New York Journal of Mathematics

New York J. Math. **24** (2018) 1101–1110.

A sharp Hardy type inequality on the sphere

Songting Yin

Abstract. We obtain a Hardy type inequality on the sphere and give the corresponding best constant. The result complements some inequalities in recent literature.

Contents

1. Introduction

The classical Hardy inequality states that, for $n \geq 3$ and all $f \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |\nabla f|^2 \, dx \geq \frac{(n-2)^2}{4} \int_{\mathbb{R}^n} \frac{f^2}{|x|^2} \, dx.$$

The constant $(n-2)^2/4$ is optimal and not attained for the Sobolev space $W^{1,2}(\mathbb{R}^n)$. There has been a lot of research concerning Hardy inequality on the Euclidean space because of its application to singular problems. See [2],[3],[8], [10],[13] and the references therein.

The validity of Hardy inequality on a manifold and its best constants allow people to obtain qualitative properties on the manifold. In [4], Carron studied the weighted L^2-Hardy inequalities on a Riemannian manifold under some geometric assumptions on the weight function and obtained

$$\int_M \rho^\alpha |\nabla f|^2 \, dV \geq \frac{(C+\alpha-1)^2}{4} \int_M \rho^\alpha \frac{f^2}{\rho^2} \, dV,$$

where the weight function ρ satisfies $|\nabla \rho| = 1$ and $\Delta \rho \geq C/\rho$. For this line of research, we refer to [7],[9],[6], [11] and so on. In particular, Kome and

Received January 23, 2018.

2010 *Mathematics Subject Classification.* Primary 26D10; Secondary 46E36.

Key words and phrases. Hardy inequality, sphere, sharp constant.

Research supported by AHNSF (1608085MA03), KLAMFJPU (SX201805), and NNSFC(11471246).

ISSN 1076-9803/2018

Özaydin obtained in [11] the following improved Hardy inequalities for the Poincaré conformal disc model:

$$\int_{\mathbb{B}^n} |\nabla f|^2 \, dV \geq \frac{(n-2)^2}{4} \int_{\mathbb{B}^n} \frac{f^2}{r^2} \, dV,$$

where $f \in C_0^\infty(\mathbb{B}^n)$ and $r = \log[(1+|x|)/(1-|x|)]$ is the geodesic distance. Furthermore, the constant $(n-2)^2/4$ is best possible.

However, there is a lack of literature discussing Hardy inequality on the sphere up to now. To our knowledge, the only papers in the literature are [1][5][14]. Recently, Xiao (see [14]) studied this issue and derived the following inequality,

$$C_1 \int_{\mathbb{S}^n} f^2 \, dV + \int_{\mathbb{S}^n} |\nabla f|^2 \, dV \geq \frac{(n-2)^2}{4} \left(\int_{\mathbb{S}^n} \frac{f^2}{d(p,x)^2} \, dV + \int_{\mathbb{S}^n} \frac{f^2}{(\pi - d(p,x))^2} \, dV \right),$$

where $d(p,x)$ is the geodesic distance from p to x on $\mathbb{S}^n$, C_1 is some positive constant, and the constant $(n-2)^2/4$ is sharp. The inequality was then generalized by Sun and Pan (see [12]) to L^p-Hardy inequality on the sphere.

In this short note we will obtain another type of Hardy inequality on the sphere and also give the corresponding sharp constant. Our main theorem is the following.

Theorem 1.1. *Let $(\mathbb{S}^n, g)$ $(n \geq 3)$ be the n-sphere with sectional curvature 1. Then for any function $f \in C^\infty(\mathbb{S}^n)$ we have*

$$\frac{n-2}{2} \int_{\mathbb{S}^n} f^2 \, dV + \int_{\mathbb{S}^n} |\nabla f|^2 \, dV \geq \frac{(n-2)^2}{4} \int_{\mathbb{S}^n} \frac{f^2}{\tan^2 d(p,x)} \, dV,$$

where p is a fixed point in $\mathbb{S}^n$ and the constant $(n-2)^2/4$ is sharp.

