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Dynamics and Julia sets of iterated elliptic functions

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Abstract. We study the parametrized family of elliptic functions of the form $F_{\Lambda,b}(z) = \wp_\Lambda(z) + b$ for $b \in \mathbb{C}$, Λ a lattice, and $\wp_\Lambda$ the Weierstrass elliptic $\wp$ function with period lattice Λ. We show that the dynamics depend on b as b varies within one fundamental region of $\mathbb{C}/\Lambda$, and on the lattice Λ. We analyze properties of the Julia sets, and bifurcations of $F_{\Lambda,b}$, focussing on real rectangular lattices; in particular the dynamical properties are more diverse than those coming from the family $\wp_\Lambda$ with Λ varying.

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1. Introduction

In this paper we show that, when iterating meromorphic functions, the connectivity of the Julia set changes when a constant is added to the Weierstrass elliptic $\wp$ function, without changing the period lattice. Given a lattice Λ, we consider maps: $F_{\Lambda,b}(z) = \wp_\Lambda(z) + b$, $b \in \mathbb{C}$, and show for example that Cantor Julia sets occur when a constant is added to $\wp_\Lambda$, even when $J(\wp_\Lambda)$ is connected. Iterated elliptic functions have been the subject of study for some time starting with [18] and [10], and now there is a significant literature on the topic (see for example [9] – [13], [15] – [19], and [25]). It is known for example that for any square lattice Λ, the Julia set of $\wp_\Lambda$ is always connected [12, 4]. The connected Julia sets vary quite a bit and depend on a classical invariant called g_2, or equivalently on the generators of the period lattice Λ. We focus on real rectangular lattices in this paper, though many statements are proved more generally. We study bifurcations that occur in parameter space paying special attention to real parameters and parameters that lie on the horizontal half lattice lines, emphasizing that the resulting dynamics are quite different from each other. For example, for Λ square, since 0 is a critical value and a pole of $\wp_\Lambda$, $b = 0$ is a very unstable parameter; every neighborhood of 0 contains b's that can move the pole to either an attracting or repelling cycle. However there is much more stability when b lies on the half lattice line as there are no poles near the critical values of $F_{\Lambda,b}$. Adding a constant to $\wp_\Lambda^n, n \geqslant 1$ was also studied in [15], from a different perspective.

In Section 2 we give preliminary definitions and background for iterated elliptic functions proving some new results relating critical values to the lattice, which are used to parametrize the dynamics in this paper. In Section 3 we introduce the parametrized family of mappings $F_{\Lambda,b}$ studied in the paper and prove some general properties of these maps. The main result in Section 3 is Theorem 3.2, which shows that for any lattice except possibly a triangular lattice (which has additional symmetries), one fundamental period for the lattice Λ provides a parameter space in which we have a representative of each conformal equivalence class of maps $F_{\Lambda,b}$. In Section 4 we study the dynamical properties of maps with real parameters b. We show that for every real rectangular lattice there are constants b such that the Julia set of $F_{\Lambda,b}$ is the whole sphere, and that same b is also an accumulation point for parameters where F_b has a super-attracting fixed point (Theorem 4.21). In Section 5 we look at a different part of parameter space along horizontal half lattice lines, and discuss bifurcations that can occur. We turn to some results about Cantor Julia sets for $F_{\Lambda,b}$ for real rectangular lattices, including square ones, in Section 6. We show that for some square lattices, whenever b lies on a horizontal half lattice line, the Julia set is a Cantor set; we also show that toral bands can occur in in the Fatou set of $F_{\Lambda,b}$.

2. Preliminary definitions and notation

By $\Lambda = [\lambda_1, \lambda_2]$ we denote the group $\Lambda = \{m\lambda_1 + n\lambda_2 \colon m, n \in \mathbb{Z}\} \subset \mathbb{C}$. If $\lambda_1, \lambda_2 \in \mathbb{C}$ are non-zero and linearly independent over $\mathbb{R}$, Λ is a *lattice*. Lattices determine double periods for elliptic functions; $z+\Lambda$ denotes a coset of $\mathbb{C}/\Lambda$ containing z. A closed, connected subset Q of $\mathbb{C}$ is a *fundamental region* for Λ if for each $z \in \mathbb{C}$, Q contains at least one point in the same Λ-orbit as z and no two points in the interior of Q are in the same Λ-orbit. If Q is a parallelogram it is called a *period parallelogram* for Λ.

Definition 2.1. Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. An *elliptic function* $f \colon \mathbb{C} \to \mathbb{C}_\infty$ is a meromorphic function in $\mathbb{C}$ which is periodic with respect to a lattice Λ.

The *Weierstrass elliptic function* is defined by

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right),$$

$z \in \mathbb{C}$. The map $\wp_\Lambda$ is an even elliptic function with poles of order 2. The derivative of the Weierstrass elliptic function is an odd elliptic function which is periodic with respect to Λ. The Weierstrass elliptic function and its derivative are related by the differential equation

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3, \tag{2.1}$$

where $g_2(\Lambda) = 60 \sum_{\lambda \in \Lambda\setminus\{0\}} \lambda^{-4}$ and $g_3(\Lambda) = 140 \sum_{\lambda \in \Lambda\setminus\{0\}} \lambda^{-6}$.

The numbers $g_2(\Lambda)$ and $g_3(\Lambda)$ are invariants of the lattice Λ in the following sense: if $g_2(\Lambda) = g_2(\Lambda')$ and $g_3(\Lambda) = g_3(\Lambda')$, then $\Lambda = \Lambda'$. Furthermore given any g_2 and g_3 such that $g_2^3 - 27g_3^2 \neq 0$ there exists a lattice Λ having $g_2 = g_2(\Lambda)$ and $g_3 = g_3(\Lambda)$ as its invariants [8]. For $\Lambda_\tau = [1, \tau]$, the functions $g_i(\tau) = g_i(\Lambda_\tau), i = 2, 3$, are analytic functions of τ in the open upper half plane $\operatorname{Im}(\tau) > 0$ ([8], Theorem 3.2). We have the following homogeneity in the invariants g_2 and g_3 [11].

Lemma 2.2. For lattices Λ and Λ', $\Lambda' = k\Lambda \Leftrightarrow$

$$g_2(\Lambda') = k^{-4}g_2(\Lambda) \quad \text{and} \quad g_3(\Lambda') = k^{-6}g_3(\Lambda).$$

A lattice Λ is said to be *real* if $\Lambda = \overline{\Lambda} := \{\overline{\lambda} :\ \lambda \in \Lambda\}$, where $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.

Proposition 2.3. [14] The following are equivalent:

1. Λ is a real lattice;
2. $\wp_\Lambda(\overline{z}) = \overline{\wp_\Lambda(z)}$;
3. $g_2, g_3 \in \mathbb{R}$.

Given any Λ, for $k \in \mathbb{C}\setminus\{0\}$, the following homogeneity property holds:

$$\wp_{k\Lambda}(ku) = \frac{1}{k^2}\wp_\Lambda(u). \tag{2.2}$$

2.1. Real rectangular period lattices for $\wp_\Lambda$. For most of this paper we assume that $\Lambda = [\lambda_1, \lambda_2]$, with $\lambda_1 > 0$ and λ_2 purely imaginary and lying in the upper half plane. Since a fundamental region Q can be chosen to be a rectangle with two sides parallel to the real axis and two sides parallel to the imaginary axis, Λ is called a *real rectangular lattice*.

Remark 2.4. 1. For any lattice Λ, $\wp_\Lambda$ has infinitely many simple critical points, one at each half lattice point, and we denote them by $\{\omega_1, \omega_2, \omega_3\} + \Lambda$, where

$$\omega_1 = \frac{\lambda_1}{2},\ \omega_2 = \frac{\lambda_2}{2},\ \omega_3 = \omega_1 + \omega_2.$$

We denote the set of all critical points by $\mathrm{Crit}(\wp_\Lambda)$.

2. $\wp_\Lambda$ has three critical values $e_j = \wp_\Lambda(\omega_j)$ satisfying, when Λ is real rectangular, $e_1 > 0$. Also, one of these hold: $e_2 < e_3 < 0$ (if $g_3 > 0$), $e_2 < 0 < e_3 < e_1$ (if $g_3 < 0$), or $e_3 = 0$ (if $g_3 = 0$). In the third case, $e_2 = -e_1$ and the lattice is called *rectangular square*.
3. Since for any lattice Λ, e_1, e_2, e_3 are the distinct zeros of Equation (2.1), we have these critical value relations:

$$\wp'_\Lambda(z)^2 = 4(\wp_\Lambda(z) - e_1)(\wp_\Lambda(z) - e_2)(\wp_\Lambda(z) - e_3). \tag{2.3}$$

Equating like terms in Equations (2.1) and (2.3), we obtain

$$e_1 + e_2 + e_3 = 0, \quad e_1e_3 + e_2e_3 + e_1e_2 = \frac{-g_2}{4}, \quad e_1e_2e_3 = \frac{g_3}{4}. \tag{2.4}$$

From Equation (2.1), we write

$$p(x) = 4x^3 - g_2x - g_3. \tag{2.5}$$

A cubic polynomial of the form (2.5) has discriminant:

$$\triangle\,(g_2, g_3) = g_2^3 - 27g_3^2. \tag{2.6}$$

4. The lattice Λ is real rectangular if and only if $\triangle(g_2, g_3) > 0$ and $g_2 > 0$. Equivalently, $\Lambda := \Lambda(g_2, g_3)$ is real rectangular if and only if (g_2, g_3) lies in the region: $\mathcal{R} = \{(g_2, g_3) \in \mathbb{R}^2 \ :\ g_2^3 - 27g_3^2 > 0\}$.
5. Λ is real rectangular square if and only if the roots of p are $0, \pm\sqrt{g_2}/2$, and then we have: $e_3 = 0$ and $e_1 = \sqrt{g_2}/2 = -e_2 > 0$.

2.1.1. Real rectangular lattice critical values. We can parametrize real rectangular lattices by their critical values $\{e_1, e_2, e_3\}$ under $\wp_\Lambda$; the invariants (g_2, g_3) they determine can be described explicitly.

Proposition 2.5. For any values $e_1 > 0$, and $e_2 < 0$ satisfying $|e_2| < 2e_1$, if we set

$$(g_2, g_3) = (4(e_1^2 + e_1e_2 + e_2^2), -4(e_1^2e_2 + e_1e_2^2)), \tag{2.7}$$

the corresponding map $\wp_\Lambda$ has critical values $\wp_\Lambda(\omega_j) = e_j$, $j = 1, 2, 3$ with $e_3 = -e_1 - e_2$. The critical value e_3 satisfies $e_2 < e_3 < e_1$. Moreover the lattice $\Lambda = \Lambda(g_2, g_3)$ is real rectangular, and all real rectangular lattices have (g_2, g_3) satisfying Equation (2.7).

Prescribed Parameter ↓	$\{e_1, e_2, e_3\}$	(g_2, g_3)	Λ-generator
Standard square	$\{1, -1, 0\}$	$(4, 0)$	γ
Center lattice	$\{\omega_1, -\omega_1, 0\}$	$(4\omega_1^2, 0)$	$2\omega_1$
e_1	$\{e_1, -e_1, 0\}$	$(4e_1^2, 0)$	$\frac{\gamma}{\sqrt{e_1}}$
g_2	$\{\frac{\sqrt{g_2}}{2}, -\frac{\sqrt{g_2}}{2}, 0\}$	$(g_2, 0)$	$\gamma\sqrt{2}g_2^{-1/4}$
Λ-generator	$\{\frac{1}{k^2}, -\frac{1}{k^2}, 0\}$	$(\frac{4}{k^4}, 0)$	$k\gamma$

TABLE 1. Parameter relationships for $\wp_\Lambda$ on a rectangular square lattice, where $\gamma \approx 2.62206$ denotes the lemniscate constant.

