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# The homotopy groups of $L_2T(m)/(p^{[\frac{m}{2}]+2},\ v_1)$ for $m > 1$

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Abstract. Let $T(m)$ be the Ravenel spectrum charaterized by the $BP_*$-homology as $BP_*[t_1, \cdots, t_m]$. Let $T(m)/(v_1)$ be the cofiber of map $v_1$ and $T(m)/(p^k, v_1)$ the cofiber of $T(m)/(v_1)$'s self-map $p^k$. In this paper we determine the homotopy groups of $L_2T(m)/(p^{[\frac{m}{2}]+2},\ v_1)$ for $m > 1$ by the Adams-Novikov spectral sequence.

Contents



## 1. Introduction

Let $T(m)$ be the Ravenel spectrum charaterized by the $BP_*$-homology as

$$BP_*T(m) = BP_*[t_1, \cdots, t_m] \subset BP_*BP.$$

$T(m)$ is a connective $p$-local ring spectrum. $T(0)$ is the $p$-local sphere spectrum, and there are maps

$$S^0_{(p)} = T(0) \to T(1) \to T(2) \to \cdots \to BP.$$

The map $T(m) \to BP$ is an equivalence below dimension $2p^{m+1} - 3$. Let $L_2$ be the Bousfield localization functor with respect to $v_2^{-1}BP_*$ (see [Rav84]). The homotopy group $\pi_*(L_2T(m))$ can be explored by the Adams-Novikov

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spectral sequence. Furthermore, one can apply the chromatic spectral sequence to determine the $E_2$-term

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*(L_2T(m))).$$

Let $T(m)/(v_1)$ be the cofiber of self map

$$v_1 : \Sigma^{2(p-1)}T(m) \to T(m)$$

and $T(m)/(p)$ the cofiber of $p : T(m) \to T(m)$. For $m = 1$, the homotopy groups $\pi_*(L_2T(1)/(v_1))$ and $\pi_*(L_2T(1)/(p))$ indicates two ways to compute $\pi_*(L_2T(1))$. At the prime 2, Shimomura [Shi95] computed the homotopy group $\pi_*(L_2T(1)/(2))$. The first author [Wan07] separately with Nakai and Shimomura [NS07] determined the homotopy group $\pi_*(L_2T(1)/(v_1))$. They also proved that the Adams-Novikov spectral sequence for $\pi_*(L_2T(1))$ has a horizontal vanishing line at the $E_4$-terms. For the odd prime cases, Wang, Liu and Yuan [LWY10] determined $\pi_*(L_2T(1)/(v_1))$. But it seems to be too difficult to work out $\pi_*(L_2T(1))$ from both ways.

For $m > 1$, let $T(m)/(p^k, v_1)$ be the cofiber of

$$p^k : T(m)/(v_1) \to T(m)/(v_1).$$

We have the following commutative diagram

$$\begin{array}{ccccc} T(m)/(v_1) & \xrightarrow{p^{k+1}} & T(m)/(v_1) & \longrightarrow & T(m)/(p^{k+1}, v_1) \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle 1} & & \downarrow \\ T(m)/(v_1) & \xrightarrow{p^{k}} & T(m)/(v_1) & \longrightarrow & T(m)/(p^{k+1}, v_1) \end{array}$$

The $3 \times 3$ lemma concludes that the fiber of

$$T(m)/(p^{k+1}, v_1) \longrightarrow T(m)/(p^k, v_1)$$

is the cofiber of

$$p : T(m)/(v_1) \longrightarrow T(m)/(v_1).$$

Thus, we can obtain a cofiber sequence

$$T(m)/(p, v_1) \xrightarrow{p^k} T(m)/(p^{k+1}, v_1) \longrightarrow T(m)/(p^k, v_1). \tag{1.1}$$

At the prime 2, the homotopy groups $\pi_*(L_2T(m)/(v_1))$ and $\pi_*(L_2T(m))$ are discussed in [IS08], [IST10]. In this paper, we study the homotopy groups of $L_2T(m)/(p^{[\frac{m}{2}]+2}, v_1)$ for odd primes, which is an important step to understand the homotopy groups $\pi_*(L_2T(m)/(v_1))$ and $\pi_*(L_2T(m))$.

## 2. Statement of results

Let $E_m(2)_* = E(2)_*[v_3, \cdots v_{m+2}]$. A $BP_*$-module structure on $E_m(2)_*$ can be induced by $f_* : BP_* \to E_m(2)_*$ where $f_*$ sends $v_i$ to $v_i$ for $i \leqslant m+2$ and to 0 for $i > m+2$. Let $E_m(2)$ be the spectrum which represents the Landweber homology theory

$$E_m(2)_*(X) = E_m(2)_* \otimes_{BP_*} BP_*(X).$$

Then

$$E_m(2)_*E_m(2) = E_m(2)_* \otimes_{BP_*} BP_*(BP) \otimes_{BP_*} E_m(2)_*.$$

The Hopf algebroid structure of $(BP_*, BP_*BP)$ implies the one on

$$(E_m(2)_*, E_m(2)_*E_m(2)).$$

Similar to the change-of-rings theorem (see [Mor78, Rav86])

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, M) = \mathrm{Ext}^{s,t}_{E(2)_*E(2)}(E(2)_*, E(2)_* \otimes M)$$

