac{1}{p'}}, \end{aligned}$$

where $B_{berg}(\cdot, \cdot)$ denotes the Bergman ball in T_Ω . By Lemma 3.5 and Corollary 3.6, we obtain that

$$(3.4) \qquad \qquad \qquad \mu(B_{berg}(x+iy, 1)) \simeq \Delta^{\nu + \frac{n}{r}}(y).$$

Then combining (3.2), (3.3) and (3.4) gives the announced estimate (3.1). □

We next deduce that the function

$$z \in T_\Omega \mapsto F(z + i\mathbf{e}) \Delta^{-\alpha}(z + i\mathbf{e})$$

belongs to the Bergman space A_ν^1 when α is sufficiently large. We distinguish two cases:

- (1) $\nu \leq -\frac{n}{r}$;
- (2) $-\frac{n}{r} < \nu \leq \frac{n}{r} - 1$.

Case 1). We suppose that $\nu \leq -\frac{n}{r}$. We take $\alpha > \frac{-\nu - \frac{n}{r}}{p}$ and we apply assertion (1) of Lemma 3.4

$$\begin{aligned} &|F(x+iy) \Delta^{-\alpha}(x+iy)| \\ &= |F(x+iy)| \left| \Delta^{-\alpha - \frac{\nu + \frac{n}{r}}{p}}(x+iy) \right| \left| \Delta^{\frac{\nu + \frac{n}{r}}{p}}(x+iy) \right| \\ &\leq |F(x+iy)| \left| \Delta^{-\alpha - \frac{\nu + \frac{n}{r}}{p}}(x+iy) \right| \left| \Delta^{\frac{\nu + \frac{n}{r}}{p}}(y) \right| \\ &\leq C_p \|F\|_{A_\nu(p, \infty)} \left| \Delta^{-\alpha - \frac{\nu + \frac{n}{r}}{p}}(x + i(y + \mathbf{e})) \right|. \end{aligned}$$

For the latter inequality, we applied estimate (3.1) of Lemma 3.13. The conclusion follows because by assertion (2) of Lemma 3.4, the function $\Delta^{-\alpha - \frac{\nu + \frac{n}{r}}{p}}(x+iy)$ is integrable on T_Ω when α is sufficiently large.

Case 2). We suppose that $-\frac{n}{r} < \nu \leq \frac{n}{r} - 1$. Since $\nu + \frac{n}{r} > 0$ and $\Delta(y + \mathbf{e}) \geq \Delta(\mathbf{e}) = 1$ by Lemma 3.1, it follows from (3.1) that the function $z \in T_\Omega \mapsto F(z + i\mathbf{e})$ is bounded on T_Ω . The conclusion easily follows.

Finally we remind that $A_\nu^1 = \{0\}$ if $\nu \leq \frac{n}{r} - 1$ (cf. e.g. [2, Proposition 3.8]). We conclude that the function $F(\cdot + i\mathbf{e})$ vanishes identically on T_Ω . An application of the analytic continuation principle then implies the identity $F \equiv 0$ on T_Ω .

(2) We suppose that $\nu > \frac{n}{r} - 1$. It suffices to show that $A_\nu(p, q)$ is a closed subspace of the Banach space $(L_\nu(p, q), \|\cdot\|_{(p,q)})$. For $p = \infty$, the interesting case is $q = \infty$ and then $A_\nu(p, q) = A_\nu^\infty$; the relevant result is easy to obtain. We next suppose that $1 < p < \infty$ and $1 \leq q \leq \infty$. In view of Lemma 3.12, every Cauchy sequence $\{f_m\}_{m=1}^\infty$ in $(A_\nu(p, q), \|\cdot\|_{A_\nu(p,q)})$ converges to a holomorphic function $f : T_\Omega \rightarrow \mathbb{C}$ on compact sets in \mathbb{C}^n contained in T_Ω . On the other hand, since the sequence $\{f_m\}_{m=1}^\infty$ is a Cauchy sequence in the Banach space $(L_\nu(p, q), \|\cdot\|_{A_\nu(p,q)})$, it converges with respect to the $L_\nu(p, q)$ -norm to a function $g \in L_\nu(p, q)$. Now by Proposition 2.6, this sequence contains a subsequence $\{f_{m_k}\}_{k=1}^\infty$ which converges μ -a.e. to g . The uniqueness of the limit implies that $f = g$ a.e. We have proved that the Cauchy sequence $\{f_m\}$ in $(A_\nu(p, q), \|\cdot\|_{A_\nu(p,q)})$ converges in $(A_\nu(p, q), \|\cdot\|_{A_\nu(p,q)})$ to the function f . \square

4. Density in Bergman-Lorentz spaces. Proof of Theorem 1.2

4.1. Density in Bergman-Lorentz spaces. We adopt the following notation given in the introduction:

$$Q_\nu = 1 + \frac{\nu}{\frac{n}{r} - 1}.$$

We shall refer to the following result. For its proof, consult [18, Corollary 3.7] and [2, Theorem 4.23].

Theorem 4.1. *Let $\nu > \frac{n}{r} - 1$.*

- (1) *The weighted Bergman projector P_γ , $\gamma > \nu + \frac{n}{r} - 1$ (resp. $\gamma \geq \nu + \frac{n}{r} - 1$) extends to a bounded operator from L_ν^p to A_ν^p for all $1 \leq p < Q_\nu$ (resp. $1 < p < Q_\nu$).*
- (2) *The weighted Bergman projector P_ν extends to a bounded operator from L_ν^p to A_ν^p for all $1 + Q_\nu^{-1} < p < 1 + Q_\nu$.*

The following corollary follows from a combination of Theorem 4.1, Theorem 2.13 and Theorem 2.14.

Corollary 4.2. *Let $\nu > \frac{n}{r} - 1$. The weighted Bergman projector P_γ , $\gamma \geq \nu + \frac{n}{r} - 1$ (resp. the Bergman projector P_ν) extends to a bounded operator from $L_\nu(p, q)$ to $A_\nu(p, q)$ for all $1 < p < Q_\nu$ (resp. for all $1 + Q_\nu^{-1} < p < 1 + Q_\nu$) and $1 \leq q \leq \infty$.*

The following proposition was proved in [2, Theorem 3.23].