Proof. Setting e_1, e_2, and e_3 as in the hypotheses, by construction we have $\sum_{j=1}^3 e_j = 0$ and $e_2 < e_3 < e_1$. The proposed value g_3 satisfies

$$\frac{g_3}{4} = -(e_1^2 e_2 + e_1 e_2^2) = (e_1 e_2) \cdot (-e_1 - e_2) = e_1 e_2 e_3,$$

and similarly

$$\begin{aligned} -\frac{g_2}{4} =& -\left(e_1^2 + e_1 e_2 + e_2^2\right) \\ =& e_1 \cdot e_2 + e_1 \cdot (-e_1 - e_2) + e_2 \cdot (-e_1 - e_2) \\ =& e_1 e_2 + e_1 e_3 + e_2 e_3. \end{aligned}$$

Using Equations (2.1) and (2.4), by uniqueness of the roots of (2.5), the result follows. The condition $|e_2| < 2e_1$ ensures that $e_3 < e_1$, so $e_1 = \wp_\Lambda(\omega_1)$ as claimed. Real lattices are characterized by having (g_2, g_3) that satisfy $\triangle(g_2, g_3) \neq 0$, and among real lattices, real rectangular are precisely those with distinct real critical values $\{e_1, e_2, e_3\}$ satisfying the properties of Equation (2.4), so the result is proved. □

Starting from the *standard square lattice* in row 2, all other entries of Table 1 follow from the homogeneity equation (2.2) for $\wp_\Lambda$, and the table shows how the various invariants for $\wp_\Lambda$ interact with each other. By definition the *center lattice* (shown in row 3 of Table 1) is the lattice (and corresponding value of g_2) for which the associated Weierstrass $\wp$ function $\wp_\Lambda$ has a superattracting fixed point at ω_1. It follows that $\omega_1 = (2/\gamma)^{-2/3} \approx 1.19787$.

For real rectangular lattices, we use the Arithmetic Geometric Mean of two nonnegative numbers A and B (this is discussed in various sources, e.g., [1]).

Definition 2.6. Given $A, B > 0$, we first set $A_0 = A$ and $B_0 = B$. We then define two sequences $\{A_n\}, \{B_n\}, n = 0, 1, \ldots$ by

$$A_{n+1} = \frac{1}{2}(A_n + B_n), \quad B_{n+1} = \sqrt{A_n B_n},$$

where for B_{n+1} we always choose the positive square root. The *Arithmetic Geometric Mean* (AGM) of Gauss, is the common limit of the two sequences, and is written $\mathcal{M}(A,B)$.

Since we restrict to real rectangular lattices here, we always assume that $e_1 > 0$. Therefore the expression under the radical sign of the following is always positive, so we define:

$$AG_1(e_1,e_2) := \mathcal{M}(\sqrt{e_1-e_2}, \sqrt{e_1-e_3})$$

and

$$AG_2(e_1,e_2) := \mathcal{M}(\sqrt{e_1-e_2}, \sqrt{e_3-e_2}).$$

For (g_2,g_3) as in Proposition 2.5 we have $\Lambda = \left[\dfrac{\pi}{AG_1(e_1,e_2)}, \dfrac{\pi i}{AG_2(e_1,e_2)}\right]$, (see [1]).

Lemma 2.7. If $\Lambda = [\lambda_1,\lambda_2]$ is a real rectangular lattice, with λ_1 real and λ_2 purely imaginary, and the corresponding critical values are $e_j, j=1,2,3$, then:

$$\begin{aligned} \frac{\pi}{\sqrt{e_1-e_2}} &\leqslant |\lambda_1| \leqslant \frac{\pi}{\sqrt{e_1-e_3}}, \\ \frac{2\pi}{\sqrt{e_1-e_2}+\sqrt{e_3-e_2}} &\leqslant |\lambda_2| \leqslant \frac{\pi}{[(e_1-e_2)(e_3-e_2)]^{\frac{1}{4}}}. \end{aligned} \tag{2.8}$$

Another important identity we use throughout is the following.

Theorem 2.8. [8] Let Λ be any lattice and $u \in \mathbb{C}$. Then for each $i \in \{1,2,3\}$,

$$\wp_\Lambda(u \pm \omega_i) = \frac{(e_i-e_j)(e_i-e_k)}{\wp_\Lambda(u)-e_i} + e_i. \tag{2.9}$$

The next definition appears in different forms; we use the definition from [7].

Definition 2.9. [7] By $\wp^n$ we denote the n-fold composition of $\wp$ with itself; we define the *postcritical set* of $\wp_\Lambda$ by $\mathcal{P}(\wp_\Lambda) = \bigcup\limits_{n>0} \wp_\Lambda^n(\mathrm{Crit}(\wp_\Lambda))$.

Lemma 2.10. [10]. If Λ is real rectangular, $\mathcal{P}(\wp_\Lambda) \subset \mathbb{R} \cup \{\infty\}$.

2.2. Fatou and Julia sets for elliptic functions. Background definitions and properties for meromorphic functions appear in ([2] – [5]) and [6]; if $S \subset \mathbb{C}_\infty$ is a set, then $\mathrm{cl}(S)$ denotes the topological closure of S.

Let $f\colon \mathbb{C} \to \mathbb{C}_\infty$ be a meromorphic function with at least two distinct poles. The *Fatou set* $F(f)$ is the set of points $z \in \mathbb{C}_\infty$ such that $\{f^n : n \in \mathbb{N}\}$ is defined and normal in some neighborhood of z. The *Julia set* is the complement of the Fatou set on the sphere, $J(f) = \mathbb{C}_\infty \backslash F(f)$. Montel's theorem implies that $J(f) = \mathrm{cl}\left(\bigcup\limits_{n\geqslant 0} f^{-n}(\infty)\right)$.

A meromorphic function is *Class S* if f has only finitely many critical and asymptotic values; for each lattice Λ, every elliptic function with period lattice Λ is of Class S. Therefore the basic dynamics are similar to those of rational maps with the exception of the poles. The first result holds for all Class S functions as was shown in ([3], Corollary 4 and Theorem 12).

Theorem 2.11. For any lattice Λ, the Fatou set of an elliptic function f_Λ with period lattice Λ has no wandering domains and no Baker domains.

In particular, all Fatou components of f_Λ are preperiodic, and because there are only finitely many critical values, we have a bound on the number of attracting periodic points that can occur.

We define the family of elliptic functions of interest in this paper. Let Λ be a lattice.

(Main Family) $$F_{\Lambda,b}(z) = \wp_\Lambda(z) + b, \text{ for } b \in \mathbb{C}.$$

The next result was proved in [11] for the Weierstrass elliptic function but since $\wp_\Lambda$ and $F_{\Lambda,b}$ have the same poles for every $b \in \mathbb{C}$, and $F_{\Lambda,b}$ is also even, the same proof works.

Theorem 2.12. For any lattice Λ, $F_{\Lambda,b}$ has no cycle of Herman rings.

Since $\wp_\Lambda$ has three distinct critical values, so does $F_{\Lambda,b}$; this limits the number of disjoint forward invariant Fatou cycles to at most three. Each of these cycles is one of four types, summarized by the following result.

Theorem 2.13. For any lattice Λ, and any $b \in \mathbb{C}$, each periodic Fatou component of $F_{\Lambda,b}$ contains one of these:

1. a linearizing neighborhood of an attracting periodic point;
2. a Böttcher neighborhood of a super-attracting periodic point;
3. an attracting Leau petal for a periodic parabolic point. The periodic point is in $J(F_{\Lambda,b})$;
4. a periodic Siegel disk containing an irrationally neutral periodic point.

The proof of Lemma 2.14 is given for $\wp_\Lambda$ in [11] but remains the same for any elliptic function.

Lemma 2.14. If Λ is any lattice and f_Λ is an elliptic function with period lattice Λ, then $J(f_\Lambda) + \Lambda = J(f_\Lambda)$, and $F(f_\Lambda) + \Lambda = F(f_\Lambda)$.

Definition 2.15. Given two elliptic functions $f = f_\Lambda$ and $g = g_{\Lambda'}$ over period lattices Λ and Λ' respectively, we say f is *conformally conjugate* to g if there exists a map $\phi(z) = \alpha z + \beta$, $\alpha \neq 0$ such that $f \circ \phi = \phi \circ g$.

3. The parametrized family of elliptic functions F_b, $b \in \mathbb{C}$

For each fixed lattice Λ, we study the dynamical and parametric planes of the one-parameter family of elliptic functions

(Main Family) $$F_{\Lambda,b}(z) := \wp_\Lambda(z) + b, \text{ for } b \in \mathbb{C},$$

which we will denote by F_b when the lattice is fixed. Clearly $\mathrm{Crit}(F_{\Lambda,b}) = \mathrm{Crit}(\wp_\Lambda)$; the critical values of $F_{\Lambda,b}$ are $\{v_i = e_i + b \ : \ i = 1, 2, 3\}$, and the critical relations from (2.4) are:

$$\sum_{i=1}^{3} v_i = 3b, \ \sum_{i \neq j} v_i v_j = 3b^2 - \frac{g_2}{4}, \ v_1 v_2 v_3 = b^3 - b\frac{g_2}{4} + \frac{g_3}{4}. \tag{3.1}$$

For each fixed lattice Λ we say that the holomorphic family of meromorphic maps F_b parametrized over $b \in A \subset \mathbb{C}$ is *reduced* if for all $b \neq b'$ in A, F_b and F_b' are not conformally conjugate.

We show that you need look no further than one period parallelogram Q for the constant b for a reduced family of maps F_b.

Proposition 3.1. Given a fixed lattice Λ, if $F_b = \wp_\Lambda + b$, then for any $\lambda \in \Lambda$, F_b is conformally conjugate to $F_{b+\lambda}$.

Proof. For $\lambda \in \Lambda$, a straightforward computation shows that the map $\phi(z) = z - \lambda$, conjugates F_b and $F_{b+\lambda}$. □

One can ask if there are conformally conjugate maps within a fundamental period.

Theorem 3.2. Suppose we have a lattice $\Lambda = [\lambda_1, \lambda_2]$, which is not triangular. If $F_b = \wp_\Lambda + b$, and if b and b' are in the interior of a fundamental region Q, then F_b is not conformally conjugate to $F_{b'}$.

Proof. Suppose that $F_b \circ \phi(z) = \phi \circ F_{b'}(z)$ for all $z \in \mathbb{C}$. The conformal conjugacy has to fix ∞ so ϕ must be of the form $\phi(z) = \alpha z + \beta$. Moreover, since 0 is a pole of $F_{b'}$, $\phi(0) = \beta$ must be a pole of F_b, so $\beta = \lambda_0 \in \Lambda$. Moreover, ϕ maps all poles to poles injectively, so we must have $\phi(\Lambda) = \alpha\Lambda + \lambda_0 = \Lambda$, or equivalently $\alpha\Lambda = \Lambda$ and since $\phi^{-1}\Lambda = \Lambda$, we have $\alpha\Lambda = \alpha^{-1}\Lambda = \Lambda = \alpha^k\Lambda$, for all $k \in \mathbb{Z}$, so $|\alpha| = 1$ and $\alpha = e^{2\pi i/p}$ for some $p \in \mathbb{N}$.

Therefore $e^{2\pi i/p}\Lambda = \Lambda$, and if $\alpha \neq 1$, by [23] (and other classical sources), $p = 2, 3, 4$ or 6.

The critical values of F_b are $e_1 + b, e_2 + b, e_3 + b$, and of $F_{b'}$ are $e_1 + b', e_2 + b', e_3 + b'$. Since ϕ must map the critical values of $F_{b'}$ to the critical values of F_b, for $j = 1, 2, 3$, $\phi(e_j + b') = e_k + b$ for some $k = 1, 2, 3$. We then have, using (2.4) and (3.1):

$$\begin{aligned} 3b &= (e_1 + b) + (e_2 + b) + (e_3 + b) \\ &= \phi(e_1 + b') + \phi(e_2 + b') + \phi(e_3 + b') \\ &= (\alpha e_1 + \alpha b' + \lambda_0) + (\alpha e_2 + \alpha b' + \lambda_0) + (\alpha e_3 + \alpha b' + \lambda_0) \\ &= 3(\alpha b' + \lambda_0), \end{aligned} \tag{3.2}$$

so $b = \alpha b' + \lambda_0$. Now from (3.2) it follows that

$$F_b(\phi(z)) = \wp_\Lambda(\alpha z + \lambda_0) + b = \wp_\Lambda(\alpha z) + \alpha b' + \lambda_0$$

and for all z, this should equal:

$$\phi(\wp_\Lambda(z) + b') = \alpha\wp_\Lambda(z) + \alpha b' + \lambda_0.$$

Therefore for all $z \in \mathbb{C}$, $\wp_\Lambda(\alpha z) = \alpha\wp_\Lambda(z)$. By Equation (2.2) and the fact that $\alpha^{-1}\Lambda = \Lambda$, this implies that $\wp_\Lambda(\alpha z) = \alpha^{-2}\wp_\Lambda(z) = \alpha\wp_\Lambda(z)$ for all z. Therefore $\alpha^3 = 1$, so $p = 1$ or 3. In the first case, $\alpha = 1$ and $\lambda_0 = 0$ or b and b' are not both in Q. Otherwise, $p = 3$, so the lattice must be triangular, and $b = e^{2\pi i/3}b' + \lambda$. This proves the result. □

Remark 3.3. We often restrict to this parameter plane domain:

$$Q = Q_\Lambda = \{b \in \mathbb{C} \ : \ -\omega_1 < \mathrm{Re}(b) \leqslant \omega_1, -\mathrm{Im}(\omega_2) < \mathrm{Im}(b) \leqslant \mathrm{Im}(\omega_2)\}.$$

We have some additional symmetries for the Julia sets of F_b that come from the analogous symmetry for $\wp_\Lambda$.