In Euclidean spaces (resp. a Riemannian manifold, the Poincaré conformal disc model), the Laplacian of the distance function (resp. some weight function) equals $(n-1)/|x|$ (resp. is not less than C/ρ, $(n-1)/r$). Thus the Hardy inequality naturally contains the term $f^2/|x|^2$ (resp. f^2/ρ^2, f^2/r^2).

Note that the Laplacian of the distance function on the sphere is

$$\Delta d(p,x) = (n-1) \cot d(p,x),$$

which explains the appearance of the term $f^2/[\tan^2 d(p,x)]$ in the theorem above. So our inequality takes a different form from those in Euclidean spaces and other Hardy type inequalities. In addition, in Theorem 1.1, the first term in the left-hand side of the inequality cannot be removed because it will lead to a contradiction if f is a nonzero constant.

It is interesting to note that, even if the coefficient $(n-2)/2$ is replaced by an arbitrary number C, the constant $(n-2)^2/4$ is still sharp.

To prove the result, we will use the symmetrty of the sphere to modify the construction of an auxiliary function that has been used in the literature and then do calculations in two hemispheres using antipodal points. Since the auxiliary function is only continuous, we use approximation by smooth functions to show sharpness of our main result. The rest of the proof is similar to the approach used in Xiao's paper [14]. See also in [11] and [15].

2. Proof of the main result

Let $r_p(x) = d(p,x)$ denote the distance function from a fixed point $p \in \mathbb{S}^n$. We follow the arguments in [11] (see also [14]) and let $f = (\sin r_p)^\alpha \varphi$ with $\alpha < 0$. Then

$$\nabla f = \varphi \nabla (\sin r_p)^\alpha + (\sin r_p)^\alpha \nabla \varphi$$

and

$$\begin{aligned} |\nabla f|^2 &= \varphi^2 |\nabla (\sin r_p)^\alpha|^2 + (\sin r_p)^{2\alpha} |\nabla \varphi|^2 + 2(\sin r_p)^\alpha \varphi \langle \nabla (\sin r_p)^\alpha, \nabla \varphi \rangle \\ &\geq \varphi^2 \alpha^2 (\sin r_p)^{2\alpha-2} \cos^2 r_p + \frac{1}{2} \langle \nabla (\sin r_p)^{2\alpha}, \nabla \varphi^2 \rangle \qquad (2.1) \\ &= \varphi^2 \alpha^2 (\sin r_p)^{2\alpha-2} \cos^2 r_p + \frac{1}{2} \mathrm{div}(\varphi^2 \nabla (\sin r_p)^{2\alpha}) - \frac{1}{2} \varphi^2 \Delta (\sin r_p)^{2\alpha}, \end{aligned}$$

where

$$\begin{aligned} \Delta(\sin r_p)^{2\alpha} &= \mathrm{div}(\nabla (\sin r_p)^{2\alpha}) \qquad (2.2) \\ &= \mathrm{div}(2\alpha (\sin r_p)^{2\alpha-1} \cos r \nabla r_p) \\ &= 2\alpha (\sin r_p)^{2\alpha-1} \cos r_p \Delta r_p + 2\alpha(2\alpha-1)(\sin r_p)^{2\alpha-2} \cos^2 r - 2\alpha (\sin r_p)^{2\alpha} \\ &= -2\alpha(n+2\alpha-1)(\sin r_p)^{2\alpha} + 2\alpha(n+2\alpha-2)(\sin r_p)^{2\alpha-2}. \end{aligned}$$

The last equality holds because $\Delta r_p = (n-1)\cot r_p$ in the sphere. Therefore, from (2.1) and (2.2), we have

$$-\alpha f^2 + |\nabla f|^2 \geq \frac{1}{2} \mathrm{div}(\varphi^2 \nabla (\sin r_p)^{2\alpha}) - \alpha(n+\alpha-2) \frac{f^2}{\tan^2 r_p}.$$