Proposition 4.3. *We suppose that $\nu, \gamma > \frac{n}{r} - 1$. Let $1 \leq p, t < \infty$. The subspace $A_\nu^p \cap A_\gamma^t$ is dense in the Banach space A_ν^p .*

Remark 4.4. If the weighted Bergman projector P_γ extends to a bounded operator on L_ν^p and if P_γ is the identity on A_γ^t , then P_γ is the identity on A_ν^p .

The next corollary is a consequence of Theorem 4.1, Proposition 4.3 and Remark 4.4.

Corollary 4.5. *Let $\nu > \frac{n}{r} - 1$.*

- (1) *For all $p \in (1+Q_\nu^{-1}, 1+Q_\nu)$, the Bergman projector P_ν is the identity on A_ν^p .*
- (2) *For all $\gamma > \nu + \frac{n}{r} - 1$ (resp. $\gamma \geq \nu + \frac{n}{r} - 1$) and for all $1 \leq p < Q_\nu$ (resp. $1 < p < Q_\nu$), the Bergman projector P_γ is the identity on A_ν^p .*

Proof. (1) Take $t = 2$ and $\gamma = \nu$ in Proposition 4.3. Then apply assertion (2) of Theorem 4.1 and Remark 4.4.

- (2) Take $t = 2$ in Proposition 4.3. Then apply assertion (1) of Theorem 4.1 and Remark 4.4.

□

We shall prove the following density result for Bergman-Lorentz spaces.

Proposition 4.6. *We suppose that $\gamma, \nu > \frac{n}{r} - 1$, $1 \leq p, t < \infty$ and $1 \leq q < \infty$. The subspace $A_\nu(p, q) \cap A_\gamma^t$ is dense in the (quasi-)Banach space $A_\nu(p, q)$ in the following three cases.*

- (1) $p = q$;
- (2) $\gamma = \nu > \frac{n}{r} - 1$, $1 + Q_\nu^{-1} < p, t < 1 + Q_\nu$ and $1 \leq q < p$;
- (3) $1 < p < Q_\nu$, $1 < t < Q_\gamma$ and $1 \leq q < p$;
- (4) $\gamma \geq \nu + \frac{n}{r} - 1$, $1 < p < Q_\nu$, $1 + Q_\gamma^{-1} < t < 1 + Q_\gamma$ and $1 \leq q < p$;
- (5) $\gamma \geq \nu$ and $p, q \in (t, \infty)$.

Proof. (1) For $p = q$, $A_\nu(p, p) = A_\nu^p$, the result is known, cf. e.g. [2, Theorem 3.23].

- (2) We suppose now that $\gamma = \nu > \frac{n}{r} - 1$, $1 + Q_\nu^{-1} < p, t < 1 + Q_\nu$ and $1 \leq q < p$. Given $F \in A_\nu(p, q)$, by Corollary 3.8, there exists a sequence $\{f_m\}_{m=1}^\infty$ of \mathcal{C}^∞ functions with compact support on T_Ω such that $\{f_m\}_{m=1}^\infty \rightarrow F$ ($m \rightarrow \infty$) in $L_\nu(p, q)$. Each f_m belongs to $L_\nu^t \cap L_\nu(p, q)$. By Corollary 4.2 and Theorem 4.1, the Bergman projector P_ν extends to a bounded operator on $L_\nu(p, q)$ and on L_ν^t respectively. So $P_\nu f_m \in A_\nu(p, q) \cap A_\nu^t$ and $\{P_\nu f_m\}_{m=1}^\infty \rightarrow P_\nu F$ ($m \rightarrow \infty$) in $A_\nu(p, q)$. Notice that $A_\nu(p, q) \subset A_\nu^p$ because $q < p$. By assertion (1) of Corollary 4.5, we obtain that $P_\nu F = F$. This finishes the proof of assertion (2).

- (3) We suppose that $1 < p < Q_\nu$, $1 < t < Q_\gamma$ and $1 \leq q < p$. Given $F \in A_\nu(p, q)$, by Corollary 3.8, there exists a sequence $\{f_m\}_{m=1}^\infty$ of \mathcal{C}^∞ functions with compact support on T_Ω such that $\{f_m\}_{m=1}^\infty \rightarrow F$ ($m \rightarrow \infty$) in $L_\nu(p, q)$. Each f_m belongs to $L_\nu^t \cap L_\nu(p, q)$. By Corollary 4.2 and Theorem 4.1, for all $s > \max\{\nu + \frac{n}{r} - 1, \gamma + \frac{n}{r} - 1\}$, the Bergman projector P_s extends to a bounded operator on $L_\nu(p, q)$ and on L_ν^t respectively. So $P_s f_m \in A_\nu(p, q) \cap A_\nu^t$ and $\{P_s f_m\}_{m=1}^\infty \rightarrow P_s F$ ($m \rightarrow \infty$) in $A_\nu(p, q)$. Notice that $A_\nu(p, q) \subset A_\nu^p$ because $q < p$.

By assertion (2) of Corollary 4.5 we obtain that $P_s F = F$. This finishes the proof of assertion (3).

- (4) Replace P_s in the previous case by P_γ .
- (5) Let l be a bounded linear functional on $A_\nu(p, q)$ such that $l(F) = 0$ for all $F \in A_\nu(p, q) \cap A_\gamma^t$. We must show that $l \equiv 0$ on $A_\nu(p, q)$.

We first prove that the holomorphic function $F_{m,\alpha}$ defined on T_Ω by

$$F_{m,\alpha}(z) = \Delta^{-\alpha} \left(\frac{z + i\mathbf{e}}{i} \right) F(z)$$

belongs to $A_\nu(p, q) \cap A_\gamma^t$ when $\alpha > 0$ is sufficiently large. By Lemma 3.4 (assertion (1)) and Lemma 3.1, we have

$$\left| \Delta^{-\alpha} \left(\frac{z + i\mathbf{e}}{i} \right) \right| \leq \Delta^{-\alpha}(\mathbf{e}) = 1.$$

So $|F_{m,\alpha}| \leq |F|$; this implies that $F_{m,\alpha} \in A_\nu(p, q)$.