Proposition 3.4. For a fixed lattice Λ, any $b \in \mathbb{C}$, and any $c \in \mathrm{Crit}(\wp_\Lambda)$, $c + z \in J(F_b)$ if and only if $c - z \in J(F_b)$.

Proof. Using Theorem 2.8 and $\mathrm{Crit}(F_{\Lambda,b}) = \mathrm{Crit}(\wp_\Lambda)$,

$$F_b(c + z) = \wp_\Lambda(c + z) + b = \wp_\Lambda(c - z) + b = F_b(c - z).$$

□

Define the horizonal half lattice line:

$$L = \{z \in \mathbb{C} : z = t + \omega_2,\ t \in \mathbb{R}\}. \tag{3.3}$$

Lemma 3.5. Assume Λ is a real lattice and fix some $b \in \mathbb{C}$.

1. Then, $F_{\Lambda,b}$ is anticonformally conjugate to $F_{\Lambda,\bar{b}}$.
2. Moreover, for Λ rectangular, if for $k \in \mathbb{Z}$, b_k denotes the reflection of b with respect to $L + k\lambda_2$, then $F_{\Lambda,b}$ is anticonformally conjugate to F_{Λ,b_k}.

Proof. Denote by $\eta(z) = \bar{z}$, that is, the complex conjugate of z; it is not hard to show that η is an anticonformal homeomorphism of the plane that implements the conjugacy. □

The next result follows from Remarks 2.4 and Table 1 (cf.[10], Theorems 8.1, 8.2). Let $\kappa = \Gamma(1/4)^2/(4\sqrt{\pi}) = \gamma/\sqrt{2}$.

Lemma 3.6. Let Λ be a real square lattice, so $e_1 = \sqrt{g_2}/2$ and $\omega_1 = \kappa/g_2^{1/4}$ for any $g_2 > 0$. We then have:

1. $e_1 = 2k\omega_1$ for some $k \in \mathbb{N}$, (and hence, the orbit of ω_1 under $\wp_\Lambda$ lands on a pole after one iteration), if and only if

$$g_2 = (4k\kappa)^{4/3}. \tag{3.4}$$

2. The critical value $e_1 = (2k + 1)\omega_1$ for some $k \in \mathbb{N}_0$, (and thus $(2k + 1)\omega_1$ is a super-attracting fixed point for $\wp_\Lambda$) if and only if

$$g_2 = (2(2k + 1)\kappa)^{4/3}. \tag{3.5}$$

4. The maps F_b for real rectangular lattices and $b \in \mathbb{R}$

Throughout this section we assume $(g_2, g_3) \in \mathcal{R}$, and Λ is the lattice associated to those invariants. We describe the dynamics for F_b for real parameters b. As in (3.3), L is the horizontal half lattice line and V denotes the vertical half lattice line: $V = \{\omega_1 + iy : y \in \mathbb{R}\}$.

Lemma 4.1. For any real rectangular lattice Λ, if $b \in \mathbb{R}$, then F_b maps $\mathbb{R}$, L, V, and the imaginary axis into $\mathbb{R}$. For all $n > 0$, $F_b^n(t) \in [v_1, \infty)$ for all $t \in \mathbb{R}$; the same is true for all $z \in L$ and $z \in V$ as long as $n \geqslant 2$.

Proof. Since Λ is real, $e_2 < e_3 < e_1$, with $e_1 > 0$ and $e_2 < 0$. For all $t \in \mathbb{R}$, $\wp_\Lambda(t) \in \mathbb{R}$ and $\wp_\Lambda(t) \geqslant e_1$. Thus $F_b^n(t) \in \mathbb{R}$ for all $n > 0$, and since $F_b(t) \geqslant v_1$ for all t, then $F_b^n(t) \geqslant v_1$. Using Theorem 2.8, for any $t \in \mathbb{R}$, $t + \omega_2 \in L$, so we have $\wp_\Lambda(t + \omega_2) \in \mathbb{R}$ and $F_b(t + \omega_2) \in \mathbb{R}$. Similarly, if we show that the imaginary axis gets mapped into $\mathbb{R}$, Theorem 2.8 will also show that points on V, which are of the form $u + \omega_1$, with u purely imaginary map under $\wp_\Lambda$ into $\mathbb{R}$. The result for purely imaginary numbers follows by Proposition 2.3(2), and the fact that $\wp_\Lambda$ is even; this implies purely imaginary numbers get mapped to real numbers for $b \in \mathbb{R}$. □

Proposition 4.2. Assume $\Lambda = [2\omega_1, 2\omega_1 i]$ is a square lattice.

1. If $b = \omega_1$ (or an odd multiple of ω_1), then $F_b^2(\omega_1) = F_b^2(\omega_2) = F_b^3(\omega_3)$; i.e., F_b has a single critical orbit.
2. If b is an odd multiple of ω_1 define $M(z) = e_1 \left(\dfrac{z + e_1}{z - e_1}\right)$. If $t \in \mathbb{R}$,
$$\begin{aligned} F_b(t) = \wp_\Lambda(t) + b &\mapsto M \circ \wp_\Lambda(\wp_\Lambda(t)) + b \mapsto \dots \\ &\mapsto (M \circ \wp_\Lambda)^n(\wp_\Lambda(t)) + b. \end{aligned}$$
Then $M^{-1} = M$, $e_1 \mapsto \infty \mapsto e_1$ and $-e_1 \mapsto 0 \mapsto -e_1$. Its fixed points are given by $e_1 \pm \sqrt{2}e_1$. Moreover, M sends the interval (e_1, ∞) onto itself: this implies that M interchanges the intervals $(e_1, e_1 + \sqrt{2}e_1]$ with $[e_1 + \sqrt{2}e_1, \infty)$. M also sends the interval $(-\infty, e_1)$ onto itself, interchanges the intervals $(e_1 - \sqrt{2}e_1, e_1]$ with $(-\infty, e_1 - \sqrt{2}e_1)$ and flips the upper and lower half planes.
3. If Λ is the center square lattice, then for any $b \in \mathbb{C}$, $F_b(v_1) = F_b(b + \omega_1) = F_b(b - \omega_1) = F_b(v_2)$, so the critical orbits of ω_1 and ω_2 coincide on the second iterate.

Proof. (1) follows from (3.1) and the symmetry of $\wp_\Lambda$ with respect to any critical point; a computation gives the result. (2) can be verified directly by writing $b = (2j + 1)\omega_1$ and using Theorem 2.8. To show (3), we have $e_1 = \omega_1 = -e_2$, and we apply Equation (2.9). □

4.1. The Schwarzian derivative. The Schwarzian derivative plays an important role in the study of the dynamics of $F_{\Lambda,b}$.

Definition 4.3. The Schwarzian derivative of a meromorphic function f is given by

$$\mathcal{S}f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

A few properties of $\mathcal{S}f$ are:

1. f is a Möbius transformation if and only if $\mathcal{S}f = 0$.
2. The Schwarzian derivative of the composition of any two functions f and g is given by

$$\mathcal{S}(f \circ g)(z) = \mathcal{S}f(g(z)) \cdot (g'(z))^2 + \mathcal{S}g(z).$$

From these properties we obtain the following result (cf. [9], [16]).

Proposition 4.4. If Λ is a real rectangular lattice, and if M is any Möbius map with real coefficients, then for all $t \in \mathbb{R}$, t not a half lattice point,

$$\mathcal{S}(M \circ \wp_\Lambda)(t) = \mathcal{S}\wp_\Lambda(t) < 0.$$

Proof. By Properties 1. and 2., it is enough to prove that if Λ is real rectangular, then for $t \in \mathbb{R}\backslash\frac{1}{2}\Lambda$, $\mathcal{S}\wp_\Lambda(t) < 0$. We have already remarked that for these lattices, $\wp_\Lambda(t) \geqslant e_1 > 0$ on $\mathbb{R}$. For any lattice Λ, we consider $\wp = \wp_\Lambda(z)$, for any $z \in \mathbb{C}$. From the differential equation in Equation (2.1) we have $2\wp'\wp'' = 12\wp^2\wp' - g_2\wp'$. Then $\wp'' = 6\wp^2 - g_2/2$; and differentiating gives $\wp''' = 12\wp\wp'$, so

$$\frac{\wp'''}{\wp'} = 12\wp, \tag{4.1}$$

which is the first term in $\mathcal{S}\wp$. We now consider the duplication formula,

$$\wp(2z) = \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 - 2\wp(z).$$

This gives immediately that

$$\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 = 4(\wp(2z) + 2\wp(z)),$$

and thus the second term becomes

$$-\frac{3}{2}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 = -6\wp(2z) - 12\wp(z). \tag{4.2}$$

Adding (4.1) and (4.2), we conclude that for any lattice Λ and any $z \in \mathbb{C}$,

$$\mathcal{S}\wp(z) = -6\wp(2z). \tag{4.3}$$

Therefore for real rectangular lattices Λ, when $z = t \in \mathbb{R}$, and $2t \notin \Lambda$, $\mathcal{S}\wp_\Lambda(t) < 0$ and $\mathcal{S}(M \circ \wp_\Lambda)(t) < 0$, since $\wp_\Lambda(2t) > 0$.

□

The following corollary follows from Equation (4.3) and holds for an arbitrary lattice Λ.

Corollary 4.5. For the Weierstrass elliptic function $\wp_\Lambda$ over the lattice Λ, $\mathcal{S}\wp_\Lambda$ is an even elliptic function over the lattice $\frac{1}{2}\Lambda$.

Assume Λ is real rectangular and the map F_b, $b \in \mathbb{R}$ has a non-repelling p-cycle. Write
$$\mathcal{C} = \{t_0, F_b(t_0), \ldots, F_b^{p-1}(t_0)\} \subset \mathbb{R}.$$
Then we consider its basin of attraction on $\mathbb{R}$, namely
$$B(\mathcal{C}) = \{x \in \mathbb{R} : F_b^k(x) \to \mathcal{C} \text{ as } k \to \infty\}.$$

We call the cycle $\mathcal{C}$ *topologically attracting* if $B(\mathcal{C})$ contains an open interval U; in this case we call $B(\mathcal{C})$ the *real attracting basin of* $\mathcal{C}$. By $B_0(\mathcal{C})$ we denote the union of components of $B(\mathcal{C})$ in $\mathbb{R}$ containing points from $\mathcal{C}$. $B_0(\mathcal{C})$ is the *real immediate (attracting) basin* of $\mathcal{C}$. For Λ real rectangular, we have $\mathrm{cl}(\mathcal{P}(F_b)) \subset \mathbb{R}$, so if $\mathcal{C}$ is non-repelling then $\mathcal{C} \subset [v_1, \infty)$ and $B(\mathcal{C}) \neq \varnothing$; i.e., $\mathcal{C}$ is topologically attracting on $\mathbb{R}$ [10]. We extend a theorem of Singer on interval maps to this setting, (proved in [9] for rhombic square lattices):

Theorem 4.6. If Λ is real rectangular and $b \in \mathbb{R}$, then:
1. the real immediate basin of a topologically attracting periodic orbit of $F_{\Lambda,b}$ contains a real critical point.
2. If $y \in \mathbb{R}$ is in a rationally neutral p-cycle for F_b then it is topologically attracting; i.e., there exists an open interval U, with possibly $y \in \partial U$, such that for every $t \in U$, $\lim_{n\to\infty} F_b^{np}(t) = y$.

The next two results appear in ([15] Proposition 2.8 and Proposition 3.8).

Lemma 4.7. For any Λ real rectangular and any $b \in \mathbb{R}$, $F_{\Lambda,b}$ has no cycles of Siegel disks.

Proposition 4.8. Let Λ be any real rectangular lattice, and $b \in \mathbb{R}$. Then either $J(F_b) = \mathbb{C}_\infty$, or there exists one real non-repelling cycle whose immediate basin of attraction contains a real critical point.

Since there are infinitely many real critical points, the following is sometimes more useful.