For an $I_2$-nil $v_2^{-1}BP_*BP$-comodule $M$, we have

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, M) = \mathrm{Ext}^{s,t}_{E_m(2)_*E_m(2)}(E_m(2)_*, E_m(2)_* \otimes M).$$

Let

$$\Gamma(2,m) = E_m(2)_*[t_{m+1}, t_{m+2}, \cdots] \otimes_{BP_*} E_m(2)_*.$$

For $E_m(2)_*(T(m) \wedge X)$, noted that

$$\begin{aligned} E_m(2)_*(T(m) \wedge X) &= E_m(2)_*[t_1, \cdots t_m] \otimes_{BP_*} E_m(2)_*(X) \\ &= E_m(2)_*E_m(2) \square_{\Gamma(2,m)} E_m(2)_*(X), \end{aligned}$$

we can obtain the change-of-rings theorem

$$\begin{aligned} &\mathrm{Ext}^{s,t}_{E_m(2)_*E_m(2)}(E_m(2)_*, E_m(2)_*(T(m) \wedge X)) \\ &\qquad = \mathrm{Ext}^{s,t}_{\Gamma(2,m)}(E_m(2)_*, E_m(2)_*(X)). \end{aligned}$$

In this paper, we will work on the Hopf algebroid $(E_m(2)_*, \Gamma(2,m))$.

Let $M_2^0 = E_m(2)_*/(p, v_1)$ and let $L(k,1) = E_m(2)_*/(p^k, v_1)$. Denote the module $\mathrm{Ext}^*_{\Gamma(2,m)}(E_m(2)_*, M)$ by $H^*(M)$ for short. The short exact sequence

$$0 \to M_2^0 \xrightarrow{p^k} L(k+1,1) \longrightarrow L(k,1) \to 0.$$

induces a long exact sequence

$$\cdots \to H^sM_2^0 \xrightarrow{p^k} H^sL(k+1,1) \longrightarrow H^sL(k,1) \xrightarrow{\delta} H^{s+1}M_2^0 \to \cdots$$

Ravenel (see [Rav86, Corollary 6.5.6]) proves that

$$H^*M_2^0 \cong \mathbb{Z}/p[v_2^{\pm 1}, v_3, \cdots, v_{m+2}] \otimes E[h^0_{m+1}, h^1_{m+1}, h^0_{m+2}, h^1_{m+2}]$$

Here $h_k^j$ corresponds to $t_i^{p^j}$. Since $\zeta_1$ and $\zeta_2$ are representatives of $h^1_{m+1}$, $h^1_{m+2}$, respectively (see Lemma 3.3, 3.4). Thus we conclude that

$$H^{*,*}M_2^0 \cong \mathbb{Z}/p[v_2^{\pm 1}, v_3, \cdots, v_{m+2}] \otimes E[h^0_{m+1}, h^0_{m+2}, \zeta_1, \zeta_2]$$

Based on $H^*M_2^0$, we compute $H^*L(k,1)$ by Bockstein spectral sequence. To state our results, we decompose the module $H^*M_2^0$ with respect to $k\,(1 \leqslant k \leqslant [\frac{m}{2}]+1)$. $H^*M_2^0$ is the direct sum of following modules

$$\begin{aligned} &(C_0(k) \oplus I_1(k) \oplus C_1(k) \oplus I_2^a(k)) \otimes E(\zeta_1), \\ &\zeta_2 C_0(k) \oplus C_2(k) \oplus I_2^b(k) \oplus C_3(k) \oplus I_3(k) \oplus I_4(k) \oplus C_4(k). \end{aligned}$$

In these modules, let $q_1=\min\{n+1,l+1\}$ and $q_2=\min\{n+1,l+1,[\frac{m}{2}]+1\}$, $0\leqslant n,l\leqslant\infty$. For convenience, $sp^n=0$ ($tp^l=0$) if $n=\infty$ ($l=\infty$). Otherwise, $p\nmid s$, $s>0$ ($p\nmid t$, $t>0$). In $sp^n-1$ ($tp^l-1$), $p\nmid s$, $s>0$ ($p\nmid t$, $t>0$). Let $D=\mathbb{Z}[v_2^{\pm},v_3,\cdots,v_m]$ and $D/p^k=\mathbb{Z}/p^k[v_2^{\pm},v_3,\cdots,v_m]$. Define conditions $A,B,C$ as follows:

$$\begin{aligned}
A:&\quad [\tfrac{m}{2}]\leqslant n\leqslant l\leqslant\infty \ \text{ or } [\tfrac{m}{2}]\leqslant l<n\leqslant\infty\\
B:&\quad n<[\tfrac{m}{2}]\leqslant l\leqslant\infty \ \text{ or } n\leqslant l<[\tfrac{m}{2}]\leqslant\infty\\
C:&\quad l<[\tfrac{m}{2}]\leqslant n\leqslant\infty \ \text{ or } l<n<[\tfrac{m}{2}]\leqslant\infty
\end{aligned}$$

Furthermore, we will use the following notations for convenience.