We next show that $F_{m,\alpha} \in A_\gamma^t$ when α is large. We obtain that

$$\begin{aligned} I &:= \int_{T_\Omega} \left| \Delta^{-\alpha} \left(\frac{x+iy+i\mathbf{e}}{i} \right) F(x+iy) \right|^t \Delta^{\gamma-\frac{n}{r}}(y) dx dy \\ &= \int_{T_\Omega} \left| \Delta^{-\alpha} \left(\frac{x+iy+i\mathbf{e}}{i} \right) F(x+iy) \right|^t \Delta^{\gamma-\nu}(y) d\mu(x+iy). \end{aligned}$$

Since $\gamma \geq \nu$, by Lemma 3.4 (assertion (1)) and Lemma 3.1, we obtain that

$$I \leq C_{m,\gamma,\nu} \int_{T_\Omega} \left| \Delta^{-\alpha t + \gamma - \nu} \left(\frac{x+iy+i\mathbf{e}}{i} \right) \right| |F(x+iy)|^t d\mu(x+iy).$$

Observing that $(|f|^t)^* = (f^*)^t$, we notice that $|F|^t \in L_\nu(\frac{p}{t}, \frac{q}{t})$. By Theorem 2.7, it suffices to show that the function $z \in T_\Omega \mapsto \Delta^{-\alpha t + \gamma - \nu} \left(\frac{x+iy+i\mathbf{e}}{i} \right)$ belongs to $L_\nu((\frac{p}{t})', (\frac{q}{t})')$ when α is large. The desired conclusion follows by Example 3.10.

So our assumption implies that

$$(4.1) \quad l(F_{m,\alpha}) = 0.$$

By the Hahn-Banach theorem, there exists a bounded linear functional \tilde{l} on $L_\nu(p, q)$ such that $\tilde{l}|_{A_\nu(p, q)} = l$ and the operator norms $\|\tilde{l}\|$ and $\|l\|$ coincide. Furthermore, by Theorem 2.7, there exists a function $\varphi \in L_\nu(p', q')$ such that

$$\tilde{l}(f) = \int_{T_\Omega} f(z)\varphi(z)d\mu(z) \quad \forall f \in L_\nu(p, q).$$

We must show that

$$(4.2) \quad \int_{T_\Omega} F(z)\varphi(z)d\mu(z) = 0 \quad \forall F \in A_\nu(p, q).$$

The equation (4.1) can be expressed in the form

$$\int_{T_\Omega} F_{m,\alpha}(z)\varphi(z)d\mu(z) = \int_{T_\Omega} \Delta^{-\alpha} \left(\frac{z + i\mathbf{e}}{i} \right) F(z)\varphi(z)d\mu(z) = 0.$$

Again by Theorem 2.7, the function $F\varphi$ is integrable on T_Ω since $F \in L_\nu(p, q)$ and $\varphi \in L_\nu(p', q')$. We also have

$$\left| \Delta^{-\alpha} \left(\frac{z + i\mathbf{e}}{i} \right) \right| \leq 1 \quad \forall z \in T_\Omega, \quad m = 1, 2, \dots$$

An application of the Lebesgue dominated theorem next gives the announced conclusion (4.2). □

We next deduce the following corollary.

Corollary 4.7. *Let $\nu > \frac{n}{r} - 1$.*

- (1) *For all $1 < p < Q_\nu$ and $1 < q < \infty$, and for every real index γ such that $\gamma \geq \nu + \frac{n}{r} - 1$ and $\gamma > \left(\frac{1}{\min(p,q)-1} - 1 \right) \left(\frac{n}{r} - 1 \right)$, the Bergman projector P_γ is the identity on $A_\nu(p, q)$.*
- (2) *For all $p \in (1 + Q_\nu^{-1}, 1 + Q_\nu)$ and $1 \leq q < \infty$, the Bergman projector P_ν is the identity on $A_\nu(p, q)$.*

Proof. (1) By assertion (1) of Corollary 4.5, for all $t \in (1 + Q_\nu^{-1}, 1 + Q_\nu)$, P_γ is the identity on A_ν^t . Next let $1 < p < Q_\nu$ and $1 < q < \infty$. Let the real index γ be such that $\gamma \geq \nu + \frac{n}{r} - 1$ and $1 + Q_\nu^{-1} < \min(p, q)$. We take t such that $1 + Q_\nu^{-1} < t < \min(p, q)$. It follows from the assertion (5) of Proposition 4.6 that the subspace $A_\nu(p, q) \cap A_\nu^t$ is dense in the Banach space $A_\nu(p, q)$. But by Corollary 4.2, P_γ extends to a bounded operator on $L_\nu(p, q)$. We conclude then that P_γ is the identity on $A_\nu(p, q)$.

- (2) By assertion (1) of Corollary 4.5, for all $t \in (1 + Q_\nu^{-1}, 1 + Q_\nu)$, P_ν is the identity on A_ν^t . Next let $1 + Q_\nu^{-1} < p < 1 + Q_\nu$. By Corollary 4.2, P_ν extends to a bounded operator on $L_\nu(p, q)$. It then suffices to show that the subspace $A_\nu(p, q) \cap A_\nu^t$ is dense in the Banach space $A_\nu(p, q)$ for some $t \in (1 + Q_\nu^{-1}, 1 + Q_\nu)$. If $1 + Q_\nu^{-1} < q < \infty$, we take t such that $1 + Q_\nu^{-1} < t < \min(p, q)$: the conclusion follows by assertion (5) of Proposition 4.6. Otherwise, if $1 \leq q \leq 1 + Q_\nu^{-1}$, by assertion (2) of the same proposition, for all $1 + Q_\nu^{-1} < p, t < 1 + Q_\nu$, the conclusion follows. □

4.2. Proof of Theorem 1.2. Combine Corollary 4.2 with Corollary 4.7.

The following corollary lifts the condition $q < p$ in assertions (2), (3) and (4) of Proposition 4.6. In fact, it gives sufficient conditions for density of $A_\nu(p, q) \cap A_\nu^t$ in $A_\nu(p, q)$ when $q > p$.

Corollary 4.8. *Let $\nu > \frac{n}{r} - 1$ and $1 < p < q < \infty$. The subspace $A_\nu(p, q) \cap A_\gamma^t$ is dense in the Banach space $A_\nu(p, q)$ in the following three cases.*

- (1) $\gamma = \nu$, $1 + Q_\nu^{-1} < p, t < 1 + Q_\nu$;
- (2) $\gamma \geq \nu + \frac{n}{r} - 1, \gamma > \frac{2-p}{p-1}(\frac{n}{r} - 1)$, $1 < p < Q_\nu, 1 < t < Q_\gamma$;
- (3) $\gamma \geq \nu + \frac{n}{r} - 1, \gamma > \frac{2-p}{p-1}(\frac{n}{r} - 1)$, $1 < p < Q_\nu, 1 + Q_\gamma^{-1} < t < 1 + Q_\gamma$.