Corollary 4.9. Under the hypotheses of Prop 4.8, if F_b has a real non-repelling cycle, then its immediate basin of attraction contains $v_1 = e_1 + b$.

Corollary 4.10. Suppose $b \in \mathbb{R}$ is such that F_b has an attracting, superattracting, or parabolic cycle $\mathcal{C}$ whose immediate basin contains $(2k+1)\omega_1$ for some $k \in \mathbb{Z}$. Then $F(F_b)$ coincides with the attracting basin of $\mathcal{C}$, and each critical orbit of F_b corresponding to ω_2 and ω_3 either eventually maps to the basin of $\mathcal{C}$ or belongs to the Julia set.

In order to study the behavior of maps associated to certain parameters b we develop some descriptive vocabulary.

Definition 4.11. Assume Λ is any real rectangular lattice, $b \in \mathbb{R}$, and $k \in \mathbb{N}$.

1. b is an *order k prepole parameter for* ω_1 if $F_b^k(\omega_1) = j\lambda_1$, for some $j \in \mathbb{Z}$;
2. b is an *order k precritical parameter for* ω_1 if $F_b^k(\omega_1) = (2j+1)\omega_1$, for some $j \in \mathbb{Z}$, and $F_b^m(\omega_1) \notin \omega_1 + \Lambda$, for $0 < m < k$. If $k = 1$, we call b *precritical.*
3. b is a *(period k) center parameter for* ω_1 if $F_b^k(\omega_1) = \omega_1$ (and k is minimal).
4. We say b is an *order k noncritical preperiodic parameter for* ω_1 if $F_b^k(\omega_1)$ is preperiodic but b is none of the above.

These definitions lead to the next proposition.

Proposition 4.12. We assume $b \in \mathbb{R}$, Λ is real rectangular, and all statements refer to the critical point ω_1 unless otherwise specified.

1. Parameters for which $J(F_b) = \mathbb{C}_\infty$:
   (a) Every order k prepole parameter b gives Julia set the whole sphere for F_b.
   (b) If g_2 is chosen as in (3.4) and $g_3 = 0$, then $b = 0$ is an order 1 prepole.
   (c) If b is a noncritical preperiodic parameter, then for some $k \in \mathbb{N}$, $F_b^k(\omega_1)$ is periodic of period $r \geqslant 1$, the cycle $\mathcal{C} = \{F_b^k(\omega_1), F_b^{k+1}(\omega_1), \ldots, F_b^{k+r-1}(\omega_1)\}$ contains no critical point, and $J(F_b) = \mathbb{C}_\infty$.
   (d) If $\omega_1 \in J(F_b)$, then $J(F_b) = \mathbb{C}_\infty$.
2. Parameters for which F_b has a super-attracting cycle:
   (a) If b is a period k center parameter, then the corresponding map F_b has a super-attracting periodic orbit that contains ω_1.
   (b) If g_2 is chosen as in (3.5), and $g_3 = 0$, then $b = 0$ is a center parameter of F_0.
   (c) Every order k precritical parameter corresponds to a map F_b with a super-attracting periodic orbit on $\mathbb{R}$ containing a real critical point of the form $(2j+1)\omega_1$.
   (d) A precritical parameter b satisfies $F_b(\omega_1) = (2j+1)\omega_1$ for some nonzero integer j. The resulting critical orbit has the form:

$$\omega_1 \mapsto v_1 = (2j+1)\omega_1 \circlearrowleft$$

   and $(2j+1)\omega_1$ is a super-attracting fixed point.

Proof. We first prove 1(d); for $b \in \mathbb{R}$ by Definition 2.9 and Lemma 4.1, we consider the orbits of the non-real critical points ω_2 and ω_3 and have that $\mathrm{cl}(\mathcal{P}(F_b)) \subset [v_2, \infty]$. If either ω_2 or ω_3 lands on a critical point in $[v_2, \infty)$, then the orbit lands on the same orbit as that of ω_1 after one more iteration, so is in $J(F_b)$. By Theorem 4.6 every Fatou component that is not super-attracting must contain an infinite forward orbit of ω_1 (no Siegel disk cycles or Herman rings occur under the hypotheses on b and Λ). If there were a super-attracting cycle, by periodicity ω_1 must land on that cycle so none exist, and any non-repelling cycle must have a basin containing v_1, which is impossible by hypothesis. Therefore ω_2 and ω_3 lie in the Julia set of F_b

along with ω_1 and the result follows from Proposition 4.8. Parts 1(a), (b), and (c) all imply $\omega_1 \in J(F_b)$ from Definition 4.11, so follow immediately.

Properties 2(a), (c), and (d) follow directly from Definition 4.11, since all real critical points map to v_1, and 2(b) follows from the assumption on b. □

Remark 4.13. (1) We can extend Definition 4.11 to define order k properties with respect to any critical point. In particular, any order k precritical parameter for ω_1 is also a period m center parameter for the critical point $c = (2j+1)\omega_1$ if $F_b^k(\omega_1) = c$, and $F_b^m(c) = c$.

(2) When $b \notin \mathbb{R}$, and b is noncritical preperiodic for ω_1, then $F(F_b)$ need not be empty, as the orbits of ω_2 and ω_3 might lie in the basin of a non-repelling orbit.

4.2. Existence of parameters with prescribed dynamics. In Figure 1 we show a reduced region from Theorem 3.2 with the lattice outlined in green; the real axis seems to cut through a homeomorphic copy of the Mandelbrot set for some lattices but this is not always the case, as shown in Figure 2 for a rectangular lattice. (The origin is in the center of each figure.) We see features of quadratic-like mappings in the parameter spaces, but the setting of elliptic functions allows us to prove the existence of prepole parameters for an arbitrary real rectangular lattice. Prepole parameters impacting the dynamics of F_b also occur in parameter space. There are infinitely many order one prepole parameters; we find it useful to distinguish the two closest to the origin, and to identify the order 1 center parameter between them. When the parameter is real, the dynamics are driven by the real critical point, so we focus on ω_1.

4.2.1. Prepole and precritical parameters for ω_1 under F_b.

Proposition 4.14 (Existence of order 1 prepole and precritical parameters for ω_1)**.** Let $(g_2, g_3) \in \mathcal{R}$ be given, and let e_1 be the critical value of the corresponding map $\wp_\Lambda$. Suppose $j \in \mathbb{N} \cup \{0\}$ satisfies either

$$j\lambda_1 \leqslant e_1 < (2j+1)\omega_1 \tag{4.4}$$

or

$$(2j+1)\omega_1 \leqslant e_1 < (j+1)\lambda_1. \tag{4.5}$$

Then for the map F_b there exist order 1 prepole parameters b_{p_j} and $b_{p_{j+1}}$ for ω_1, exactly one of which is in $(-\omega_1, \omega_1]$; in addition there is exactly one order 1 precritical parameter, b_c in $(-\omega_1, \omega_1]$. These parameters are arranged as follows.

1. $-\omega_1 < b_{p_j} \leqslant 0 < b_c \leqslant \omega_1 < b_{p_{j+1}}$ if (4.4) holds.
2. $b_{p_j} \leqslant -\omega_1 < b_c \leqslant 0 < b_{p_{j+1}} \leqslant \omega_1$ if (4.5) holds.
3. $|b_c - b_{p_j}| = |b_c - b_{p_{j+1}}| = \omega_1$.

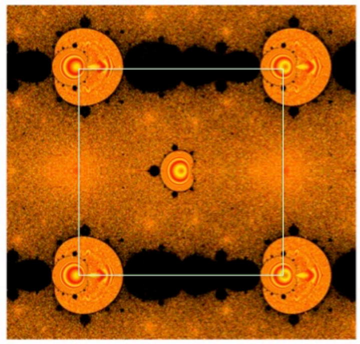

FIGURE 1. b-space for F_b using $(g_2, g_3) \approx (5.7395, 0)$

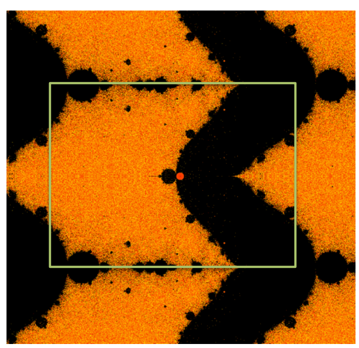

FIGURE 2. b-space for F_b using $(g_2, g_3) = (7, -3)$

Proof. Assume first $j\lambda_1 \leqslant e_1 < (2j+1)\omega_1$ for some integer $j \geqslant 0$ (Equation (4.4)). Then there exists an order 1 prepole parameter $b \in (-\omega_1, \omega_1]$ such that $F_b(\omega_1) = e_1 + b = q\lambda_1$ for some integer q if and only if

$$b = q\lambda_1 - e_1; \tag{4.6}$$

from the assumption, we see that choosing $q = j$ gives

$$b_{p_j} := j\lambda_1 - e_1,$$

and $-\omega_1 < b_{p_j} \leqslant 0$ as claimed. Any other choice of integer q would yield a pole parameter b outside the interval $(-\omega_1, 0]$. (Equation (4.4) implies that $b_{p_j} = 0$ is possible.)

An order 1 precritical parameter $b_c \in (-\omega_1, \omega_1]$ satisfies $F_b(\omega_1) = e_1 + b_c = (2q+1)\omega_1$ if and only if

$$b_c := (2q+1)\omega_1 - e_1 \leqslant \omega_1; \tag{4.7}$$

again choosing $q = j$ gives a unique $b_c \in (-\omega_1, \omega_1]$ and $b_c > 0$ by Equation (4.4). Clearly $b_c - b_{p_j} = \omega_1$.

If $b_{p_{j+1}} = (j+1)\lambda_1 - e_1$, then clearly $0 < b_c \leqslant \omega_1 < b_{p_{j+1}}$ and since $\lambda_1 = 2\omega_1$, $|b_c - b_{p_j}| = |b_c - b_{p_{j+1}}| = \omega_1$ as claimed.

The case when $(2j+1)\omega_1 \leqslant e_1 < (j+1)\lambda_1$, i.e., when Equation (4.5) holds is similar. In particular, we choose $q = j+1$ in Equation (4.6) so that $b_{p_{j+1}} \in (0, \omega_1]$ and $q = j$ in Equation (4.7) to obtain $b_c \in (-\omega_1, 0]$ as claimed.

□

We denote by b_p the unique order 1 prepole parameter in $(-\omega_1, \omega_1]$ (from Proposition 4.14). We simplify the notation with the next definition.

Definition 4.15. For any real rectangular lattice $\Lambda = [\lambda_1, \lambda_2]$, and for any $j \in \mathbb{Z}$, we define $p_j := b_{p_j} = j\lambda_1 - e_1$, where e_1 is the positive real critical value of $\wp_\Lambda$ associated with $\omega_1 > 0$.

We have the following consequence of the previous results.

Corollary 4.16. Assume Λ is real rectangular.

1. For any $b = p_j, j \in \mathbb{Z}$ as in Definition 4.15, we have $J(F_b) = \mathbb{C}_\infty$.
2. If $b = b_c + q\lambda_1, q \in \mathbb{Z}$, then there is a super-attracting fixed point in $F(F_b)$.

Proof. It suffices to consider $F_{p_j} = \wp_\Lambda(t) + p_j$, for $p_j \in (-\omega_1, \omega_1]$ by Proposition 3.1. The result follows from Proposition 4.8 since there cannot be any non-repelling cycles. Similarly, $F_{b_c}((2j+1)\omega_1) = (2j+1)\omega_1$ so we have a fixed critical point and Proposition 3.1 gives the result. □

4.2.2. Parabolic parameters for $\boldsymbol{\omega_1}$. We turn to the existence of parameters which correspond to maps F_b with parabolic fixed points, which we call *parabolic parameters*. For any integer j, set $I_j = [j\lambda_1, (j+1)\lambda_1)$.

Lemma 4.17. Suppose we have a real rectangular lattice $\Lambda = [\lambda_1, \lambda_2]$ satisfying $e_1 \in I_j$; then there exists a unique parameter value $b_{+1} \in (-\omega_1, \omega_1]$ with the property that the map $F_{b_{+1}} = \wp_\Lambda + b_{+1}$ has a fixed point $s_1 \in I_j$ such that $F'_{b_{+1}}(s_1) = 1$.