$$\begin{aligned}
&\widehat{t}_i:=t_{m+i},\quad \widehat{c}_{i,j}:=c_{m+i,j},\quad \widehat{v}_i:=v_{m+i},\quad \widehat{h}_i^j:=h_{m+i}^j,\quad \widehat{b}_i^j:=b_{m+i}^j\\
&\widehat{\widehat{t}}_i:=t_{2m+i},\quad \widehat{\widehat{b}}_i^k:=b_{2m+i}^k\quad \widehat{\widehat{c}}_{i,j}:=c_{2m+i,j},\quad \omega:=p^m
\end{aligned}\tag{2.1}$$

$$\begin{aligned}
&C_0(k)=C_0^1(k)\oplus C_0^2(k)\oplus C_0^3(k)\\
&\quad C_0^1(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\,|\,0\leqslant l<n\leqslant\infty,q_1\leqslant k\}\\
&\quad C_0^2(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\,|\,0\leqslant n\leqslant l\leqslant\infty,q_1\leqslant k\}\\
&\quad C_0^3(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\,|\,q_1>k\}\\
&I_1(k)=I_1^1(k)\oplus I_1^2(k)\\
&\quad I_1^1(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0|\,0\leqslant l<n\leqslant\infty,q_1\leqslant k\}\\
&\quad I_1^2(k)=D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0|\,0\leqslant n\leqslant l\leqslant\infty,q_1\leqslant k\}\\
&C_1^a(k)=C_1^1(k)\oplus C_1^2(k)\oplus C_1^3(k)\oplus C_1^4(k)\\
&\quad C_1^1(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0|\,0\leqslant n\leqslant l\leqslant\infty,q_1\leqslant k\}\\
&\quad C_1^2(k)=D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0|\,0\leqslant l<n\leqslant\infty,q_1\leqslant k\}\\
&\quad C_1^3(k)=D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0|\,q_1>k\}\\
&\quad C_1^4(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0|\,q_1>k\}\\
&C_1^b(k)=\zeta_2C_0(k)=\oplus_{i=5}^8C_1^i(k)\\
&\quad C_1^5(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2|\,A,q_2\leqslant k\}\\
&\quad C_1^6(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2|\,B,q_2\leqslant k\}\\
&\quad C_1^7(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2|\,C,q_2\leqslant k\}\\
&\quad C_1^8(k)=D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2|\,q_2>k\}\\
&I_2^a(k)=D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^m-1}\widehat{h}_1^0\widehat{h}_2^0|\,q_1\leqslant k\}
\end{aligned}$$

$$I_2^b(k) = I_2^1(k) \oplus I_2^2(k) \oplus I_2^3(k)$$
$$I_2^1(k) = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2 |\, A, q_2 \leqslant k\}$$
$$I_2^2(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_2 |\, B, q_2 \leqslant k\}$$
$$I_2^3(k) = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_2 |\, C, q_2 \leqslant k\}$$
$$C_2(k) = \oplus_{i=1}^{10} C_2^i(k)$$
$$C_2^1 = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2 |\, B, q_2 \leqslant k\}$$
$$C_2^2 = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2 |\, C, q_2 \leqslant k\}$$
$$C_2^3 = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_2 |\, A, q_2 \leqslant k\}$$
$$C_2^4 = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_2 |\, C, q_2 \leqslant k\}$$
$$C_2^5 = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_2 |\, A, q_2 \leqslant k\}$$

$$C_2^6 = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_2 |\, B, q_2 \leqslant k\}$$
$$C_2^7 = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0 |\, q_2 > k\}$$
$$C_2^8 = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2 |\, q_2 > k\}$$
$$C_2^9 = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_2 |\, q_2 > k\}$$
$$C_2^{10} = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_2 |\, q_2 > k\}$$
$$I_3(k) = \oplus_{i=1}^{6} I_3^i(k)$$
$$I_3^1(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_1\zeta_2 |\, B, q_2 \leqslant k\}$$
$$I_3^2(k) = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_1\zeta_2 |\, C, q_2 \leqslant k\}$$
$$I_3^3(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_1\zeta_2 |\, A, q_2 \leqslant k\}$$
$$I_3^4(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_2 |\, C, q_2 \leqslant k\}$$
$$I_3^5(k) = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_1\zeta_2 |\, A, q_2 \leqslant k\}$$
$$I_3^6(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_2 |\, B, q_2 \leqslant k\}$$
$$C_3(k) = \oplus_{i=1}^{6} C_3^i(k)$$
$$C_3^1(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_1\zeta_2 |\, C, q_2 \leqslant k\}$$
$$C_3^2(k) = D/p\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_1\zeta_2 |\, B, q_2 \leqslant k\}$$
$$C_3^3(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_2 |\, A, q_2 \leqslant k\}$$
$$C_3^4(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_1\zeta_2 |\, q_2 > k\}$$

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$$I_4(k) = I_4^1(k) \oplus I_4^2(k) \oplus I_4^3(k)$$
$$I_4^1(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_1\zeta_2|\, C, q_2 \leqslant k\}$$
$$I_4^2(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_1\zeta_2|\, B, q_2 \leqslant k\}$$
$$I_4^3(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_1\zeta_2|\, A, q_2 \leqslant k\}$$
$$C_4(k) = D/p\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_1\zeta_2|\, q_2 > k\}.$$

Based on the modules above, we introduce the following submodules of $H^*L(k, v_1)$, $1 \leqslant k \leqslant [\frac{m}{2}] + 2$.