Proof. We resume the proof of assertion (2) (resp. assertions (3) and (4)) of Proposition 4.6 up to the equality $P_\gamma F = F$ (resp. $P_\nu F = F$). In the proof of Proposition 4.6, this equality followed from the inclusion $A_\nu(p, q) \subset A_\nu^p$ (since $q < p$) and assertion (1) (resp. assertion (2)) of Corollary 4.5. Here we use assertion (1) (resp. assertion (2)) of Corollary 4.7. \square

5. Real interpolation between Bergman-Lorentz spaces.

Proof of Theorem 1.3.

In this section, we prove Theorem 1.3. In the sequel, we write

$$L_0 = L_\nu^{p_0} \text{ (resp. } L_0 = L_\nu(p_0, q_0)), L_1 = L_\nu(p_1, q_1), \bar{L} = L_0 + L_1$$

and

$$A_0 = A_\nu^{p_0} \text{ (resp. } A_0 = A_\nu(p_0, q_0)), A_1 = A_\nu(p_1, q_1), \bar{A} = A_0 + A_1.$$

We shall use the following lemma.

Lemma 5.1. *Let $0 < \theta < 1$, $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1, q \leq \infty$. Define the exponent p by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. We have the identification with equivalence of (quasi-)norms:*

$$(5.1) \quad (L_0, L_1)_{\theta, q} \cap (A_0 + A_1) = A_\nu(p, q) \cap (A_0 + A_1).$$

Moreover, the identity $A_\nu(p, q) \cap (A_0 + A_1) = A_\nu(p, q)$ holds if there exists $\gamma > \frac{n}{r} - 1$ such that P_γ is the identity on $A_\nu(p, q)$ and extends to a bounded operator on L_i , $i = 0, 1$.

The space on the left side of (5.1) is equipped with the (quasi-)norm induced by the real interpolation space $(L_0, L_1)_{\theta, q}$ and the space on the right side is equipped with the (quasi-)norm induced by the Lorentz space $L_\nu(p, q)$.

Proof of the lemma. We have

$$\begin{aligned} (L_0, L_1)_{\theta, q} \cap (A_0 + A_1) &= (L_0, L_1)_{\theta, q} \cap ((A_0 + A_1) \cap Hol(T_\Omega)) \\ &= \left((L_0, L_1)_{\theta, q} \cap Hol(T_\Omega) \right) \cap (A_0 + A_1) \\ &= (L_\nu(p, q) \cap Hol(T_\Omega)) \cap (A_0 + A_1) \\ &= A_\nu(p, q) \cap (A_0 + A_1), \end{aligned}$$

where the third equality follows from assertion b) of Theorem 2.13.

For the second assertion, suppose that there exists $\gamma > \frac{n}{r} - 1$ such that P_γ is the identity on $A_\nu(p, q)$ and extends to a bounded operator on L_i , $i = 0, 1$. Then $P_\gamma F = F$ for all $F \in A_\nu(p, q)$. Now, since $L_\nu(p, q) \subset L_0 + L_1$, there exist

$a_i \in L_i$ ($i = 0, 1$) such that $F = a_0 + a_1$. Then $F = P_\gamma F = P_\gamma a_0 + P_\gamma a_1$ with $P_\gamma a_i \in A_i$ ($i = 0, 1$). This shows that $A_\nu(p, q) \subset A_0 + A_1$: the conclusion follows. \square

We next prove assertions (1) and (2) of Theorem 1.3. In view of the previous lemma, it follows from the hypotheses and from Theorem 4.1, Corollary 4.5, Corollary 4.2 and Corollary 4.7 that it suffices to prove the identity

$$(A_0, A_1)_{\theta, q} = (L_0, L_1)_{\theta, q} \cap (A_0 + A_1),$$

with equivalence of norms. We shall prove this identity for all $1 \leq q \leq \infty$. More precisely, we prove the following theorem.

Theorem 5.2. *Let $\nu > \frac{n}{r}$, $1 \leq p_0 < p_1 < \infty$, $1 \leq q_0, q_1, q \leq \infty$ and $0 < \theta < 1$. Suppose that there exists $\gamma > \frac{n}{r} - 1$ such that P_γ is the identity on $A_\nu(p, q)$ (resp. on $A_0 + A_1$) and extends to a bounded operator on L_i , $i = 0, 1$. Then if $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, we have*

$$(5.2) \quad (A_0, A_1)_{\theta, q} = (L_0, L_1)_{\theta, q} \cap (A_0 + A_1) = A_\nu(p, q) \cap (A_0 + A_1) = A_\nu(p, q).$$

Proof of Theorem 5.2. The second identity of (5.2) was given by Lemma 5.1. By Definition 2.10 (the definition of real interpolation spaces), we have at once

$$(A_0, A_1)_{\theta, q} \hookrightarrow (L_0, L_1)_{\theta, q}$$

with $1 \leq q \leq \infty$. Since $(A_0, A_1)_{\theta, q} \subset A_0 + A_1$, we conclude that

$$(A_0, A_1)_{\theta, q} \hookrightarrow (L_0, L_1)_{\theta, q} \cap (A_0 + A_1).$$

We next show the converse, i.e. $(L_0, L_1)_{\theta, q} \cap (A_0 + A_1) \hookrightarrow (A_0, A_1)_{\theta, q}$. We must show that there exists a positive constant C such that

$$(5.3) \quad \|F\|_{(A_0, A_1)_{\theta, q}} \leq C \|F\|_{(L_0, L_1)_{\theta, q}} \quad \forall F \in (L_0, L_1)_{\theta, q} \cap (A_0 + A_1).$$

We recall that, given a compatible couple (X_0, X_1) of (quasi-)Banach spaces, if we write $\bar{X} = X_0 + X_1$, we have

$$(5.4) \quad \|F\|_{(X_0, X_1)_{\theta, q}} = \left(\int_0^\infty \left(t^{-\theta} K(t, F, \bar{X}) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

if $q < \infty$ (resp.