Proof. The interval I_j is a fundamental period interval for $\wp_\Lambda|_\mathbb{R}$ and $\wp'_\Lambda|_\mathbb{R}$. We showed that there exists a parameter $b_c = (2j+1)\omega_1 - e_1 \in (-\omega_1, \omega_1]$ such that $F_{b_c}(\omega_1)$ is a fixed critical point. Since $\wp'_\Lambda$ is monotone, real analytic, increasing on $(j\lambda_1, (j+1)\lambda_1)$ for each $j \in \mathbb{Z}$, and

$$\wp'_\Lambda : (j\lambda_1, (j+1)\lambda_1) \to (-\infty, \infty),$$

with $\wp'_\Lambda((2j+1)\omega_1) = 0$, there exists a unique $s_1 \in ((2j+1)\omega_1, (j+1)\lambda_1)$ such that $\wp'_\Lambda(s_1) = 1$. The corresponding parameter b, which makes s_1 a fixed point of F_b, is then chosen to be $b_{+1} = s_1 - \wp_\Lambda(s_1)$.

It remains to show that $b_{+1} \in (-\omega_1, \omega_1]$. We consider the function $t - \wp_\Lambda(t)$ on $(j\lambda, (j+1)\lambda)$; its derivative $1 - \wp'_\Lambda(t)$ is positive on $(j\lambda, s_1)$, 0 at s_1, and negative on $(s_1, (j+1)\lambda)$. Therefore s_1 is a maximum point, with maximum value b_{+1}, so

$$-\omega_1 < b_c = (2j+1)\omega_1 - \wp_\Lambda(\omega_1) = (2j+1)\omega_1 - \wp_\Lambda((2j+1)\omega_1) < b_{+1}.$$

If $b_p > 0$, then $-\omega_1 < b_c < b_{+1} < b_p \leqslant \omega_1$ (by definition of b_p) and the result is proved. Otherwise $b_p \leqslant 0$, and Equation (4.4) holds so $j\lambda_1 \leqslant e_1 < (2j+1)\omega_1$. Since $\wp_\Lambda$ has its minimum at $(2j+1)\omega_1$ on $(j\lambda_1, (j+1)\lambda_1)$, then $\wp_\Lambda(s_1) > e_1$. Moreover $(2j+1)\omega_1 < s_1 < (j+1)\lambda_1$ by construction. These inequalities give:

$$\begin{aligned} -\omega_1 < b_{+1} &= s_1 - \wp_\Lambda(s_1) \\ &< (j+1)\lambda_1 - e_1 \\ &< (j+1)\lambda_1 - j\lambda_1 \\ &< \lambda_1. \end{aligned} \tag{4.8}$$

In this case if $b_{+1} > \omega_1$, then we replace it by $\tilde{b}_1 = b_{+1} - \lambda_1 \in (-\omega_1, \omega_1)$. Then $F_{\tilde{b}_1}(s_1 - \lambda_1) = \wp_\Lambda(s_1 - \lambda_1) + b_{+1} - \lambda_1 = \wp_\Lambda(s_1) + s_1 - \wp_\Lambda(s_1) - \lambda_1 = s_1 - \lambda_1$; also $F'_{\tilde{b}_1}(s_1 - \lambda_1) = 1$ so the result is proved. □

In the interval $(-\omega_1, \omega_1]$, we always find b_{+1} such that $b_c < b_{+1}$ by Lemma 4.17. We have a similar lemma for the existence and placement of a parabolic parameter b_{-1}.

Lemma 4.18. Suppose we have a real rectangular lattice $\Lambda = [\lambda_1, \lambda_2]$ as above. Then there exists a unique parameter value $b_{-1} \in (-\omega_1, \omega_1]$ with the property that the map $F_{b_{-1}}$has a fixed point $s_{-1} \in (j\lambda_1, (j+1)\lambda_1)$ such that $F'_{b_{-1}}(s_{-1}) = -1$.

Proof. The proof is essentially the same as the proof of Lemma 4.17 since there is a unique $s_{-1} \in (j\lambda_1, (2j+1)\omega_1)$ such that $\wp'_\Lambda(s_{-1}) = -1$. It remains to show that $b_{-1} \in (-\omega_1, \omega_1]$. Since $s_{-1} < (2j+1)\omega_1 < s_1$ (by monotonicity of $\wp'_\Lambda$) we established that $t - \wp_\Lambda(t)$ is increasing there, so $b_{-1} < b_c$ follows.

By symmetry of both $\wp_\Lambda$ and $\wp'_\Lambda$ about critical points (and using $\wp'_\Lambda$ is an odd function while $\wp_\Lambda$ is even),

$$s_1 - (2j+1)\omega_1 = (2j+1)\omega_1 - s_{-1}, \quad \text{and} \quad \wp_\Lambda(s_1) = \wp_\Lambda(s_{-1})$$

and therefore

$$\begin{aligned} b_{-1} &= s_{-1} - \wp_\Lambda(s_{-1}) \\ &= (2j+1)\lambda_1 - s_1 - \wp_\Lambda(s_1) > (2j+1) - (2j+1) - \omega_1 = -\omega_1, \end{aligned}$$

since $\wp_\Lambda(s_1) < s_1 + \omega_1$ and $s_1 < (j+1)\lambda_1$, so the result is proved.

□

4.2.3. Higher order precritical and prepole parameters for ω_1. Recall that an order 2 precritical parameter is b such that $F_b^2(\omega_1)$ is a critical point, and an order 2 prepole has $F_b^2(\omega_1)$ a lattice point. Equivalently, $F_b(v_1)$ is a critical or lattice point respectively.

For the next result we shift our focus to a fundamental region in parameter space on the interval: $U_j = (p_j, p_{j+1}]$, chosen such that $e_1 \in I_j$ (so $p_j = j\lambda_1 - e_1$). We set $v_b = F_b(\omega_1)$.

Proposition 4.19. For any real rectangular lattice Λ, there exists some $T_0 > 0$ dependent on Λ, such that if $t > T_0$, there is a $b_t \in U_j$ such that $F_{b_t}(v_{b_t}) = t$.

Before giving the proof we mention an important consequence of this result.

Theorem 4.20 (Order 2 precritical and prepole parameters)**.** If $t = \omega_1 + \lambda > T_0 > 0$, $\lambda \in \Lambda$ real, then

$$\omega_1 \mapsto v_{b_t} \mapsto t \mapsto v_{b_t}$$

so $F_{b_t}(v_{b_t})$ lies in a super-attracting period 2 orbit, making b_t an order 2 precritical parameter for ω_1. Moreover if $t = \lambda > T_0, \lambda \in \Lambda$, then

$$\omega_1 \mapsto v_{b_t} \mapsto t = \lambda \mapsto \infty$$

so b_t is an order 2 prepole parameter.

Proof. (*of Proposition* 4.19*)* We showed in Lemma 4.18 that $p_j < b_{-1}$, and we note that if $b \in (p_j, b_{-1})$ then $F_b(\omega_1) < F_{b_{-1}}(\omega_1)$; i.e., $v_b < v_{b_{-1}}$, and v_b decreases in b to $j\lambda_1$ as b decreases to p_j from the right.

Given t, if we find a point $c_t > 0$ such that

$$t + e_1 = \wp_\Lambda(c_t) + c_t \tag{4.9}$$

the result follows, because we then set $b_t = c_t - e_1$, so that the orbit of ω_1 under F_{b_t} is:

$$\omega_1 \mapsto e_1 + b_t = v_{b_t} = e_1 - e_1 + c_t \mapsto \wp_\Lambda(c_t) + c_t - e_1 = t$$

as claimed.

To show Equation (4.9) has a solution, we note that $\wp'_\Lambda$ is monotone increasing on every interval I_j, $j \in \mathbb{Z}$, and $\wp'_\Lambda + 1 < 0$ on $(j\lambda_1, s_{-1}] \subset I_j$; therefore $\wp_\Lambda(t) + t$ is monotone decreasing from ∞ to $\wp_\Lambda(s_{-1}) + s_{-1}$ on $(j\lambda_1, s_{-1}]$. So as long as $t > T_0 = \wp_\Lambda(s_{-1}) + s_{-1} - e_1$, there exists a unique $c_t \in (j\lambda_1, s_{-1})$ on which $\wp_\Lambda(c_t) + c_t = t + e_1$, which is Equation (4.9), so the result is shown.

□

We next show that higher order prepole and precritical parameters accumulate on the prepole parameters p_j. We give the proof for order 3 precritical parameters.

Theorem 4.21 (Order 3 precritical parameters for ω_1)**.** Given any real rectangular lattice Λ, there are infinitely many order 3 precritical parameters that have $b_p := p_j \in (-\omega_1, \omega_1]$ as a limit point. The parameter b_p is a two-sided accumulation point for these precritical parameters.

Proof. Given a lattice, Λ, consider $e_1 > 0$, and ω_1 determined by Λ. We define the map:

$$S_2(b) = F_b^2(v_1) = F_b^3(\omega_1).$$

If $S_2(b) = (2j+1)\omega_1$ for some integer j, then the parameter b is order 3 precritical for ω_1(because $F_b^3(\omega_1)$ is a critical point).

On $\mathbb{C}_\infty$, for $R > 0$, let $B_R(\infty) = \{z : |z| > R\}$. Take a (planar) ball of the form $B_\epsilon(b_p) \subset \mathbb{C}$, then $S_2 : B_\epsilon(b_p) \backslash \{b_p\} \to \mathbb{C}_\infty$; we have that S_2 is meromorphic for ϵ small.

Therefore there exists some large R such that $B_R(\infty) \backslash \{\infty\} \subset S_2(B_\epsilon(b_p))$. We choose any $\gamma_j := (2j+1)\omega_1 \in B_R(\infty) \cap \mathbb{R}^+$. Then there exists some real parameter $b \in B_\epsilon(b_p)$ mapping to γ_j.

By choosing a sequence $\epsilon_m = 2^{-m}$, starting with m large enough, we obtain the result.

□

Essentially the same proof shows that order 3 prepole parameters accumulate on b_p as well, by choosing $\gamma_j = (2j)\omega_1 = j\lambda_1 \in B_R(\infty) \cap \mathbb{R}^+$.

4.2.4. Noncritical preperiodic parameters for $\boldsymbol{\omega_1}$. Our standing assumption is that $(g_2, g_3) \in \mathcal{R}$; we find the nonnegative integer j such that $e_1 \in I_j$. The parameter b is noncritical preperiodic for ω_1 means by definition that ω_1 is not periodic, but it terminates in a cycle not containing a critical point. A noncritical preperiodic parameter implies that $J(F_b) = \mathbb{C}_\infty$ by Proposition 4.12. We now turn to the existence of these parameters.

Lemma 4.22. For any $q \in \mathbb{N}$, there is a branch of the multi-valued function $\wp_\Lambda^{-1}(q\lambda_1 + e_1)$, with a value η_q such that the parameter

$$b_q^+ = \eta_q - e_1 \in (0, \omega_1].$$

Similarly there is a branch of $\wp_\Lambda^{-1}(q\lambda_1 + e_1)$, with a value denoted by γ_q such that

$$b_q^- = \gamma_q - e_1 \in (-\omega_1, 0].$$

Proof. We first note that since $q\lambda_1 + e_1 > e_1$ for any $q \in \mathbb{N}$, there will be exactly 2 real values of $\wp_\Lambda^{-1}(q\lambda_1 + e_1)$ in each periodic interval $I_j = [j\lambda_1, (j+1)\lambda_1)$, since $\wp_\Lambda|_\mathbb{R} : I_j \to [e_1, \infty)$ is two-to-one except at the critical point $j\lambda_1 + \omega_1$. Since $\wp_\Lambda$ maps each interval $I_j^+ = [j\lambda_1 + \omega_1, (j+1)\lambda_1)$ and $I_j^- = (j\lambda_1, j\lambda_1 + \omega_1]$ injectively onto $[e_1, \infty)$, there is exactly one value $t_{j,q} \in I_j^+$ such that $\wp_\Lambda(t_{j,q}) = q\lambda_1 + e_1$; also there is one value $s_{j,q} \in I_j^-$ such that $\wp_\Lambda(s_{j,q}) = q\lambda_1 + e_1$.

If $j_0\lambda_1 \leqslant e_1 < j_0\lambda_1 + \omega_1$, we choose $\eta_q = s_{j_0,q}$ and $\gamma_q = t_{(j_0-1),q}$. Then setting $b_q^+ = \eta_q - e_1$ gives the first result and $b_q^- = \gamma_q - e_1$ gives the second.

We obtain a similar result if $j_0\lambda_1 + \omega_1 \leqslant e_1 < (j_0+1)\lambda_1 + \omega_1$.