$$\widetilde{C}_0(k) = \widetilde{C}_0^1(k) \oplus \widetilde{C}_0^2(k) \oplus \widetilde{C}_0^3(k)$$
$$\widetilde{C}_0^1(k) = D/p^{l+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}/p^{l+1}|\, 0 \leqslant l < n \leqslant \infty, q_1 \leqslant k\}$$
$$\widetilde{C}_0^2(k) = D/p^{n+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}/p^{n+1}|\, 0 \leqslant n \leqslant l \leqslant \infty, q_1 \leqslant k\}$$
$$\widetilde{C}_0^3(k) = D/p^k\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}/p^k|\, q_1 > k\}$$
$$\widetilde{C}_1(k) = \widetilde{C}_1^1(k) \oplus \widetilde{C}_1^2(k) \oplus \widetilde{C}_1^3(k) \oplus \widetilde{C}_1^4(k)$$
$$\widetilde{C}_1^1(k) = D/p^{n+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0/p^{n+1}|\, 0 \leqslant n \leqslant l \leqslant \infty, q_1 \leqslant k\}$$
$$\widetilde{C}_1^2(k) = D/p^{l+1}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0/p^{l+1}|\, 0 \leqslant l < n \leqslant \infty, q_1 \leqslant k\}$$
$$\widetilde{C}_1^3(k) = D/p^k\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0/p^k|\, q_1 > k\}$$
$$\widetilde{C}_1^4(k) = D/p^k\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0/p^k|\, q_1 > k\}$$
$$\widetilde{\zeta_2 C_0}(k) = \oplus_{i=5}^8 \widetilde{C}_1^i(k)$$
$$\widetilde{C}_1^5(k) = D/p^{[\frac{m}{2}]+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2/p^{[\frac{m}{2}]+1}|\, A, q_2 \leqslant k\}$$
$$\widetilde{C}_1^6(k) = D/p^{n+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2/p^{n+1}|\, B, q_2 \leqslant k\}$$
$$\widetilde{C}_1^7(k) = D/p^{l+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2/p^{l+1}|\, C, q_2 \leqslant k\}$$
$$\widetilde{C}_1^8(k) = D/p^k\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2/p^k|\, q_2 > k\}$$
$$\widetilde{C}_2(k) = \oplus_{i=1}^{10} \widetilde{C}_2^i(k)$$
$$\widetilde{C}_2^1 = D/p^{n+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2/p^{n+1}|\, B, q_2 \leqslant k\}$$
$$\widetilde{C}_2^2 = D/p^{l+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2/p^{l+1}|\, C, q_2 \leqslant k\}$$
$$\widetilde{C}_2^3 = D/p^{[\frac{m}{2}]+1}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_2/p^{[\frac{m}{2}]+1}|\, A, q_2 \leqslant k\}$$
$$\widetilde{C}_2^4 = D/p^{l+1}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_2/p^{l+1}|\, C, q_2 \leqslant k\}$$

$$\widetilde{C}_2^5 = D/p^{[\frac{m}{2}]+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_2/p^{[\frac{m}{2}]+1}|\, A, q_2\leqslant k\}$$
$$\widetilde{C}_2^6 = D/p^{n+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_2/p^{n+1}|\, B, q_2\leqslant k\}$$
$$\widetilde{C}_2^7 = D/p^{k}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0/p^{k}|\, q_2> k\}$$
$$\widetilde{C}_2^8 = D/p^{k}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2/p^{k}|\, q_2> k\}$$
$$\widetilde{C}_2^9 = D/p^{k}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_2/p^{k}|\, q_2> k\}$$
$$\widetilde{C}_2^{10} = D/p^{k}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_2/p^{k}|\, q_2> k\}$$

$$\widetilde{C}_3(k) = \oplus_{i=1}^6\widetilde{C}_3^i(k)$$
$$\widetilde{C}_3^1(k) = D/p^{l+1}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_1\zeta_2/p^{l+1}|\, C, q_2\leqslant k\}$$
$$\widetilde{C}_3^2(k) = D/p^{n+1}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_1\zeta_2/p^{n+1}|\, B, q_2\leqslant k\}$$
$$\widetilde{C}_3^3(k) = D/p^{[\frac{m}{2}]+1}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_2/p^{[\frac{m}{2}]+1}|\, A, q_2\leqslant k\}$$
$$\widetilde{C}_3^4(k) = D/p^{k}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{h}_1^0\zeta_1\zeta_2/p^{k}|\, q_2> k\}$$
$$\widetilde{C}_3^5(k) = D/p^{k}\{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{h}_2^0\zeta_1\zeta_2/p^{k}|\, q_2> k\}$$
$$\widetilde{C}_3^6(k) = D/p^{k}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_2/p^{k}|\, q_2> k\}$$
$$\widetilde{C}_4(k) = D/p^{k}\{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{h}_1^0\widehat{h}_2^0\zeta_1\zeta_2/p^{k}|\, q_2> k\}$$

The modules $\widetilde{C}_i(k)\,(0\leqslant i\leqslant 4)$ and $\widetilde{\zeta_2 C_0}(k)$ form basic building blocks of $H^*L(k,v_1)$. Noted that the first non-trivial Adams-Novikov differential is $d_{2p-1}$ and for $s>4$ $\mathrm{Ext}^s_{BP_*BP}(BP_*, BP_*L_2T(m)/(p^k,v_1))=0$ , we conclude that the Adams-Novikov spectral sequence for $\pi_*(L_2T(m)/(p^k,v_1))$ collapses. Thus
$$H^*L(k,v_1)\cong \pi_*(L_2T(m)/(p^k,v_1)).$$
The main theorem of this paper is as follows.