$$(5.5) \quad \|F\|_{(X_0, X_1)_{\theta, \infty}} = \sup_{t>0} t^{-\theta} K(t, F, \bar{X})$$

if $q = \infty$) with

$$K(t, F, \bar{X}) = \inf \{ \|a_0\|_{X_0} + t \|a_1\|_{X_1} : F = a_0 + a_1, a_0 \in X_0, a_1 \in X_1 \}.$$

When we compare (5.4) (resp. (5.5)) for $\bar{X} = \bar{A}$ and $\bar{X} = \bar{L}$, the estimate (5.3) will follow if we can prove that

$$K(t, F, \bar{A}) \leq CK(t, F, \bar{L}) \quad (F \in A_\nu(p, q) \text{ (resp. } F \in A_0 \cap A_1)).$$

We have $F = P_\gamma F$ and P_γ extends to a bounded operator from L_i to A_i , $i = 0, 1$. So we obtain the inclusion

$$\{(a_0, a_1) \in A_0 \times A_1 : F = a_0 + a_1\} \\ \supset \{(P_\gamma a_0, P_\gamma a_1) : (a_0, a_1) \in L_0 \times L_1, F = a_0 + a_1\},$$

which implies that

$$K(t, F, \bar{A}) \\ \leq \inf \{ \|P_\gamma a_0\|_{L_0} + t \|P_\gamma a_1\|_{L_1} : F = a_0 + a_1, a_0 \in L_0, a_1 \in L_1 \} \\ \leq \inf \{ \|P_\gamma\|_0 \|a_0\|_{L_0} + t \|P_\gamma\|_1 \|a_1\|_{L_1} : F = a_0 + a_1, a_0 \in L_0, a_1 \in L_1 \}$$

where $\|\cdot\|_i$ denotes the operator norm on L_i ($i = 0, 1$). We have shown that

$$K(t, F, \bar{A}) \leq \left(\max_{i=0,1} \|P_\gamma\|_i \right) K(t, F, \bar{L}).$$

Remind that $\max_{i=0,1} \|P_\gamma\|_i < \infty$. This completes the proof. □

We now prove assertions (3) and (4) of Theorem 1.3. We combine the Wolff reiteration theorem (Theorem 2.20) with the first assertion of the theorem. Here $X_1 = A_\nu^1$ (resp. $X_1 = A_\nu(p_0, q_0)$) and $X_4 = A_\nu(p_1, q_1)$. We take $X_2 = A_\nu(p_2, r)$ and $X_3 = A_\nu(p_3, q)$ with $1 + Q_\nu^{-1} < p_2 < p_3 < Q_\nu, 1 \leq r < \infty$ and $1 \leq q < \infty$. By assertion (1) of the theorem, we have

$$X_2 = (X_1, X_3)_{\varphi, r} \quad \text{with} \quad \frac{1}{p_2} = \frac{1 - \varphi}{p_0} + \frac{\varphi}{p_3}$$

and

$$X_3 = (X_2, X_4)_{\psi, q} \quad \text{with} \quad \frac{1}{p_3} = \frac{1 - \psi}{p_2} + \frac{\psi}{p_1}.$$

By Theorem 2.20, we conclude that

$$X_2 = (X_1, X_4)_{\rho, r}, \quad X_3 = (X_1, X_4)_{\theta, q}$$

with

$$\rho = \frac{\varphi\psi}{1 - \varphi + \varphi\psi}, \quad \theta = \frac{\psi}{1 - \varphi + \varphi\psi}.$$

In other words, in the present context, we have

$$A_\nu(p_2, r) = (A_\nu^1, A_\nu(p_1, q_1))_{\rho, r}, \quad A_\nu(p_3, q) = (A_\nu^1, A_\nu(p_1, q_1))_{\theta, q}$$

(resp.

$$A_\nu(p_2, r) = (A_\nu(p_0, q_0), A_\nu(p_1, q_1))_{\rho, r}, \quad A_\nu(p_3, q) = (A_\nu(p_0, q_0), A_\nu(p_1, q_1))_{\theta, q})$$

with

$$\rho = \frac{\varphi\psi}{1 - \varphi + \varphi\psi}, \quad \theta = \frac{\psi}{1 - \varphi + \varphi\psi}.$$

To conclude, given $1 + Q_\nu^{-1} < p < Q_\nu$, take for instance $p_3 = p$.

Remark 5.3. Theorem 5.2 has a more general form. Let $\overline{X} = (X_0, X_1)$ be a compatible couple of (quasi-)Banach spaces and let $\overline{Y} = (Y_0, Y_1)$ be a sub-couple of (X_0, X_1) , i.e. each Y_i is a closed subspace of X_i for each $i = 0, 1$. It is known [15, 17] that for all $0 < \theta < 1, 1 \leq q \leq \infty$, we have

$$(Y_0, Y_1)_{\theta, q} = (X_0, X_1)_{\theta, q} \cap (Y_0 + Y_1)$$

if there exists a positive constant C such that

$$K(t, a, \overline{Y}) \leq CK(t, a, \overline{X}) \quad \forall t > 0 \quad \forall a \in Y_0 + Y_1.$$

In [15], such a subcouple \overline{Y} is called a K -subcouple of \overline{X} .

In this vein, the following corollary is a consequence of Theorem 5.2.

Corollary 5.4. *Let $\nu > \frac{n}{r} - 1$.*

- (1) *For all $1 < p_1 < Q_\nu, 1 < q_1 < \infty$ and $0 < \theta < 1$, the real interpolation space*

$$(A_\nu^1, A_\nu(p_1, q_1))_{\theta, q}$$

identifies with

$$A_\nu(p, q) \cap (A_\nu^1 + A_\nu(p_1, q_1)), \quad \frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}, \quad 1 \leq q \leq \infty$$

with equivalence of norms.

- (2) *For all $1 < p_0 < p_1 < Q_\nu, 1 < q_0, q_1 < \infty$ (resp. $1 + Q_\nu^{-1} < p_0 < p_1 < 1 + Q_\nu, 1 \leq q_0, q_1 < \infty$) the real interpolation space*

$$(A_\nu(p_0, q_0), A_\nu(p_1, q_1))_{\theta, q}$$

identifies with

$$A_\nu(p, q) \cap (A_\nu(p_0, q_0) + A_\nu(p_1, q_1)), \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1}, \quad 1 \leq q \leq \infty$$

with equivalence of norms.

Proof. We apply Theorem 5.2 after the following two remarks.