□

Proposition 4.23. (Noncritical preperiodic parameters accumulate on b_p.) Let Λ be a fixed real rectangular lattice, and suppose $e_1 \in I_j$. Denote by $b_p \in (-\omega_1, \omega_1]$ either p_j or p_{j+1}. Then for any large enough integer $q > j$, we can choose a branch of $\wp_\Lambda^{-1}(q\lambda_1 + e_1)$ with value $\eta_q \in I_j$ such that the parameter

$$b_q = \eta_q - e_1 \in (0, \omega_1]$$

gives rise to the map F_{b_q} with a preperiodic critical point for ω_1. The orbit of ω_1 under F_{b_q} is:

$$\omega_1 \mapsto v_1 \mapsto \zeta_q,$$

with ζ_q a repelling fixed point for F_{b_q}. Moreover, b_p is a limit point for the b_q's, as $q \to \infty$.

Proof. We assume that e_1 satisfies Equation (4.4); the proof when Equation (4.5) holds is similar. The hypotheses imply that $b_p = j\lambda_1 - e_1 \in (-\omega_1, 0]$, and $b_c > b_p$ (if $b_p < 0$; if $b_p = 0$ replace b_c by $b_c + \lambda_1$ in what follows.) We can write P_j^+ for the restriction of $\wp_\Lambda^{-1}$ to I_j^+, and P_j^- for the restriction of $\wp_\Lambda^{-1}$ to I_j^-; we then choose the inverse $\gamma_q = P_j^-(q\lambda_1 + e_1)$ that yields b_q^- from Lemma 4.22. Then $b_c > b_q^- > b_p$, and

$$|b_q^- - b_p| = |\gamma_q - j\lambda_1| \searrow 0$$

as $q \to \infty$. This follows since $\wp_\Lambda$ decreases monotonically from ∞ to e_1 on $(j\lambda_1, j\lambda_1 + \omega_1]$, so b_p is an accumulation point since $b_q^- \searrow b_p$.

$$F_{b_q^-}(\omega_1) = \wp_\Lambda(\omega_1) + P_j^-(q\lambda_1 + e_1) - e_1 = P_j^-(q\lambda_1 + e_1),$$

and

$$\begin{aligned} F_{b_q^-}(P_j^-(q\lambda_1 + e_1)) &= \wp_\Lambda(P_j^-(q\lambda_1 + e_1)) + P_j^-(q\lambda_1 + e_1) - e_1 \\ &= q\lambda_1 + P_j^-(q\lambda_1 + e_1), \end{aligned}$$

and since $q\lambda_1$ is a lattice point, by periodicity we have

$$F_{b_q^-}(q\lambda_1 + P_j^-(q\lambda_1 + e_1)) = q\lambda_1 + P_j^-(q\lambda_1 + e_1).$$

Therefore the point $\zeta_q = q\lambda_1 + P_j^-(q\lambda_1 + e_1)$ is a repelling fixed point since $F'_{b_q^-}$ decreases to $-\infty$ as $t \searrow j\lambda_1$, and ζ_q gets closer to lattice points of the form $(j+q)\lambda_1$ as q increases.

□

4.3. Examples. We illustrate some of the preceding results with examples. For the first several examples we use the center square lattice with $g_2 = (2\kappa)^{4/3} \approx 5.7395$, with $\omega_1 = e_1$ and $\kappa = \Gamma(1/4)^2/(4\sqrt{\pi})$. For $F_b(z) = \wp_\Lambda(z) + b$, we have a center parameter at $b_c = 0$; also $b_p = \omega_1$ since there is an order 1 prepole parameter at each endpoint of the interval $(-\omega_1, \omega_1]$. For the map $F_{b_c} = \wp_\Lambda$, two critical points terminate at the same super-attracting fixed point while ω_3 is a prepole. We use the notation for branches of inverses of $\wp_\Lambda$: P_j^+ and P_j^-, from Proposition 4.23.

1. *A square lattice with all critical points terminating in repelling fixed points.* Using $b_M = P_{-1}^+(\lambda_1) \approx -0.6642$, we have the critical orbit:

$$v_1 = \omega_1 + P_{-1}^+(\lambda_1) \mapsto 3\omega_1 + P_{-1}^+(\lambda_1) = p \mapsto p \approx 2.9294,$$

(using Theorem 2.8); p is a repelling fixed point and $b_M \in (-\omega_1, b_{-1})$. We know that $F_{b_M}(v_2) = F_{b_M}(v_1)$ so ω_2 terminates in a repelling orbit. More surprising is that v_3 is also preperiodic. In this example we have:

$$v_3 = b_M \mapsto P_0^+(\lambda_1) \approx 1.7315,$$

a repelling fixed point.

If a parameter has all critical points terminating in repelling cycles, we call it a *Misiurewicz parameter*. Even for a square lattice, in general one cannot expect ω_3 to terminate in a repelling cycle when ω_1 and ω_2 do.

2. Using Theorem 4.20 and choosing $t = 3\omega_1$ we obtain an order 1 precritical parameter $b_* \approx -0.7123$ such that

$$\omega_1 \mapsto e_1 + b_* = v_1 \leftrightarrow 3\omega_1.$$

In parameter plane, b_* lies between $b_p - 2\omega_1$ and b_{-1} and is a center parameter for $3\omega_1$. Thus $b_* < 0$ and $0 < v_1 < \omega_1$, but $F_{b_*}(v_1) = \wp_\Lambda(v_1) + b_* = 3\omega_1$.

3. *$J(F_b)$ is a Cantor set for a square lattice for some $b \in \mathbb{R}$.* We use the values $(g_2, g_3) = (1, 0)$ and we set $b = \omega_1 - e_1 = \kappa - 1/2$, so there is a super-attracting fixed point at ω_1. Using the approximations from Lemma 2.7, we have that $1/2 = e_1 < e_2 + b < 1 < b < e_1 + b = \omega_1$. We know that there is an attracting basin for the fixed point at $\omega_1 = \kappa \approx 1.854$, and numerical estimates show that its immediate basin of attraction contains all the critical values. In particular, it is enough to show it contains $v_2 = \omega_1 - 1 \approx .854$, which is equivalent to showing that

$$|F_b(\omega_1 - 1) - \omega_1| = |\wp_\Lambda(\omega_1 - 1) - 1/2| < 1.$$

This can be shown using Theorem 2.8. Once we know that all critical values are in the immediate attracting basin of an attracting fixed point, $J(F_b)$ is a Cantor set by [12], as shown on the left in Figure 5. When $b = 0$ it is known that $J(F_0)$ is connected [4].

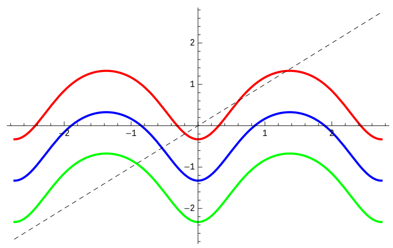

FIGURE 3. Three graphs of F_b restricted to L (the function ℓ_a), with $b = a + \omega_2$, showing $a = 0$ (blue), $a < 0$ (green), and $a > 0$ (red).

5. Dynamical properties of F_b with b on the half lattice line

In this section we show that for parameters b lying on the half lattice line L, the dynamics vary from those on $\mathbb{R}$ as the parameter moves along L. We continue to assume that $\Lambda = [\lambda_1, \lambda_2]$, with $\lambda_1 > 0$ and λ_2 purely imaginary. We consider parameters from the principal horizontal half period line defined in Equation (3.3): $L = \{b \in \mathbb{C} : b = t + \omega_2, t \in \mathbb{R}\}$. The line L contains all critical points of the form $\omega_2 + n\lambda_1$ and $\omega_3 + m\lambda_1$, $m, n \in \mathbb{Z}$.

Lemma 5.1. For any real rectangular lattice, and any parameter $b \in L$, the function F_b maps L into L.

Proof. Set $b = a + \omega_2$ for some $a \in \mathbb{R}$. Since $F_b(t+\omega_2) = \wp_\Lambda(t+\omega_2)+a+\omega_2$, it is enough to show that $\wp_\Lambda(t + \omega_2)$ is real for any $t \in \mathbb{R}$. This follows from Theorem 2.8 and the assumption that Λ is real.

□

Lemma 5.2. For any parameter $b \in L$, F_b maps $\mathbb{R}$ into L and the line $V = \{\omega_1 + iy \ : \ y \in \mathbb{R}\}$ and $-V$ into L.

Proof. $\wp_\Lambda$ takes $\mathbb{R}$ and L to $\mathbb{R}$, and $\wp_\Lambda$ maps V and $-V$ into $\mathbb{R}$ [8], so F_b maps $\mathbb{R}, V$, and $-V$ into L when $b \in L$ □

Remark 5.3. 1. When Λ is real square, it follows from Proposition 4.2(2), that:

$$\wp_\Lambda(t + \omega_2) = e_2 \left(\frac{\wp_\Lambda(t) + e_2}{\wp_\Lambda(t) - e_2} \right), \ t \in \mathbb{R}. \tag{5.1}$$

2. From Lemma 5.1, for $b \in L$, the map F_b can be decomposed into its real and imaginary parts, with the imaginary part the constant value ω_2: writing $b = (a, \omega_2)$ and $z = (t, \omega_2)$ we have

$$F_b(z) = (\ell_a(t), \omega_2),$$

where

$$\ell_a(t) = \wp_\Lambda(t + \omega_2) + a, \quad t \in \mathbb{R}.$$

Since the postcritical set determines the dynamics of F_b, the usefulness of looking at ℓ_a is shown in the next two lemmas. Several graphs of ℓ_a for different values of a are shown in Figure 3.

Lemma 5.4. Given F_b as above, with $b = a + \omega_2$ and $a \in \mathbb{R}$, $\mathrm{cl}(\mathcal{P}(F_b)) \subset L$.

Proof. The points in the postcritical set coming from ω_2 and ω_3 clearly remain on L under iteration. Moreover $\wp_\Lambda$ maps V, $-V$, and $\mathbb{R}$ into L by Lemma 5.2; since ω_1 lies on $V \cap \mathbb{R}$, the result follows. □

Lemma 5.5. Given F_b as above, $b = a + \omega_2$, and $a \in \mathbb{R}$, for $z_o = t_o + \omega_2$, $t_o \in \mathbb{R}$, we have that $F_b(z_o) = z_o$ if and only if $\ell_a(t_o) = t_o$. Moreover, ℓ_a is periodic on $\mathbb{R}$ of period λ_1.

Proof. We have $\ell_a(t_o) = t_o = \wp_\Lambda(t_o + \omega_2) + a$ if and only if $t_o + \omega_2 = \wp_\Lambda(t_o + \omega_2) + a + \omega_2 = F_b(t_o + \omega_2)$ if and only if $F_b(t_o + \omega_2) = t_o + \omega_2$. Since $\wp_\Lambda(t + \lambda_1) = \wp_\Lambda(t)$, the second statement follows. □

5.1. Properties of the auxiliary map ℓ_a. Based on the discussion above, we shift our focus to the real numbers to study the dynamics when $b \in L$. We note that for $a = 0$, ℓ_0 is just the map $\wp_\Lambda|_L$, and by periodicity, restricting the map to a fundamental region,

$$\ell_0 : (-\omega_1, \omega_1] \to [e_2, e_3],$$

since $\ell_0(-\omega_1) = \wp_\Lambda(-\omega_1 + \omega_2) = \ell_0(\omega_1) = e_3$. Since Λ is real rectangular, $e_2 < 0$ and $e_3 > e_2$ so the maximum value occurs at the two endpoints of the interval. There is a critical point of ℓ_0 at 0 which is a minimum, since $\ell_0(0) = \wp_\Lambda(\omega_2) = e_2 < 0$. The maximum value e_3 will be positive, negative or zero depending on g_3 being negative, positive, or 0 respectively.

For ℓ_a, with $a \in \mathbb{R}$, the maxima, minima, and critical points occur at the same points in the interval, independent of a. We assume $a \in [-\omega_1, \omega_1]$, and set $\mathcal{I}_a = [e_2 + a, e_3 + a]$, so $\ell_a : \mathbb{R} \to \mathcal{I}_a$ or by periodicity, we can write: $\ell_a : [-\omega_1, \omega_1] \to \mathcal{I}_a$.

The range of values for the derivative of ℓ_a can be easily computed. The map ℓ_a' can be written as $\ell'(t)$ since it does not depend on a.

Proposition 5.6. For Λ real rectangular, $b \in L$, and F_b, ℓ_a as above, the function $\ell'(t) = \wp_\Lambda'(t + \omega_2)$ is a real analytic, periodic, and odd function, which maps onto the interval $[-\sqrt{\eta}, \sqrt{\eta}]$ with $\eta = -g_3 + (g_2/3)^{3/2} > 0$.