**Theorem 2.1.** *If $1\leqslant k\leqslant [\frac{m}{2}]+2$, then the homology group $H^*L(k,v_1)$ and the homotopy group $\pi_*(L_2T(m)/(p^k,v_1))$ are isomorphic to the direct sum of*
$$\left(\widetilde{C}_0(k)\oplus\widetilde{C}_1(k)\right)\otimes E[\zeta_1]$$
*and*
$$\widetilde{\zeta_2 C_0}(k)\oplus\widetilde{C}_2(k)\oplus\widetilde{C}_3(k)\oplus\widetilde{C}_4(k)$$

As a special case, if $k=[\frac{m}{2}]+2$, we can obtain the following corollary.

**Corollary 2.2.** *The homology group $H^*L([\frac{m}{2}]+2,v_1)$ and then the homotopy group $\pi_*(L_2T(m)/(p^{[\frac{m}{2}]+2},v_1))$ are isomorphic to the direct sum of*
$$\left(\widetilde{C}_0\left([\tfrac{m}{2}]+2\right)\oplus\widetilde{C}_1([\tfrac{m}{2}]+2)\right)\otimes E[\zeta_1]$$

*and*

$$\widetilde{\zeta_2 C_0}\left([\tfrac{m}{2}]+2\right)\oplus \widetilde{C}_2\left([\tfrac{m}{2}]+2\right)\oplus \widetilde{C}_3\left([\tfrac{m}{2}]+2\right)\oplus \widetilde{C}_4\left([\tfrac{m}{2}]+2\right).$$

## 3. Some elements in the cobar complex

The structure maps $\eta_R$ and $\Delta$ of $(E_m(2)_*, \Gamma(2,m))$ are induced from those of $(BP_*, BP_*BP)$. Let $v_i$ be the Hazewinkel's generators. In $(BP_*, BP_*BP)$, $\eta_R$ and $\Delta$ are defined as follows:

$$\eta_R(m_n) = \sum_{i+j=n} m_i t_j^{p^i}, \tag{3.1}$$

$$\sum_{i+j=n} m_i \Delta(t_j)^{p^i} = \sum_{i+j+k=n} m_i t_j^{p^i} \otimes t_k^{p^{i+j}}. \tag{3.2}$$

$$v_n = pm_n - \sum_{i=1}^{n-1} m_i v_{n-i}^{p^i} \tag{3.3}$$

Furthermore, for the map $\Delta$, we have the following lemma

**Lemma 3.1.** *In the Hopf algebroid $(BP_*, BP_*BP)$, for $k \geqslant -1, n \geqslant 0$,*

$$\Delta(t_n^{p^{k+1}}) = \sum_{i=1}^{n-1} v_i^{p^{k+1}} b_{n-i}^{k+i} + \sum_{i+j=n} t_i^{p^{k+1}} \otimes t_j^{p^{i+k+1}} - pb_n^k, \tag{3.4}$$

*where $b_n^{-1} = 0$, $b_n^k$ $(n \geqslant 1, k \geqslant 0)$ can be defined inductively by*

$$pb_n^k = \sum_{i=1}^{n-1} v_i^{p^{k+1}} b_{n-i}^{k+i} + \sum_{i+j=n} t_i^{p^{k+1}} \otimes t_j^{p^{i+k+1}} - \Delta(t_n)^{p^{k+1}}. \tag{3.5}$$

*Furthermore, for all $k \geqslant 1$, we have*

$$b_n^k \equiv (b_n^{k-1})^p \mod (p). \tag{3.6}$$

To prove this lemma, recall some basic notations and properties of $\Delta$ first. Let $I = (i_1, i_2, \cdots, i_m)$ be a finite (possibly empty) sequence of positive integers. Let $|I| = m$ and $\|I\| = \sum i_t$. Given sequences $I$ and $J$, let $IJ$ denote the sequence $(i_1, \cdots, i_m, j_1, \cdots, j_n)$. Then we have $|IJ| = |I| + |J|$ and $\|IJ\| = \|I\| + \|J\|$. For each sequence $I$, there is a symmetric integral polynomial of degree $p^{\|I\|}$ in any number of variables satisfying

(1) $w_\emptyset = \sum_t x_t$

(2) Let $K = (k_1, k_2, \cdots, k_m)$, $K'' = (k_1+k_2, \cdots, k_m)$, $K''' = (k_2, \cdots, k_m)$.

$$w_K = \tfrac{1}{p}(w_{K''} - w_{K'''}^{p^{k_1}}) \tag{3.7}$$

(3)

$$\sum_t x_t^{p^{\|K\|}} = \sum_{IJ=K} p^{|J|} w_J^{p^{\|I\|}} \tag{3.8}$$

(4) Let $w_n = w_n(x_1, x_2, \cdots)$ be the symmetric integral polynomial of degree $p^n$ such that

$$w_0 = \sum x_t \text{ and } \sum_t x_t^{p^n} = \sum_t p^j w_j^{p^{n-j}}.$$

Then

$$w_I \equiv w_{|I|}^{p^{\|I\|-|I|}} \mod (p). \tag{3.9}$$

Define $v_I$ by $v_\emptyset = 1$ and $v_I = v_{i_1}(v_{I'})^a$ where $a = p^{i_1}$ and $I' = (i_2, i_3, \cdots)$. Let $M_n = \{t_i \otimes t_{n-i}^{p^i} \,|\, 0 \leqslant i \leqslant n\}$ and let

$$\Delta_n = M_n \cup \bigcup_{|J|>0} \{v_J w_J(\Delta_{n-\|J\|})\}.$$

From Theorem 4.3.13 in Ravenel [Rav86], one can obtain

$$\Delta(t_n) = w_\emptyset(\Delta_n). \tag{3.10}$$

Our proof of Lemma 3.1 is based on Equation (3.10).