Integrating both sides of the inequality above on $\mathbb{S}^n$ and letting $\alpha = -\frac{n-2}{2}$, we deduce that

$$\frac{n-2}{2} \int_{\mathbb{S}^n} f^2 \, dV + \int_{\mathbb{S}^n} |\nabla f|^2 \, dV \geq \frac{(n-2)^2}{4} \int_{\mathbb{S}^n} \frac{f^2}{\tan^2 r_p} \, dV.$$

In what follows, we show that the constant $(n-2)^2/4$ above is sharp. The argument is borrowed from [15] (see also [14]).

Let $\eta : R \to [0,1]$ be a smooth function such that $0 \leq \eta \leq 1$ and

$$\eta(t) = \begin{cases} 1, & t \in [-1,1]; \\ 0, & |t| \geq 2. \end{cases}$$

Let $H(t) = 1 - \eta(t)$, and for sufficiently small $\varepsilon > 0$, define

$$f_\varepsilon(r) = \begin{cases} 0, & r = 0; \\ H\left(\frac{r}{\varepsilon}\right)\tan^{\frac{2-n}{2}} r, & 0 < r \le \frac{\pi}{2}; \\ H\left(\frac{\pi - r}{\varepsilon}\right)\tan^{\frac{2-n}{2}}(\pi - r), & \frac{\pi}{2} \le r < \pi; \\ 0, & r = \pi. \end{cases}$$

Observe that $f_\varepsilon(r)$ can be approximated by smooth functions on the sphere $\mathbb{S}^n$. Thus we have

$$\begin{aligned} &\int_{\mathbb{S}^n} f_\varepsilon^2\, dV = \int_{B_p(\frac{\pi}{2})} f_\varepsilon^2\, dV + \int_{B_q(\frac{\pi}{2})} f_\varepsilon^2\, dV \qquad (2.3) \\ =&\mathrm{Vol}(\mathbb{S}^{n-1}) \int_\varepsilon^{\frac{\pi}{2}} H^2\left(\frac{r_p}{\varepsilon}\right) \tan^{2-n} r_p (\sin r_p)^{n-1}\, dr \\ &+ \mathrm{Vol}(\mathbb{S}^{n-1}) \int_{\frac{\pi}{2}}^{\pi-\varepsilon} H^2\left(\frac{\pi - r_p}{\varepsilon}\right) \tan^{2-n}(\pi - r_p)(\sin(\pi - r_p))^{n-1}\, dr \\ =&\mathrm{Vol}(\mathbb{S}^{n-1}) \int_\varepsilon^{\frac{\pi}{2}} H^2\left(\frac{r_p}{\varepsilon}\right) \tan^{2-n} r_p (\sin r_p)^{n-1}\, dr \\ &+ \mathrm{Vol}(\mathbb{S}^{n-1}) \int_\varepsilon^{\frac{\pi}{2}} H^2\left(\frac{r_q}{\varepsilon}\right) \tan^{2-n} r_q (\sin r_q)^{n-1}\, dr \\ =&2\mathrm{Vol}(\mathbb{S}^{n-1}) \int_\varepsilon^{\frac{\pi}{2}} H^2\left(\frac{r_p}{\varepsilon}\right) \tan^{2-n} r_p (\sin r_p)^{n-1}\, dr \\ \le&2\mathrm{Vol}(\mathbb{S}^{n-1}) \int_\varepsilon^{\frac{\pi}{2}} r_p^{2-n} r_p^{n-1}\, dr = \left(\frac{\pi^2}{4} - \varepsilon^2\right) \mathrm{Vol}(\mathbb{S}^{n-1}), \end{aligned}$$

where q is the antipodal point of p and $r_q(x) = d(q, x) = \pi - r_p(x)$ denotes the distance function from q.