Proof. This follows from classical identities in ([8], Chapter 2.23). □

The next result follows from Theorem 2.8 and is a generalization of Proposition 4.2 (2) to real rectangular lattices.

Proposition 5.7. For Λ a real rectangular lattice, for all $b \in \mathbb{C}$, and $z \in L$, writing $z = t + \omega_2$, we have $F_b(z) = M \circ \wp_\Lambda(t) + b$, where M is the Möbius transformation defined by:

$$M(z) = e_2 \left(\frac{z + e_2 + \frac{g_3}{4e_2^2}}{z - e_2} \right).$$

Moreover for $a \in \mathbb{R}$, we have $\ell_a(t) = M \circ \wp_\Lambda(t) + a$.

Proof. The proofs of the two parts are almost identical so we prove the second statement. We use Theorem 2.8 and rewrite the numerator using the identities given in (2.4) to see that:

$$\begin{aligned} \ell_a(t) &= \frac{2e_2^2 + \frac{g_3}{4e_2}}{\wp_\Lambda(t) - e_2} + e_2 + a \\ &= e_2 \left(\frac{\wp_\Lambda(t) + e_2 + \frac{g_3}{4e_2^2}}{\wp_\Lambda(t) - e_2} \right) + a \\ &= M \circ \wp_\Lambda(t) + a, \end{aligned} \tag{5.2}$$

□

The map M preserves the real line, interchanges the upper and lower half planes and permutes e_2 with ∞ and 0 with $-(e_2 + g_3/(4e_2^2))$.

It is of interest to determine when we obtain attracting cycles for F_b. We have transformed the question into one for maps on the real line (ℓ_a), so we can use the Schwarzian derivative. Using Proposition 4.4 we prove the following result.

Proposition 5.8. If Λ is any real rectangular lattice, then for any $b \in \mathbb{C}$, for all $z \in L$, $\mathcal{S}F_b(z) < 0$. Equivalently $\mathcal{S}\ell_a(t) < 0$ for all $t \in \mathbb{R}$.

Proof. Since $e_1 > 0$ for real rectangular lattices Λ, for all real t, $\wp_\Lambda(t) > 0$, so we have that $\mathcal{S}\wp_\Lambda(t) = -6\wp_\Lambda(2t) < 0$. Then for $z \in L$, writing $z = t + \omega_2$, we have that $\mathcal{S}F_b(z) = \mathcal{S}(M \circ \wp_\Lambda)(t) = \mathcal{S}\wp_\Lambda(t) = -6\wp_\Lambda(2t) < 0$ by Proposition 4.4. □

Using Proposition 5.8, we can apply the result from ([9], Theorem 4.1) to obtain the following result. Write $[z]$ for the coset $z + \Lambda$.

Theorem 5.9. If Λ is a real rectangular lattice, and $b = a + \omega_2$, $a \in \mathbb{R}$, then:

1. The immediate basin of an attracting periodic orbit of F_b on L contains an element of either $[\omega_2] \cap L$ or $[\omega_3] \cap L$ (or both).
2. If $z_0 = t_0 + \omega_2$, (t_0 real) is in a rationally neutral p-cycle for F_b then it is topologically attracting in the sense that there is an open interval in L that is attracted to z_0, and a critical point in its immediate attracting basin that contains an element of either $[\omega_2] \cap L$ or $[\omega_3] \cap L$ (or both).

5.2. Center and parabolic parameters on L. In Definition 4.11 we defined order k prepole and (period m) center parameters in terms of ω_1; we carry those definitions over verbatim for ω_2 and ω_3. We first show that there are no prepole parameters along L.

Proposition 5.10. For Λ real rectangular, there are no prepole parameters $b \in L$.

Proof. We show that if $b \in L$, then none of ω_1, ω_2, or ω_3 land on a pole under F_b. For any $z \in \mathbb{R} \cup L$, $F_b(z) = u + a + \omega_2$, with $u = \wp_\Lambda(z) \in \mathbb{R}$, and $a \in \mathbb{R}$. Since ω_2 is purely imaginary, $F_b(z) \in L$, which contains no poles. Therefore $F_b^n(z) \in L$ as well for each n, and hence $F_b^n(z) \notin \Lambda$. Since $\omega_1 \in \mathbb{R}, \omega_2, \omega_3 \in L$, there are no prepole parameters. □

Many bifurcations in b-space depend on the lattice Λ, but the next result shows there exist center parameters $b \in L$ for any lattice Λ.

Theorem 5.11 (Fixed center parameters for ω_2 and ω_3)**.** For any real rectangular lattice Λ the parameters on L given by $b_j = \omega_j - e_j$ each give F_{b_j} with a super-attracting fixed point at ω_j, $j = 2, 3$.

Proof. The point ω_1 is a super-attracting fixed point for $\ell_{(\omega_1 - e_3)}$, and 0 is a fixed critical point for ℓ_{-e_2}. The result follows from Lemma 5.5. □

The connectivity of the resulting Julia sets in Theorem 5.11 depends on the lattice; we develop this idea further below.

5.2.1. Parabolic parameters on L. Whenever $\eta \geqslant 1$ from Proposition 5.6, we obtain parabolic behavior for some parameters on L.

Proposition 5.12 (Existence of parabolic parameters)**.** If $(g_2, g_3) \in \mathcal{R}$ and $\triangle(g_2, g_3 + 1) > 0$, in the set $(-\omega_1, \omega_1] + \omega_2 \subset L$, there exist parameters $b_j \in L$, $j = 1, 2$ for which F_{Λ, b_j} has a parabolic fixed point in each periodic interval on L with multiplier $(-1)^j$. In particular in the interval $(-\omega_1, \omega_1]$, there are exactly 2 parameters a_1^+, a_2^+ for which F_b, $b = a_j^+ + \omega_2$, $j = 1, 2$ has a fixed point with derivative 1, and two parameters a_1^-, a_2^- for which F_b, $b = a_j^- + \omega_2$, $j = 1, 2$ has a fixed point with derivative -1.

Proof. If $(g_2, g_3) \in \mathcal{R}$ then $\triangle(g_2, g_3) > 0$. If in addition $\triangle(g_2, g_3 + 1) > 0$, then by Proposition 5.6 we have $-g_3 + (g_2/3)^{3/2} = \eta > 1$. By periodicity on each line segment of L of length $2\omega_1$, the maximum value of ℓ' is $\sqrt{\eta} > 1$, and the minimum value is $-\sqrt{\eta} < -1$. By the Intermediate Value Theorem there exists a point $t_0 \in (0, \omega_1)$ such that $\ell'(t_0) = 1 < \sqrt{\eta}$; since ℓ' is an odd function, we have $\ell'(-t_0) = -1$. Therefore setting $a_1 = t_0 - \ell_0(t_0)$, we have $\ell_{a_1}(t_0) = t_0$ with multiplier 1. Similarly $a_{-1} = -t_0 - \ell_0(-t_0) = -t_0 - \ell_0(t_0)$, we have $\ell_{a_{-1}}(-t_0) = -t_0$ with multiplier -1.

Since $\ell' = 0$ at the endpoints and midpoint of $(-\omega_1, \omega_1]$, the Intermediate Value Theorem guarantees the existence of two points $t_1^+ < t_2^+$ in $[0, \omega_1]$ such that $\ell' = 1$, and two points $t_1^- < t_2^-$ in $(-\omega_1, 0]$ such that $\ell' = -1$. Each of these four points (or a translation by $\pm\lambda_1$) becomes a parabolic fixed point of ℓ_a for the appropriate choice of parameter $a_j^{\pm} \in (-\omega_1, \omega_1]$. Specifically, we choose $a_j^+ = (t_j^+ + k\lambda_1) - \wp_\Lambda(t_j^+)$ for some $k \in \mathbb{Z}$ such that $a_j^+ \in (-\omega_1, \omega_1]$. This is possible since there is a representative of $(t_j^+ - \wp_\Lambda(t_j^+)) + \Lambda$ in every interval of L of length $2\omega_1$. □

Remark 5.13. When $\triangle(g_2, g_3 + 1) = 0$, every fundamental interval along L contains exactly two parabolic parameters, each corresponding to a map with fixed point; one with multiplier 1 and the other with multiplier -1.

Unlike the case when b is real, we next show that for some real rectangular lattices no parabolic parameters exist.

6. Maps F_b with Cantor Julia set

We recall that for any square lattice, $J(\wp_\Lambda)$ is connected [11]. In addition for any example of a lattice Λ for which the connectivity of $J(\wp_\Lambda)$ is known, it is connected. In this section we show that adding a constant changes the connectivity. As always we consider Λ to be a real rectangular lattice. The next result shows that for some lattices Λ, every $b \in L$ yields a map with a Cantor Julia set. Since we write parameters $b \in L$ as $b = a + \omega_2$, a real, we denote a line segment along L by $[\alpha_1, \alpha_2] + \omega_2$ where $\alpha_1, \alpha_2 \in \mathbb{R}$.

Theorem 6.1. Let $\Lambda = \Lambda(g_2, g_3)$ be any real rectangular lattice and suppose that $(g_2, g_3) \in \mathcal{R}$ also satisfies: $\triangle(g_2, g_3 + 1) < 0$. Then for any $b \in L$, $J(F_b)$ is a Cantor set.

Proof. We write $b = a + \omega_2$. The conditions on the pair (g_2, g_3) imply that $|\ell_a'(t)| < 1$ for all $t \in \mathbb{R}$ by Proposition 5.6. Since $\ell_a : \mathbb{R} \to [e_2 + a, e_3 + a]$, let $p = \max\{e_3 + a, e_2 + a + 2\omega_1\}$, $I = [e_2 + a, p]$, and consider $\ell_a : I \to I$. By the Contraction Mapping Theorem, there exists a unique fixed point $t_0 \in I$, and all points in I converge under iteration to t_0. Since I contains a fundamental period of ℓ_a, then all points $t \in L$, and therefore all points in $\pm V$ are attracted to the fixed point at t_0. The Fatou set is open, so by ([12], Corollary 3.11 and Theorem 3.12), we have a double toral band, and a Cantor Julia set (see Definition 6.9 below). □

The region in $\mathbb{R}$ where (g_2, g_3) satisfies the hypotheses of Theorem 6.1 is shown in yellow on the left in Figure 4. A typical Julia set obtained by choosing (g_2, g_3) satisfying the hypotheses of Theorem 6.1, with a generic value of b on L is shown in Figure 4. We can generalize some of these results to arbitrary lattices.

Theorem 6.2. If Λ is a real rectangular lattice and F_b, $b \in L$ has an attracting fixed point whose basin of attraction contains $[0, \omega_1] + \omega_2$, then $J(F_b)$ is a Cantor set.

Proof. Let z_0 satisfy $F_b(z_0) = z_0$ with $|F_b'(z_0)| < 1$. Then $z_0 \in L$ by Lemma 5.4. Let $\mathcal{A}$ denote the immediate attracting basin of z_0, and set $\mathcal{A}_{\mathcal{L}} = \mathcal{A} \cap L$.

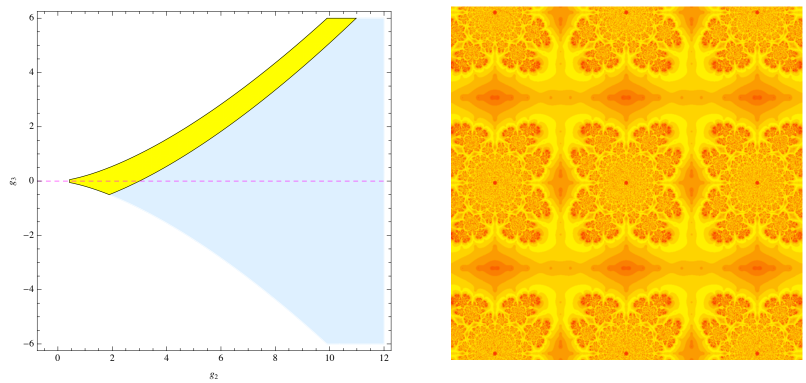

FIGURE 4. Pairs (g_2, g_3) satisfying $\triangle(g_2, g_3) > 0$ and $\triangle(g_2, g_3 + 1) < 0$ are in yellow on the left, and a Cantor Julia set for F_b using $(g_2, g_3) = (6, 2)$ satisfying: $\triangle(g_2, g_3) > 0$, $\triangle(g_2, g_3 + 1) < 0$, and $b \in L$ is shown on the right.