**Proof of Lemma 3.1.** Let $K = (k)$, Equation (3.8) implies

$$w_{(k)} = \tfrac{1}{p}\left(\sum x_t^{p^k} - w_\emptyset^{p^k}\right) \tag{3.11}$$

Let $b_n^k$ $(k \geqslant 0, n \geqslant 1)$ be defined as

$$b_n^k = \sum_J v_J^{p^{k+1}} w_{(k+1,J)}(\Delta_{n-\|J\|}). \tag{3.12}$$

This definition is equivalent to Equation (3.5). By induction on $n$. For $n = 1$,

$$\begin{aligned} b_1^k &= \tfrac{1}{p}\left(1 \otimes t_1^{p^{k+1}} + t_1^{p^{k+1}} \otimes 1 - (1 \otimes t_1 + t_1 \otimes 1)^{p^{k+1}}\right) \\ &= w_{(k+1)}(\Delta_1) = \sum_J v_J^{p^{k+1}} w_{(k+1,J)}(\Delta_{1-\|J\|}) \qquad \text{by (3.11)} \end{aligned}$$

From $n < m$ to $n = m$.

$$\begin{aligned} pb_m^k &= p\sum_J v_J^{p^{k+1}} w_{(k+1,J)}(\Delta_{m-\|J\|}) \quad \text{by (3.12)} \\ &= p\sum_{|J|>0} v_J^{p^{k+1}} w_{(k+1,J)}(\Delta_{m-\|J\|}) + pw_{(k+1)}(\Delta_m) \\ &= p\sum_{i=1}^{m-1}\sum_{J'} v_{(i,J')}^{p^{k+1}} w_{(k+1,i,J')}(\Delta_{m-i-\|J'\|}) + \sum_{i=1}^{m-1}\sum_{J'} v_{(i,J')}^{p^{k+1}} w_{(i,J')}^{p^{k+1}}(\Delta_{m-i-\|J'\|}) \\ &\quad + \sum_{i+j=m} t_i^{p^{k+1}} \otimes t_j^{p^{i+k+1}} - \Delta(t_m)^{p^{k+1}} \quad \text{by (3.11)} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{m-1} v_i^{p^{k+1}} \sum_{J'} v_{J'}^{p^{i+k+1}} (pw_{(k+1,i,J')} + w_{(i,J')}^{p^{k+1}})(\Delta_{m-i-\|J'\|}) \\
&\quad + \sum_{i+j=m} t_i^{p^{k+1}} \otimes t_j^{p^{i+k+1}} - \Delta(t_m)^{p^{k+1}} \\
&= \sum_{i=1}^{m-1} v_i^{p^{k+1}} \sum_{J'} v_{J'}^{p^{i+k+1}} w_{(k+1+i,J')}(\Delta_{m-i-\|J'\|}) \\
&\quad + \sum_{i+j=m} t_i^{p^{k+1}} \otimes t_j^{p^{i+k+1}} - \Delta(t_m)^{p^{k+1}} \quad \text{by } (3.7) \\
&= \sum_{i=1}^{m-1} v_i^{p^{k+1}} b_{m-i}^{k+i} + \sum_{i+j=m} t_i^{p^{k+1}} \otimes t_j^{p^{i+k+1}} - \Delta(t_m)^{p^{k+1}} \quad \text{by } (3.12)
\end{aligned}
$$

This completes the proof of equivalence of two definitions.

Next, we will prove Equation (3.4). By induction, if $k = -1$,

$$
\begin{aligned}
&\Delta(t_n) = w_\emptyset(\Delta_n) \\
&= \sum_{i+j=n} t_i \otimes t_j^{p^i} + \sum_{|J|>0} v_J w_J(\Delta_{n-\|J\|}) \\
&= \sum_{i=1}^{n-1} \sum_{J'} v_{(i,J')} w_{(i,J')}(\Delta_{n-i-\|J'\|}) + \sum_{i+j=n} t_i \otimes t_j^{p^i} \quad J = (i, J') \\
&= \sum_{i=1}^{n-1} v_i \sum_{J'} v_{J'}^{p^i} w_{(i,J')}(\Delta_{n-i-\|J'\|}) + \sum_{i+j=n} t_i \otimes t_j^{p^i} \\
&= \sum_{i=1}^{n-1} v_i b_{n-i}^{i-1} + \sum_{i+j=n} t_i \otimes t_j^{p^i} \quad \text{by } (3.12)
\end{aligned}
$$