On the other hand, we have

$$\begin{aligned} \int_{B_p(\frac{\pi}{2})} \frac{f_\varepsilon^2}{\tan^2 r_p}\, dV =&\mathrm{Vol}(\mathbb{S}^{n-1}) \int_\varepsilon^{\frac{\pi}{2}} H^2\left(\frac{r_p}{\varepsilon}\right) \tan^{-n} r_p (\sin r_p)^{n-1}\, dr \\ \ge&\mathrm{Vol}(\mathbb{S}^{n-1}) \int_{2\varepsilon}^{\frac{\pi}{2}} H^2\left(\frac{r_p}{\varepsilon}\right) \tan^{-n} r_p (\sin r_p)^{n-1}\, dr \\ =&\mathrm{Vol}(\mathbb{S}^{n-1}) \int_{2\varepsilon}^{\frac{\pi}{2}} \tan^{-n} r_p (\sin r_p)^{n-1} dr, \end{aligned}$$

and

$$\begin{aligned} &\int_{B_q(\frac{\pi}{2})} \frac{f_\varepsilon^2}{\tan^2 r_p}\, dV \\ &\quad =\mathrm{Vol}(\mathbb{S}^{n-1}) \int_{\frac{\pi}{2}}^{\pi-\varepsilon} H^2\left(\frac{\pi - r_p}{\varepsilon}\right) \tan^{-n}(\pi - r_p)(\sin(\pi - r_p))^{n-1}\, dr \end{aligned}$$

$$=\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\varepsilon}^{\frac{\pi}{2}} H^2\left(\frac{r_q}{\varepsilon}\right)\tan^{-n} r_q(\sin r_q)^{n-1}\,dr$$
$$\geq\mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{\frac{\pi}{2}} \tan^{-n} r_q(\sin r_q)^{n-1}\,dr.$$

Therefore, combining the above two inequalities, we obtain

$$\int_{\mathbb{S}^n}\frac{f_\varepsilon^2}{\tan^2 r_p}\,dV \geq 2\mathrm{Vol}(\mathbb{S}^{n-1})\int_{2\varepsilon}^{\frac{\pi}{2}} \tan^{-n} r_p(\sin r_p)^{n-1}\,dr. \tag{2.4}$$

Next we are going to estimate the integral

$$\int_{\mathbb{S}^n}|\nabla f_\varepsilon|^2\,dV=\int_{B_p(\frac{\pi}{2})}|\nabla f_\varepsilon|^2\,dV+\int_{B_q(\frac{\pi}{2})}|\nabla f_\varepsilon|^2\,dV.$$

A straightforward calculation yields

$$\left(\int_{B_p(\frac{\pi}{2})}|\nabla f_\varepsilon|^2\,dV\right)^{\frac12}$$
$$=\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac12}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\left|H'\left(\frac{r_p}{\varepsilon}\right)\frac{1}{\varepsilon}\tan^{\frac{2-n}{2}} r_p\right.\right.$$
$$\left.\left.+\frac{2-n}{2}H\left(\frac{r_p}{\varepsilon}\right)\tan^{-\frac{n}{2}} r_p\sec^2 r_p\right|^2(\sin r_p)^{n-1}\,dr\right)^{\frac12}$$
$$\leq\frac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac12}}{\varepsilon}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\left|H'\left(\frac{r_p}{\varepsilon}\right)\right|^2\tan^{2-n} r_p(\sin r_p)^{n-1}\,dr\right)^{\frac12}$$
$$+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac12}\left(\int_{\varepsilon}^{\frac{\pi}{2}}H^2\left(\frac{r_p}{\varepsilon}\right)\tan^{-n} r_p(\sin r_p)^{n-1}\,dr\right)^{\frac12}$$
$$+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac12}\left(\int_{\varepsilon}^{\frac{\pi}{2}}H^2\left(\frac{r_p}{\varepsilon}\right)\tan^{-n+4} r_p(\sin r_p)^{n-1}\,dr\right)^{\frac12}$$
$$=\frac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac12}}{\varepsilon}\left(\int_{\varepsilon}^{2\varepsilon}\left|H'\left(\frac{r_p}{\varepsilon}\right)\right|^2\tan^{2-n} r_p(\sin r_p)^{n-1}\,dr\right)^{\frac12}$$
$$+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac12}\left(\int_{\varepsilon}^{\frac{\pi}{2}}H^2\left(\frac{r_p}{\varepsilon}\right)\tan^{-n} r_p(\sin r_p)^{n-1}\,dr\right)^{\frac12}$$
$$+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac12}\left(\int_{\varepsilon}^{\frac{\pi}{2}}H^2\left(\frac{r_p}{\varepsilon}\right)\tan^{-n+4} r_p(\sin r_p)^{n-1}\,dr\right)^{\frac12}$$
$$\leq\frac{\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac12}}{\varepsilon}\max_{t\in[0,2]}H'(t)\left(\int_{\varepsilon}^{2\varepsilon}r_p\,dr\right)^{\frac12}$$