By hypothesis, symmetry about 0, and periodicity, we have that $\mathcal{A}_{\mathcal{L}} = L$, since the entire interval $[-\omega_1, \omega_1] + \omega_2 \subset F(F_b)$. By the proof of Lemma 5.4 we have that $\ell_a(\pm V) \subset \mathcal{A}$, if $b = a + \omega_2$, hence $\pm V \subset \mathcal{A}$ as well. The Fatou set is open, so by ([12], Corollary 3.11 and Theorem. 3.12) we have a double toral band and a Cantor Julia set.

$\square$

We generalize the conditions for $J(F_b)$ to be a Cantor set once more.

Proposition 6.3. Let Λ be any real rectangular lattice. For any fixed $t_0 \in \mathbb{R}$, choosing $a = t_0 - \wp_\Lambda(t_0 + \omega_2) \in \mathbb{R}$ gives t_0 as a fixed point of $\ell_a(t)$. When t_0 is attracting for ℓ_a, setting $p = \max\{e_3 + a, e_2 + a + 2\omega_1\}$, and letting $U = [e_2 + a, p]$, if ℓ_a contains U in its attracting basin, then for $b = a + \omega_2$, $J(F_b)$ is a Cantor set.

Proof. We have that $\ell_a(t_0) = \wp_\Lambda(t_0 + \omega_2) + a = \wp_\Lambda(t_0 + \omega_2) - \wp_\Lambda(t_0 + \omega_2) + t_0 = t_0$. The derivative at the fixed point is exactly $\wp'_\Lambda(t_0 + \omega_2) \in \mathbb{R}$; assume it is attracting. Set $p = \max\{e_3 + a, e_2 + a + 2\omega_1\}$, and let $U = [e_2 + a, p]$; consider $\ell_a : U \to U$. If U is in the basin of attraction of t_0, then all points $t \in L$ are as well. As above, it follows that all points in $\pm V$ are attracted to the fixed point at t_0, and $J(F_b)$ is a Cantor set.

$\square$

We now turn to the existence of maps F_b which satisfy the hypotheses of Theorem 6.2 but not those of Theorem 6.1.

Example 6.4. We start with a rectangular lattice Λ determined by the invariants $(g_2, g_3) = (5, 1)$, so that $\triangle(g_2, g_3) = 98 > 0$ and $\triangle(g_2, g_3 + 1) = 17 > 0$; so Theorem 6.1 does not apply. One can check that $e_2 = -1$ for this lattice, and since $g_3 > 0$, we have that $e_3 < 0$. By choosing $a = 1$, Theorem 5.11 implies that 0 is a super-attracting fixed point for ℓ_a.

By symmetry of the map ℓ_a about critical points, it is enough to check that on $[0, \omega_1]$, applying (5.2),

$$\ell_a(t) = \frac{2e_2^2 + \frac{g_3}{4e_2}}{\wp_\Lambda(t) - e_2} + e_2 + a = \frac{2 - 1/4}{\wp_\Lambda(t) + 1} < t.$$

This will imply that all points in L iterate to the fixed point at 0. Even though the maximum value of $\ell_a' > 1$, this is straightforward to check. We have:

$$\ell_a(t) = \frac{7}{4}\left(\wp_\Lambda(t) + 1\right)^{-1} < \frac{7}{4}\left(1 + \frac{1}{t^2} + \frac{t^2}{4}\right)^{-1},$$

using the Laurent series expansion of $\wp_\Lambda$ about 0, and truncating it after the first two terms (since $g_2, g_3 > 0$ this provides a lower bound for $\wp_\Lambda(t)$ and an upper bound for $\ell_a(t)$).

Since

$$\left(1 + \frac{1}{t^2} + \frac{t^2}{4}\right)^{-1} = \left(\frac{4t^2 + 4 + t^4}{4t^2}\right)^{-1} = \frac{4t^2}{4t^2 + 4 + t^4},$$

it suffices to show $7t^2 < t(t^2 + 2)^2$, for $t > 0$ and this can easily be shown.

Remark 6.5. For a real square lattice and b of the form $b = a + \omega_2$, $a \in \mathbb{R}$, if we choose $a = e_1$, ℓ_a, can be written as:

$$\ell_a(t) = \frac{2e_1^2}{e_1 + \wp_\Lambda(t)}. \tag{6.1}$$

Then a sufficient condition for $J(F_b)$ to be a Cantor set for a square lattice is given in Theorem 6.7.

Proposition 6.6. For Λ real rectangular square, the point 0 is a super-attracting fixed point of $\ell_{e_1}(t)$. When 0 is the only fixed point for ℓ_{e_1} on the interval $[0, \omega_1]$, then $J(F_{e_1+\omega_2})$ is Cantor. It is necessary that $e_1 < (\gamma/2)^{2/3}$ for the condition to be satisfied.

Proof. By Theorem 5.11, $a = -e_2 = e_1$ will yield a super-attracting fixed point at 0. If 0 is the only fixed point of ℓ_a, then we have $\ell_a(t) < t$ near 0, and therefore for all $t \in (0, \omega_1]$. Clearly if there is some t such that $\ell_a(t) = t$ we have a fixed point, and if $\ell_a(t) > t$ for some $t \in (0, \omega_2)$, then by the Intermediate Value Theorem we have a fixed point in between 0 and t. If $\ell_a(t) < t$ on $(0, \omega_1]$, then $\ell_a(\omega_1) < \omega_1$. By Table 1, this means $e_1 < \frac{\gamma}{2\sqrt{e_1}}$, or $e_1 < (\gamma/2)^{2/3} \approx 1.19787$.

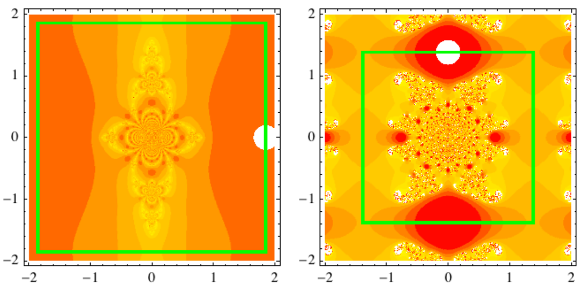

FIGURE 5. Cantor Julia sets, on the left using $b \in \mathbb{R}$ and $(g_2, g_3) = (1, 0)$, and on the right using $b \in L$ and $(g_2, g_3) = (3.24, 0)$. The attracting fixed point is marked in white and the period lattice is shown in green in both.

Since $\ell_a(t) < t$ on $(0, \omega_1]$, then $\ell_a^n(t) < \ell_a^{n-1}(t)$ for all $n \in \mathbb{N}$, and therefore the decreasing sequence $\{\ell_a^n(t)\}$ must converge to a fixed point which has to be 0. Then all $t \in L$ and V are attracted to 0 and $J(F_b)$ is a Cantor set. □

Using the idea for the proof above we obtain a continuum of examples with Cantor Julia set; the right picture of Figure 5 shows $J(F_b)$ from Theorem 6.7, using $e_1 = .9$.

Theorem 6.7. If $\wp_\Lambda$ has a real square period lattice Λ and $e_1 \leqslant 1$, then setting $b = e_1 + \omega_2$ we have that $J(F_b)$ is a Cantor set.

Proof. We show that Theorem 6.2 can be applied to yield the result. By Equation (6.1), $\ell_a(t) = \dfrac{2e_1^2}{e_1 + \wp_\Lambda(t)}$ and has 0 as a super-attracting fixed point. Since $\ell_a(t) < t$ for small $t > 0$, it suffices to show that $\ell_a(t) < t$ for all $t > 0$. As above we approximate $\wp_\Lambda$ from below using its Laurent series expansion, all of whose nonzero coefficients are positive if $g_2 > 0$, and we have from Table 1, that $g_2 = 4e_1^2$. Therefore $\wp_\Lambda(t) > \frac{1}{t^2} + \frac{e_1^2 t^2}{5}$, so it is enough to show that

$$\frac{10e_1 t^2}{e_1^2 t^4 + 5e_1 t^2 + 5} < t. \tag{6.2}$$

Reducing Equation (6.2) to showing the quadratic polynomial $r(t) = 5e_1 t^2 - 10e_1 t + 5$ is positive as above, this holds when $e_1 \leqslant 1$ for all $t > 0$. □

This next result gives rise to a large number of examples, not satisfying the hypotheses of any of the previous results, of maps F_b on square lattices with Cantor Julia sets.

Proposition 6.8. Assume Λ is the center square lattice, and set $A = (-\omega_1, 0)$. Suppose $a \in A$ is such that there exists an attracting fixed point

t_0 for ℓ_a, and $t_0 \in A$. Then t_0 is the only fixed point in A; if t_0 is the only fixed point for ℓ_a on $\mathcal{I}_a = [-\omega_1 + a, a]$, then $J(F_{a+\omega_2})$ is a Cantor set.

Proof. Since $t_0 \in A$, the interval $B = A \cap \mathcal{I}_a = (-\omega_1, a) \neq \emptyset$. Since $\wp'_\Lambda(z) < 0$ on $A + \omega_2$, we have $\ell'(t_0) < 0$. Therefore ℓ_a is strictly decreasing on A and therefore t_0 is the only fixed point of ℓ_a in A. By Theorem 4.6 there must be a critical point in the immediate basin of attraction of t_0, so since $-\omega_1$ is the only critical point in the domain and range of ℓ_a, it iterates to t_0. But $\ell_a(-\omega_1) = a$, so a is also in the immediate attracting basin and therefore the entire interval B is.

For all $t \in \mathcal{I}_a \backslash A$, we have $t \in (-2\omega_1, -\omega_1)$ and $\ell'_a(t) > 0$, so the sequence $\{\ell_a^n(t)\}$ increases until $\ell_a^{n_o}(t) \in B$ for some n_o. For $n > n_o$, $\{\ell_a^n(t)\}$ is attracted to t_0. Therefore the entire interval $\mathcal{I}_a$ is in the attracting basin of t_0 and by Proposition 6.3, $J(F_b)$ is a Cantor set.

□

Toral band Fatou components. A fundamental region for an elliptic function can be identified with the torus $\mathbb{C}/\Lambda$; we consider Fatou components on a torus, and have the following definition from ([11], Definition 5.1 and Proposition 5.2).

Definition 6.9. A Fatou component A_0 of the map F_b is a *toral band* if A_0 contains an open subset U which is simply connected in $\mathbb{C}$, but U projects to a topological band around the torus $\mathbb{C}/\Lambda$ containing a homotopically nontrivial curve. We say A_0 is a *double toral band* if $U \subset A_0$ contains a simple closed loop which forms the boundary of a fundamental region for Λ.

It is clear that when $J(F_b)$ is a Cantor set we have a double toral band but other types of toral bands can occur. We refer to a Fatou component A_0 as a single toral band if it is a toral band but not a double toral band.

Proposition 6.10. [11] For any lattice Λ, an elliptic function f_Λ has a toral band if and only if there is a component of the Fatou set which is not completely contained in the interior of one fundamental region Q.

Example 6.11. We show the existence of a map F_b with a toral band but such that $J(F_b)$ is not a Cantor set using numerical estimates from Lemma 2.7. For this example, $(g_2, g_3) = (7, -3)$. The critical values for the associated map $\wp_\Lambda$ are $(e_1, e_2, e_3) = (1, -1.5, .5)$, by Proposition 2.5.

We choose $b \in L$ given by: $b = \omega_2 - e_2 = \omega_2 + 1.5$ so that we have a super-attracting fixed point at ω_2. We have a second attracting fixed point on L; this is the fixed point contained in a toral band. If we denote by (η_1, η_2, η_3), the real parts of (v_1, v_2, v_3) which all lie on L, we have: $\eta_2 = 0 < \eta_2 = 2 < \eta_3 = 2.5$, and calculating the first few terms in the AGM sequence for λ_1, we see that $\eta_3 < \lambda_1$, and it then follows that $\omega_1 < \eta_2 < \eta_3 < \lambda_1$, and the function ℓ_a defined earlier, using $a = 1.5$, is concave down on the interval (ω_1, λ_1) and maps L periodically onto $[0, 2]$. From here it is not too difficult to show there is an attracting fixed point $p = \alpha + \omega_2$, with $\omega_1 < \alpha < \eta_2$, and with both v_2 and v_3 in its attracting basin. This gives us the existence of a toral band; the additional attracting fixed point at ω_2 implies that $J(F_b)$ is not a Cantor set. The parameter space for the example is shown on the right in Figure 2.

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