- (1) By assertion (2) of Corollary 4.5 and assertion (1) of Corollary 4.7 respectively, for $\gamma > \nu + \frac{n}{r} - 1$ sufficiently large, the weighted Bergman projector P_γ is the identity on A_ν^1 and $A_\nu(p_1, q_1)$. Hence P_γ is the identity on $A_\nu^1 + A_\nu(p_1, q_1)$. Next, by assertion (1) of Theorem 4.1 and Corollary 4.2 respectively, P_γ also extends to a bounded operator on L_ν^1 and $L_\nu(p_1, q_1)$.
- (2) By assertion (1) (resp. assertion (2)) of Corollary 4.7, for $\gamma > \nu + \frac{n}{r} - 1$ sufficiently large, the weighted Bergman projector P_γ (resp. the weighted Bergman projector P_ν) is the identity on $A_\nu(p_0, q_0)$ and $A_\nu(p_1, q_1)$. Hence P_γ is the identity on $A_\nu(p_0, q_0) + A_\nu(p_1, q_1)$. Next, by Corollary 4.2, P_γ (resp. P_ν) also extends to a bounded operator on $L_\nu(p_0, q_0)$ and $L_\nu(p_1, q_1)$.

□

As announced in assertion (2) of Remark 3.11, we next prove the following theorem.

Theorem 5.5. *Let $\nu > \frac{n}{r} - 1$. Then $A_\nu(p_0, q_0) \neq A_\nu(p_1, q_1)$ unless $p_0 = p_1, q_0 = q_1$, in the following two cases:*

- (a) $1 < p_0, p_1 < Q_\nu$ and $1 < q_0, q_1 < \infty$;
- (b) $1 + Q_\nu^{-1} < p_0, p_1 < 1 + Q_\nu$ and $1 \leq q_0, q_1 < \infty$.

Moreover, $A_\nu(p, q)$ is strictly contained in $A_\nu(p, \infty)$ in the following two cases:

- (a) $1 < p < Q_\nu$ and $1 < q < \infty$;
- (b) $1 + Q_\nu^{-1} < p < 1 + Q_\nu$ and $1 \leq q < \infty$.

Proof. We rely on the following theorem.

Theorem 5.6. [16, Theorem 2 of section 3] *Let (A_0, A_1) be a compatible couple of Banach spaces such that $A_0 \cap A_1$ is not closed in $A_0 + A_1$. Let $0 < \theta, \eta < 1, 1 \leq p, q \leq \infty$. Then $(A_0, A_1)_{\theta, p} \neq (A_0, A_1)_{\eta, q}$ unless $\theta = \eta, p = q$.*

Let us prove the first assertion of Theorem 5.5. According to assertion (2) of Theorem 1.3, for all $p^{(0)}$ and $p^{(1)}$ satisfying $1 < p^{(0)} < p_0, p_1 < p^{(1)} < Q_\nu$ in case (a) (resp. $1 + Q_\nu^{-1} < p^{(0)} < p_0, p_1 < p^{(1)} < 1 + Q_\nu$ in case (b)), there are $0 < \theta, \eta < 1$ such that $A_\nu(p_0, q_0) = (A_\nu^{p^{(0)}}, A_\nu^{p^{(1)}})_{\theta, q_0}$ and $A_\nu(p_1, q_1) = (A_\nu^{p^{(0)}}, A_\nu^{p^{(1)}})_{\eta, q_1}$. By Theorem 5.6, it suffices to show that the subspace $A_\nu^{p^{(0)}} \cap A_\nu^{p^{(1)}}$ is not closed in the space $A_\nu^{p^{(0)}} + A_\nu^{p^{(1)}}$ when $1 < p^{(0)} < p^{(1)} < \infty$. In view of Proposition 4.3, we recall that the subspace $A_\nu^{p^{(0)}} \cap A_\nu^{p^{(1)}}$ is dense in the space $A_\nu^{p^{(1)}}$, and in view of Example 3.10, we point out that $A_\nu^{p^{(0)}} \cap A_\nu^{p^{(1)}}$ is strictly contained in $A_\nu^{p^{(1)}}$. Now let $F \in A_\nu^{p^{(1)}} \setminus \{A_\nu^{p^{(0)}} \cap A_\nu^{p^{(1)}}\}$. There exists a sequence $\{F_n\}_{n=1}^\infty$ of elements of $A_\nu^{p^{(0)}} \cap A_\nu^{p^{(1)}}$ which converges to F in $A_\nu^{p^{(1)}}$. Moreover, since

$$\|F_n - F\|_{A_\nu^{p^{(0)}} + A_\nu^{p^{(1)}}} \leq \|F_n - F\|_{A_\nu^{p^{(1)}}},$$

we also have the convergence $\{F_n\}_{n=1}^\infty \rightarrow F$ in $A_\nu^{p^{(0)}} + A_\nu^{p^{(1)}}$. So F belongs to the closure of $A_\nu^{p^{(0)}} \cap A_\nu^{p^{(1)}}$ in $A_\nu^{p^{(0)}} + A_\nu^{p^{(1)}}$ and does not belong to $A_\nu^{p^{(0)}} \cap A_\nu^{p^{(1)}}$: we have reached the conclusion that $A_\nu^{p^{(0)}} \cap A_\nu^{p^{(1)}}$ is not closed in $A_\nu^{p^{(0)}} + A_\nu^{p^{(1)}}$.

We next prove the second assertion of Theorem 5.5. It follows from Proposition 2.8 that $A_\nu(p, q) \subset A_\nu(p, q')$ for all $1 \leq p < \infty, 1 \leq q < q' \leq \infty$. Hence, in view of the first part of the theorem, we obtain that $A_\nu(p, q) \subsetneq A_\nu(p, q')$ if $1 < p < Q_\nu, 1 < q < q' < \infty$ (resp. $1 + Q_\nu^{-1} < p < 1 + Q_\nu, 1 \leq q < q' < \infty$). In both cases, we conclude that $A_\nu(p, q) \subsetneq A_\nu(p, q') \subset A_\nu(p, \infty)$. \square

Remark 5.7. (1) Theorem 5.5 takes the following form in the one-dimensional case, $n = r = 1, \nu > 0$. For all $1 < p_0, p_1 < \infty, 1 \leq$

$q_0, q_1 < \infty$, the property $A_\nu(p_0, q_0) \neq A_\nu(p_1, q_1)$ holds unless $p_0 = p_1, q_0 = q_1$. Moreover, $A_\nu(p, q) \subsetneq A_\nu(p, \infty)$ for all $1 < p < \infty$ and $1 \leq q < \infty$.