$$
\begin{aligned}
&+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr\right)^{\frac{1}{2}}\\
&+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4}r_p(\sin r_p)^{n-1}\,dr\right)^{\frac{1}{2}}\\
=&\sqrt{\frac{3}{2}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\max_{t\in[0,2]}H'(t)+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr\right)^{\frac{1}{2}}\\
&+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4}r_p(\sin r_p)^{n-1}\,dr\right)^{\frac{1}{2}},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\int_{B_q(\frac{\pi}{2})}|\nabla f_\varepsilon|^2\,dV\right)^{\frac{1}{2}}\\
=&\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\frac{\pi}{2}}^{\pi-\varepsilon}\left|H'\left(\frac{\pi-r_p}{\varepsilon}\right)\frac{-1}{\varepsilon}\tan^{\frac{2-n}{2}}(\pi-r_p)\right.\right.\\
&\left.\left.-\frac{2-n}{2}H\left(\frac{\pi-r_p}{\varepsilon}\right)\tan^{-\frac{n}{2}}(\pi-r_p)\sec^2(\pi-r_p)\right|^2(\sin(\pi-r_p))^{n-1}\,dr\right)^{\frac{1}{2}}\\
=&\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\left|H'\left(\frac{r_q}{\varepsilon}\right)\frac{-1}{\varepsilon}\tan^{\frac{2-n}{2}}r_q\right.\right.\\
&\left.\left.-\ \frac{2-n}{2}H\left(\frac{r_q}{\varepsilon}\right)\tan^{-\frac{n}{2}}r_q\sec^2 r_q\right|^2(\sin r_q)^{n-1}\,dr\right)^{\frac{1}{2}}\\
\leq&\sqrt{\frac{3}{2}}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\max_{t\in[0,2]}H'(t)+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_q(\sin r_q)^{n-1}\,dr\right)^{\frac{1}{2}}\\
&+\frac{n-2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4}r_q(\sin r_q)^{n-1}\,dr\right)^{\frac{1}{2}}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\int_{\mathbb{S}^n}|\nabla f_\varepsilon|^2\,dV \qquad (2.5)\\
\leq&3\mathrm{Vol}(\mathbb{S}^{n-1})(\max_{t\in[0,2]}H'(t))^2+\frac{(n-2)^2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_q(\sin r_q)^{n-1}\,dr\\
&+\sqrt{\frac{3}{2}}(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})\max_{t\in[0,2]}H'(t)\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_q(\sin r_q)^{n-1}\,dr\right)^{\frac{1}{2}}
\end{aligned}
$$

$$+\sqrt{\frac{3}{2}}(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})\max_{t\in[0,2]}H'(t)\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4}r_q(\sin r_q)^{n-1}\,dr\right)^{\frac{1}{2}}$$
$$+\frac{(n-2)^2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_q(\sin r_q)^{n-1}dr\right)^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4}r_q(\sin r_q)^{n-1}\,dr\right)^{\frac{1}{2}}$$
$$+\frac{(n-2)^2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4}r_q(\sin r_q)^{n-1}\,dr.$$