If $k \geqslant 0$

$$
\begin{aligned}
&\Delta(t_n^{p^{k+1}}) = (w_\emptyset(\Delta_n))^{p^{k+1}} \quad \text{by } (3.10) \\
&= \sum_{i+j=n} t_i^{p^{k+1}} \otimes t_j^{p^{i+k+1}} + \sum_{|J|>0} v_J^{p^{k+1}} w_J^{p^{k+1}} (\Delta_{n-\|J\|}) - pw_{(k+1)}(\Delta_n) \\
&\qquad \text{by } (3.11)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1} \sum_{J'} v_{(i,J')}^{p^{k+1}} w_{(i,J')}^{p^{k+1}} (\Delta_{n-i-\|J'\|}) - pw_{(k+1)}(\Delta_n) \\
&\quad + \sum_{i+j=n} t_i^{p^{k+1}} \otimes t_j^{p^{i+k+1}} \quad J = (i, J')
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1}\sum_{J'} v_{(i,J')}^{p^{k+1}}\Big(w_{(k+1+i,J')}(\Delta_{n-i-\|J'\|})\\
&\quad - pw_{(k+1,i,J')}(\Delta_{n-i-\|J'\|})\Big) - pw_{(k+1)}(\Delta_n) + \sum_{i+j=n} t_i^{p^{k+1}}\otimes t_j^{p^{i+k+1}}\\
&\qquad\qquad\text{by (3.7)}\\
&= \sum_{i=1}^{n-1} v_i^{p^{k+1}}\sum_{J'} v_{J'}^{p^{i+k+1}} w_{(k+1+i,J')}(\Delta_{n-i-\|J'\|})\\
&\quad - \sum_{|(i,J')|>0} pv_{(i,J')}^{p^{k+1}} w_{(k+1,i,J')}(\Delta_{n-i-\|J'\|}) - pw_{(k+1)}(\Delta_n)\\
&\quad + \sum_{i+j=n} t_i^{p^{k+1}}\otimes t_j^{p^{i+k+1}} \quad \text{by (3.7)}\\
&= \sum_{i=1}^{n-1} v_i^{p^{k+1}} b_{n-i}^{k+i} - pb_n^k + \sum_{i+j=n} t_i^{p^{k+1}}\otimes t_j^{p^{i+k+1}} \quad \text{by (3.12)}
\end{aligned}
$$

This proves Equation (3.4) for $k \geqslant 0$.

For the final claim, the equation

$$w_I \equiv w_{|I|}^{p^{\|I\|-|I|}} \mod (p)$$

implies that

$$
\begin{aligned}
b_n^k &= \sum_J v_J^{p^{k+1}} w_{(k+1,J)}(\Delta_{n-\|J\|})\\
&\equiv \sum_J v_J^{p^{k+1}} w_{|J|+1}^{p^{\|J\|-|J|+k}}(\Delta_{n-\|J\|}) \qquad \mod (p).\\
(b_n^{k-1})^p &= \left(\sum_J v_J^{p^k} w_{(k,J)}(\Delta_{n-\|J\|})\right)^p\\
&\equiv \left(\sum_J v_J^{p^k} w_{|J|+1}^{p^{\|J\|-|J|+k-1}}(\Delta_{n-\|J\|})\right)^p \qquad \mod (p)\\
&\equiv \sum_J v_J^{p^{k+1}} w_{|J|+1}^{p^{\|J\|-|J|+k}}(\Delta_{n-\|J\|}) \qquad \mod (p)
\end{aligned}
$$

This completes the proof. □

Based on these formulas, we can compute $\eta_R$ and $\Delta$ in $(E_m(2)_*, \Gamma(2,m))$. With the notations defined in Equation (2.1), their structures are shown in the lemma below.

**Lemma 3.2.** *In the Hopf algebroid* $(E_m(2)_*, \Gamma(2,m))$, $\eta_R(v_i)$ $(1 \leqslant k \leqslant m+5)$ *and* $\Delta(t_n^{p^j})$ $(j \geqslant 0)$ *are given as follows:*

$$
\begin{aligned}
\eta_R(v_k) &= v_k \quad (1 \leqslant k \leqslant m)\\
\eta_R(\widehat{v}_1) &= \widehat{v}_1 + p\widehat{t}_1\\
\eta_R(\widehat{v}_2) &= \widehat{v}_2 + p\widehat{t}_2 \mod (v_1)\\
\eta_R(\widehat{v}_3) &= \widehat{v}_3 + v_2\widehat{t}_1^{p^2} - v_2^{p\omega}\widehat{t}_1 + p\widehat{t}_3 \mod (p^2, v_1)\\
\eta_R(\widehat{v}_4) &= \widehat{v}_4 + v_3\widehat{t}_1^{p^3} + v_2\widehat{t}_2^{p^2} - v_3^{p\omega}\widehat{t}_1\\
&\quad - v_2^{p^2\omega}\widehat{t}_2 + p\widehat{t}_4 \mod (p^2, v_1)\\
\eta_R(\widehat{v}_5) &= \widehat{v}_5 + v_2\widehat{t}_3^{p^2} - v_2^{p^3\omega}\widehat{t}_3 + v_4\widehat{t}_1^{p^4} - v_4^{p\omega}\widehat{t}_1\\
&\quad + v_3\widehat{t}_2^{p^3} - v_3^{p^2\omega}\widehat{t}_2 - v_2\widehat{u}_{3,2} + p\widehat{t}_5 \mod (p^2, v_1)\\
\Delta(t_n^{p^j}) &= t_n^{p^j} \otimes 1 + 1 \otimes t_n^{p^j} + \sum_{i=2}^{n-m-1} v_i^{p^j} b_{n-i}^{i+j-1}\\
&\quad - pb_n^{j-1} \quad (\mathit{for}\ m+1 \leqslant n \leqslant 2m+1) \mod (v_1)\\
\Delta(\widehat{\widehat{t}}_2^{p^j}) &= \widehat{\widehat{t}}_2^{p^j} \otimes 1 + 1 \otimes \widehat{\widehat{t}}_2^{p^j} + \widehat{t}_1^{p^j} \otimes \widehat{t}_1^{p^{j+1}\omega}\\
&\quad + \sum_{i=2}^{m+1} v_i^{p^j} \widehat{b}_{m+2-i}^{i+j-1} - p\widehat{\widehat{b}}_2^{j-1} \mod (v_1)\\
\Delta(\widehat{\widehat{t}}_3^{p^j}) &= \widehat{\widehat{t}}_3^{p^j} \otimes 1 + 1 \otimes \widehat{\widehat{t}}_3^{p^j} + \widehat{t}_1^{p^j} \otimes \widehat{t}_2^{p^{j+1}\omega}\\
&\quad + \widehat{t}_2^{p^j} \otimes \widehat{t}_1^{p^{j+2}\omega} + \sum_{i=2}^{m+2} v_i^{p^j} \widehat{b}_{m+3-i}^{i+j-1} - p\widehat{\widehat{b}}_3^{j-1} \mod (v_1)
\end{aligned}
$$