(2) Still in the one-dimensional case, Theorem 1.3 gives the following corollary.

Corollary 5.8 ($n = r = 1$). *Let $\nu > 0$. For all $0 < \theta < 1, 1 \leq q < \infty$, the real interpolation space*

$$(A_\nu(p_0, q_0), A_\nu(p_1, q_1))_{\theta, q}$$

identifies with $A_\nu(p, q)$ with equivalence of norms in the following two cases.

- (1) $p_0 = q_0 = 1, 1 < p_1 < \infty, 1 \leq q_1 \leq \infty$;
- (2) $1 < p_0 < p_1 < \infty, 1 \leq q_0, q_1 \leq \infty$.

6. Back to the density in Bergman-Lorentz spaces.

In this section, we shall prove the following more general result than Proposition 4.6 and Corollary 4.8.

Theorem 6.1. *Let $\gamma, \nu > \frac{n}{r} - 1$. Then the subspace $A_\nu(p, q) \cap A_\gamma^t$ is dense in the Banach space $A_\nu(p, q)$ in the following two cases:*

- (1) $1 < p < Q_\nu, 1 \leq t < \infty, 1 < q < \infty$;
- (2) $1 + Q_\nu^{-1} < p < 1 + Q_\nu, 1 \leq t < \infty, 1 \leq q < \infty$.

Proof. According to Proposition 4.3, the subspace $A_\nu^p \cap A_\gamma^t$ is dense in the Banach space A_ν^p for all $p, t \in [1, \infty)$.

Let $F \in A_\nu(p, q)$. Given $\alpha > 0$ and $m = 1, 2, \dots$, let

$$F_{m,\alpha}(z) = F\left(z + \frac{i\mathbf{e}}{m}\right) \Delta^{-\alpha}\left(\frac{\frac{z}{m} + i\mathbf{e}}{i}\right).$$

We claim that

- 1. $F_{m,\alpha} \in A_\nu(p, q)$ with $\|F_{m,\alpha}\|_{A_\nu(p,q)} \leq \|F\|_{A_\nu(p,q)}$;
- 2. $\lim_{m \rightarrow \infty} \|F - F_{m,\alpha}\|_{A_\nu(p,q)} = 0$;
- 3. for α large enough, $F_{m,\alpha} \in A_\gamma^t$.

For claim 1, by Lemma 3.4 and Lemma 3.1, observe that if $z = x + iy$,

$$\left| \Delta^{-\alpha}\left(\frac{\frac{z}{m} + i\mathbf{e}}{i}\right) \right| \leq \Delta^{-\alpha}\left(\frac{y}{m} + \mathbf{e}\right) \leq \Delta^{-\alpha}(\mathbf{e}) = 1.$$

We recall from [2, Theorem 3.7] that

$$\|F_{m,\alpha}\|_{A_\nu^\sigma} \leq \|F\|_{A_\nu^\sigma} \quad (F \in A_\nu^\sigma)$$

for all $\sigma \in [1, \infty)$. Applying Theorem 1.3 to the linear operator $F \mapsto F_{m,\alpha}$ with $p_0 = 1, 1 < p_1 < Q_\nu, 1 \leq q_1 \leq \infty$ (resp. $1 + Q_\nu^{-1} < p_0, p_1 < 1 + Q_\nu, 1 \leq q_0, q_1 \leq \infty$), we obtain that

$$\|F_{m,\alpha}\|_{A_\nu(p,q)} \leq \|F\|_{A_\nu(p,q)}$$

for all $1 < p < Q_\nu$, $1 < q < \infty$ (resp. $1 + Q_\nu^{-1} < p < 1 + Q_\nu$, $1 \leq q < \infty$).

For claim 2, it was shown in [2, Theorem 3.23] that

$$\lim_{m \rightarrow \infty} \|F - F_{m,\alpha}\|_{A_\nu^\sigma} = 0 \quad (F \in A_\nu^\sigma)$$

for all $\sigma \in [1, \infty)$. Applying again Theorem 1.3, we reach the announced conclusion.

Finally, to prove claim 3, we obtain by Lemma 3.13 and Lemma 3.1 that there are positive constants $C = C(p)$ and $C(m, p, \nu)$ such that for all $x + iy \in T_\Omega$ and all $F \in A_\nu(p, q)$ we have

$$\begin{aligned} |F(x + i(y + \frac{\mathbf{e}}{m}))| &\leq C \Delta^{-\frac{\nu + \frac{n}{r}}{p}}(y + \frac{\mathbf{e}}{m}) \|F\|_{A_\nu(p,q)} \\ &\leq C(m, p, \nu) \|F\|_{A_\nu(p,q)}. \end{aligned}$$

In particular, the function $F(\cdot + \frac{i\mathbf{e}}{m})$ is bounded. It now suffices to show that the function $z \mapsto \Delta^{-\alpha}(\frac{z + i\mathbf{e}}{i})$ belongs to A_γ^t when α is sufficiently large. This follows by assertion (2) of Lemma 3.4. \square

Remark 6.2. It follows from assertion (2) of the previous theorem that when $n = 1$ (and then $r = 1$), the subspace $A_\nu(p, q) \cap A_\gamma^t$ is dense in the Banach space $A_\nu(p, q)$ for all $\gamma, \nu > 0$, $1 < p < \infty$ and $1 \leq q, t < \infty$.

7. Open questions.

7.1. Statement of Question 1. In [4], the following conjecture was stated.

Conjecture. Let $\nu > \frac{n}{r} - 1$. Then the Bergman projector P_ν admits a bounded extension to L_ν^p if and only if

$$p'_\nu < p < p_\nu := \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1} - \frac{(1 - \nu)_+}{\frac{n}{r} - 1}.$$

This conjecture was completely settled recently for the case of tube domains over Lorentz cones ($r = 2$). The proof of this result is a combination of results of [3] and [10] (cf. [6]; cf. also [9] for the particular case where $\nu = \frac{n}{2}$). In this case, Theorem 1.3 can be extended as follows.