Since $f_\varepsilon(r)$ can be approximated by smooth functions on the sphere $\mathbb{S}^n$, it follows from (2.3)-(2.5) that

$$C:=\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}}\frac{\int_{\mathbb{S}^n}|\nabla f|^2\,dV+\frac{n-2}{2}\int_{\mathbb{S}^n}f^2\,dV}{\int_{\mathbb{S}^n}\frac{f^2}{\tan^2 r_p}\,dV}$$
$$\le\frac{\int_{\mathbb{S}^n}|\nabla f_\varepsilon|^2\,dV+\frac{n-2}{2}\int_{\mathbb{S}^n}f_\varepsilon^2\,dV}{\int_{\mathbb{S}^n}\frac{f_\varepsilon^2}{\tan^2 r_p}\,dV}\le\frac{\frac{n-2}{2}(\frac{\pi^2}{4}-\varepsilon^2)}{2\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr}$$
$$+\frac{3(\max_{t\in[0,2]}H'(t))^2}{2\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr}+\frac{(n-2)^2}{4}\frac{\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_q(\sin r_q)^{n-1}\,dr}{\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr}$$
$$+\frac{\sqrt{\frac{3}{2}}(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})\max_{t\in[0,2]}H'(t)\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_q(\sin r_q)^{n-1}\,dr\right)^{\frac{1}{2}}}{2\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr}$$
$$+\frac{\frac{(n-2)^2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4}r_q(\sin r_q)^{n-1}dr}{2\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr}$$
$$+\frac{\sqrt{\frac{3}{2}}(n-2)\mathrm{Vol}(\mathbb{S}^{n-1})\max_{t\in[0,2]}H'(t)\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4}r_q(\sin r_q)^{n-1}dr\right)^{\frac{1}{2}}}{2\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr}$$
$$+\frac{\frac{(n-2)^2}{2}\mathrm{Vol}(\mathbb{S}^{n-1})\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_q(\sin r_q)^{n-1}\,dr\right)^{\frac{1}{2}}\left(\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4}r_q(\sin r_q)^{n-1}\,dr\right)^{\frac{1}{2}}}{2\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr}$$
$$:=I+II+III+IV+V+VI+VII.$$

Note that

$$\lim_{\varepsilon\to 0}\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr=+\infty.$$

Also, by L'Hospital rule,

$$\lim_{\varepsilon\to 0}\frac{\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_q(\sin r_q)^{n-1}\,dr}{\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n}r_p(\sin r_p)^{n-1}\,dr}=1,$$

and

$$\lim_{\varepsilon\to 0}\frac{\int_{\varepsilon}^{\frac{\pi}{2}}\tan^{-n+4} r_q(\sin r_q)^{n-1}\,dr}{\int_{2\varepsilon}^{\frac{\pi}{2}}\tan^{-n} r_p(\sin r_p)^{n-1}\,dr}=0.$$

This implies that

$$I=II=IV=V=VI=VII=0,$$

and

$$C\le\frac{(n-2)^2}{4}.$$

Thus the constant $(n-2)^2/4$ is sharp and the proof of Theorem 1.1 is complete.

3. Two corollaries

Recall that the first eigenvalue of $\mathbb{S}^n$ is $\lambda_1=n$. From Theorem 1.1 it is then not difficult to obtain the following result.

Corollary 3.1. *Let* $(\mathbb{S}^n,g)$ *be the n-sphere as in Theorem* 1.1. *Then*

$$\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}}\frac{\int_{\mathbb{S}^n} f^2\cos^2 d(p,x)\,dV}{\int_{\mathbb{S}^n} f^2\sin^2 d(p,x)\,dV}\le\frac{6n-4}{(n-2)^2}$$

for any $p\in\mathbb{S}^n$

Proof. By Theorem 1.1, we have

$$\frac{n-2}{2}+\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}}\frac{\int_{\mathbb{S}^n}|\nabla f|^2\,dV}{\int_{\mathbb{S}^n} f^2\,dV}\ge\frac{(n-2)^2}{4}\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}}\frac{\int_{\mathbb{S}^n}\frac{f^2}{\tan^2 d(p,x)}\,dV}{\int_{\mathbb{S}^n} f^2\,dV}.$$

Since $\lambda_1=n$, this means that

$$\frac{6n-4}{(n-2)^2}\ge\inf_{f\in C^\infty(\mathbb{S}^n)\setminus\{0\}}\frac{\int_{\mathbb{S}^n}\frac{f^2}{\tan^2 d(p,x)}\,dV}{\int_{\mathbb{S}^n} f^2\,dV}$$

for any smooth function f. Replacing f by $f\sin d(x,p)$, we obtain the desired inequality. □

Another consequence of Theorem 1.1 is the following.

Corollary 3.2. *Let* $(\mathbb{S}^n,g)$ *be the n-sphere as in Theorem* 1.1. *Then*

$$\frac{n}{2}\int_{\mathbb{S}^n} f^2\,dV+\int_{\mathbb{S}^n}|\nabla f|^2\,dV\ge\frac{n(n-2)}{4}\int_{\mathbb{S}^n} f^2\cos^2 d(p,x)\,dV$$

for any $p\in\mathbb{S}^n$ *and any smooth function* f *in* $\mathbb{S}^n$.

Proof. Set $u=f\sin d(x,p)$, $f\in C^\infty(\mathbb{S}^n)$. Then by Theorem 1.1,

$$\frac{n-2}{2}\int_{\mathbb{S}^n} f^2\sin^2 d(p,x)\,dV+\int_{\mathbb{S}^n}|\sin d(p,x)\nabla f+f\cos d(p,x)\nabla d(p,x)|^2\,dV$$
$$\ge\frac{(n-2)^2}{4}\int_{\mathbb{S}^n} f^2\cos^2 d(p,x)\,dV.$$

By the Cauchy-Schwarz inequality and the fact that $|\nabla d(p,x)| = 1$ in $\mathbb{S}^n \backslash \{p,q\}$, we have

$$\begin{aligned}
&|\sin d(p,x)\nabla f + f\cos d(p,x)\nabla d(p,x)|^2\\
=&\sin^2 d(p,x)|\nabla f|^2 + f^2\cos^2 d(p,x) + 2f\sin d(p,x)\cos d(p,x)\langle\nabla f, \nabla d(p,x)\rangle\\
\leq&\sin^2 d(p,x)|\nabla f|^2 + f^2\cos^2 d(p,x) + \cos^2 d(p,x)|\nabla f|^2 + f^2\sin^2 d(p,x)\\
=&|\nabla f|^2 + f^2.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\frac{n-2}{2}\int_{\mathbb{S}^n} f^2\sin^2 d(p,x)\,dV + \int_{\mathbb{S}^n}(|\nabla f|^2 + f^2)\,dV\\
\geq&\frac{(n-2)^2}{4}\int_{\mathbb{S}^n} f^2\cos^2 d(p,x)\,dV.
\end{aligned}$$

Another simple calculation then yields the desired inequality. □

References

[1] Abolarinwa, Abimbola; Apata, Timothy. L^p-Hardy–Rellich and uncertainty principle inequalities on the sphere. *Adv. Oper. Theory* **3** (2018), no. 4, 745–762. MR3856170, Zbl 06946375, doi: 10.15352/aot.1712-1282. 1102

[2] Baras, Pierre; Goldstein, Jerome A. The heat equation with a singular potential. *Trans. Amer. Math. Soc.* **284** (1984), no. 1, 121–139. MR0742415, Zbl 0556.35063, doi: 10.2307/1999277. 1101

[3] Brezis, Haim; Vázquez, Juan Luis. Blow-up solutions of some nonlinear elliptic problems. *Rev. Mat. Univ. Complut. Madrid* **10** (1997), no. 2, 443–469. MR1605678, Zbl 0894.35038, doi: 10.5209/rev_REMA.1997.v10.n2.17459. 1101