Theorem 7.1. *We restrict to the particular case of tube domains over Lorentz cones ($r = 2$). Let $\nu > \frac{n}{2} - 1$.*

- (1) *For all $1 < p_1 < Q_\nu$, $1 \leq q_1 \leq \infty$ and $0 < \theta < 1$, the real interpolation space*

$$(A_\nu^1, A_\nu(p_1, q_1))_{\theta, q}$$

identifies with $A_\nu(p, q)$, $\frac{1}{p} = 1 - \theta + \frac{\theta}{p_1}$, $1 < q < \infty$ with equivalence of norms.

- (2) *For all $1 < p_0 < p_1 < Q_\nu$ (resp. $p'_\nu < p_0 < p_1 < p_\nu$), $1 \leq q_0, q_1 \leq \infty$ and $0 < \theta < 1$, the real interpolation space*

$$(A_\nu(p_0, q_0), A_\nu(p_1, q_1))_{\theta, q}$$

identifies with $A_\nu(p, q)$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $1 < q < \infty$ (resp. $1 \leq q < \infty$) with equivalence of norms.

- (3) For all $Q_\nu \leq p_1 < p_\nu$, $1 \leq q_1 \leq \infty$, the Bergman-Lorentz spaces $A_\nu(p, q)$, $p'_\nu < p < Q_\nu$, $1 \leq q < \infty$ are real interpolation spaces between A_ν^1 and $A_\nu(p_1, q_1)$ with equivalence of norms.
- (4) For all $1 < p_0 \leq p'_\nu$, $Q_\nu \leq p_1 < p_\nu$, $1 \leq q_0, q_1 \leq \infty$, the Bergman-Lorentz spaces $A_\nu(p, q)$, $p'_\nu < p < Q_\nu$, $1 \leq q < \infty$ are real interpolation spaces between $A_\nu(p_0, q_0)$ and $A_\nu(p_1, q_1)$ with equivalence of norms.

In the upper rank case ($r \geq 3$), we always have $(1 - \nu)_+ = 0$ and the conjecture has the form

$$p'_\nu < p < p_\nu := \frac{\nu + \frac{2n}{r} - 1}{\frac{n}{r} - 1}.$$

The best result so far towards the above conjecture is Theorem 1.2. If we could show that P_ν is of restricted weak type (p_1, p_1) for some $1 + Q_\nu < p_1 < p_\nu$ (resp. $p'_\nu < p_1 < 1 + Q_\nu^{-1}$), then by the Stein-Weiss interpolation theorem (Theorem 2.20), we would obtain that P_ν admits a bounded extension to L^p_ν for all $1 + Q_\nu \leq p < p_1$ (resp. $p'_\nu < p_1 < 1 + Q_\nu^{-1}$). This would improve Theorem 1.2.

Question 1. Prove the existence of an exponent $p_1 \in (1 + Q_\nu, p_\nu)$ (resp. $p_1 \in (p'_\nu, 1 + Q_\nu^{-1})$) such that P_ν is of restricted weak type (p_1, p_1) . That is, there exists a positive constant C_{p_1} such that for each measurable subset E of T_Ω , we have

$$\sup_{\lambda > 0} \lambda^{p_1} \mu_{P_\nu \chi_E}(\lambda) \leq C_{p_1} \mu(E).$$

Remark 7.2. This question, in the suggested range, is equivalent to the conjecture. For Lorentz cones ($r = 2$), the latter property holds for all $p_1 \in (p'_\nu, 1 + Q_\nu^{-1}) \cup (1 + Q_\nu, p_\nu)$. In this case, it would be interesting, and stronger than the conjecture, to prove that the Bergman projector P_ν is of restricted weak type (p, p) at the end-points $p = p_\nu$ or $p = p'_\nu$. Recall that when $n = r = 1$, the Bergman projector is of weak type for $p = 1$; for the unit disc, this result goes back to [19].

7.2. Question 2. Can Theorem 6.1 be extended to some (all) exponents $1 + Q_\nu \leq p < \infty$? One could expect that $A_\nu(p, q) \cap A^t_\gamma$ should be dense in $A_\nu(p, q)$ with no restriction on the indices.

7.3. Question 3. This question is twofold. It concerns real interpolation spaces between two Bergman-Lorentz spaces and it is induced by Theorem 1.3 and Corollary 5.4.

- (1) We suppose that $n \geq 3$, $r \geq 2$. Can assertions (3) and (4) of Theorem 1.3 be extended to some (all) values $p \in (1, 1 + Q_\nu^{-1}]$ or $p \in [Q_\nu, 1 + Q_\nu)$?

- (2) Is the Bergman-Lorentz space $A_\nu(p, \infty)$ a real interpolation space between two different Bergman-Lorentz spaces for some (all) $1 \leq p < \infty$? By Corollary 5.8, this question arises even in the one-dimensional case.

Related is the following question induced by the second part of Lemma 5.1. Let $0 < \theta < 1$. We consider two cases:

- (1) $p_0 = q_0 = 1$, $1 < p_1 < \infty$, $1 \leq q_1 \leq \infty$;
 (2) $1 < p_0 < p_1 < \infty$, $1 \leq q_0, q_1 \leq \infty$.

Define the exponent p by $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Does the inclusion

$$A_\nu(p, q) \subset A_\nu(p_0, q_0) + A_\nu(p_1, q_1)$$

hold for all $1 \leq q \leq \infty$? In particular, even in the one-dimensional case, is the inclusion $A_\nu(p, \infty) \subset A_\nu(p_0, q_0) + A_\nu(p_1, q_1)$ valid?

7.4. Question 4. This question is also twofold. It is induced by Remark 3.11, Proposition 2.8 and Theorem 5.5.

- (1) Prove that the subspace $A_\nu(p, q)$ is strictly contained in the space $A_\nu(p, \infty)$ for all $1 \leq p < \infty$, $1 \leq q < \infty$. It is likely that the function $\Delta^{-\frac{\nu+\frac{2n}{r}-1}{p}}(z + ie)$ belongs to $A_\nu(p, \infty)$. By assertion (2) of Lemma 3.4, it does not belong to A_ν^p .
 (2) More generally, let $1 < p_0, p_1 < \infty$ and $1 \leq q_0, q_1 \leq \infty$. Prove that $A_\nu(p_0, q_0) \neq A_\nu(p_1, q_1)$ unless $p_0 = p_1$, $q_0 = q_1$.

8. Final remark

Many results in this paper make sense as well in the quasi-normed setting, that is, $0 < p < \infty$, $0 < q \leq \infty$. The interested reader would state and prove them in the general setting.

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