[4] Carron, Gilles. Inégalités de Hardy sur les variétés Riemanniennes non-compactes. *J. Math. Pures Appl.* **76** (1997), no. 10, 883–891. MR1489943, Zbl 0886.58111, doi: 10.1016/S0021-7824(97)89976-X. 1101

[5] Dai, Feng; Xu, Yuan. The Hardy-Rellich inequality and uncertainty principle on the sphere. *Constr. Approx.* **40** (2014), no. 1, 141–171. MR3229998, Zbl 1319.46026, arXiv:1212.3887, doi: 10.1007/s00365-014-9235-5. 1102

[6] D'Ambrosio, Lorenzo; Dipierro, Serena. Hardy inequalities on Riemannian manifolds and applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31** (2014), no. 3, 449–475. MR3208450, Zbl 1317.46022, arXiv:1210.5723, doi: 10.1016/j.anihpc.2013.04.004. 1101

[7] Du, Feng; Mao, Jing. Hardy and Rellich type inequalities on metric measure spaces. *J. Math. Anal. Appl.* **429** (2015), no. 1, 354–365. MR3339079, Zbl 1316.26022, doi: 10.1016/j.jmaa.2015.04.021. 1101

[8] García Azorero, Jesús P.; Peral Alonso, Ireneo. Hardy inequalities and some critical elliptic and parabolic problems. *J. Differential Equations* **144** (1998), no. 2, 441–476. MR1616905, Zbl 0918.35052, doi: 10.1006/jdeq.1997.3375. 1101

[9] Grillo, Gabriele. Hardy and Rellich-type inequalities for metrics defined by vector fields. *Potential Anal.* **18** (2003), no. 3, 187–217. MR1953228, Zbl 1032.46049, doi: 10.1023/A:1020963702912. 1101

[10] Kombe, Ismail. The linear heat equation with highly oscillating potential. *Proc. Amer. Math. Soc.* **132** (2004), no. 9, 2683–2691. MR2054795, Zbl 1059.35045, doi: 10.1090/S0002-9939-04-07392-7. 1101

[11] Kombe, Ismail; Özaydin, Murad. Improved Hardy and Rellich inequalities on Riemannian manifolds. *Trans. Amer. Math. Soc.* **361** (2009), no. 12, 6191–6203. MR2538592, Zbl 1178.26013, arXiv:math/0702047, doi: 10.1090/S0002-9947-09-04642-X. 1101, 1102, 1103

[12] Sun, Xiaomei; Pan, Fan. Hardy type inequalities on the sphere. *J. Inequal. Appl.* **2017**, no. 148, 1–8. MR3665809, Zbl 1367.26036, doi: 10.1186/s13660-017-1424-x. 1102

[13] Vazquez, Juan Luis; Zuazua, Enrike. The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. *J. Funct. Anal.* **173** (2000), no. 1, 103-153. MR1760280, Zbl 0953.35053, doi: 10.1006/jfan.1999.3556. 1101

[14] Xiao, Yingxiong. Some Hardy inequalities on the sphere. *J. Math. Inequal.* **10** (2016), no. 3, 793–805. MR3565153, Zbl 1348.26014, doi: 10.7153/jmi-10-64. 1102, 1103

[15] Yang, Qiaohua; Su, Dan; Kong, Yinying. Hardy inequalities on Riemannian manifolds with negative curvature. *Commun. Contemp. Math.* **16** (2014), no. 2, 1350043, 24 pp. MR3195155, Zbl 1294.26019, doi: 10.1142/S0219199713500430. 1103

(Songting Yin) Department of Mathematics and Computer Science, Tongling University, Tongling, 244000 Anhui, China, and Key Laboratory of Applied Mathematics, Putian University, Fujian 351100, China.
yst419@163.com

This paper is available via http://nyjm.albany.edu/j/2018/24-53.html.