*where*

$$
\begin{aligned}
p\widehat{u}_{3,2} &= (\widehat{v}_3 + v_2\widehat{t}_1^{p^2} - v_2^{p\omega}\widehat{t}_1)^{p^2} - \widehat{v}_3^{p^2}\\
&\quad - v_2^{p^2}\widehat{t}_1^{p^4} + v_2^{p^3\omega}\widehat{t}_1^{p^2}.
\end{aligned}
$$

*Thus in* $\Gamma(2, m)$

$$
\begin{aligned}
v_2\widehat{t}_1^{p^2} &= v_2^{p\omega}\widehat{t}_1 - p\widehat{t}_3 \mod (p^2, v_1)\\
v_2\widehat{t}_2^{p^2} &= v_2^{p^2\omega}\widehat{t}_2 - v_3 v_2^{p^2\omega - p}\widehat{t}_1^{p} + v_3^{p\omega}\widehat{t}_1 - p\widehat{t}_4 \mod (p^2, v_1)\\
v_2\widehat{t}_3^{p^2} &= v_2^{p^3\omega}\widehat{t}_3 - v_4\widehat{t}_1^{p^4} + v_4^{p\omega}\widehat{t}_1 - v_3\widehat{t}_2^{p^3} + v_3^{p^2\omega}\widehat{t}_2\\
&\quad - p\widehat{t}_5 \mod (p^2, v_1)
\end{aligned}
\tag{3.13}
$$

**Proof.** From Equations (3.1) and (3.3), we have $\eta_R(v_i)\,(1 \leqslant k \leqslant m+5)$ by induction. In $\Gamma(2, m)$, if $i \geqslant m+3$, then $v_i = 0$. Thus from $\eta_R(\widehat{v}_3)$, $\eta_R(\widehat{v}_4)$ and $\eta_R(\widehat{v}_5)$, we can obtain the equivalence relations in (3.13). In the computations, noted that $\widehat{u}_{3,2} \equiv 0 \mod (p^2, v_1)$.

The computations of $\Delta$ are directly from Lemma 3.1. □

Apply the structure map of $\eta_R$ and $\Delta$ in $(E_m(2)_*, \Gamma(2,m))$, we can construct two elements $\zeta_1$, $\zeta_2$ with the following properties. The proofs are shown in Section 5.

**Lemma 3.3.** *In the cobar complex $\Omega^1_{\Gamma(2,m)}(E_m(2)_*/(p^{[\frac{m}{2}]+2}, v_1))$, there is a cocycle $\zeta_1$ for each $m$.*

$$\zeta_1 = \begin{cases} \widehat{t}_2^p + \cdots & \text{if } m \text{ is even,} \\ \widehat{t}_1^p + \cdots & \text{if } m \text{ is odd.} \end{cases}$$

**Lemma 3.4.** *In the cobar complex $\Omega^1_{\Gamma(2,m)}(E_m(2)_*)/(p^{[\frac{m}{2}]+2}, v_1))$, there is a cochain $\zeta_2$. If $m$ is even, then*

$$\zeta_2 = \widehat{t}_1^p + \cdots, \qquad d\zeta_2 = p^{[\frac{m}{2}]+1}\widehat{t}_2^p \otimes \widehat{t}_1^p$$

*If $m$ is odd, then*

$$\zeta_2 = \widehat{t}_2^p + \cdots, \qquad d\zeta_2 = p^{[\frac{m}{2}]+1}\widehat{t}_1^p \otimes \widehat{t}_2^p$$

## 4. The connecting homomorphisms

Recall that the short exact sequence

$$0 \to M_2^0 \xrightarrow{p^k} L(k+1,1) \longrightarrow L(k,1) \to 0$$

induces a long exact sequence

$$\cdots \to H^{s,*}M_2^0 \xrightarrow{p^k} H^{s,*}L(k+1,1) \to H^{s,*}L(k,1) \xrightarrow{\delta_s} H^{s+1,*}M_2^0 \to \cdots.$$

Let $q_1 = \min\{n+1, l+1\}$ and $q_2 = \min\{n+1, l+1, [\frac{m}{2}]+1\}$. The connecting homomorphisms of this long exact sequence will be explored.

**Lemma 4.1.** *For the connecting homomorphism $\delta_0 : H^0L(k,1) \longrightarrow H^1M_2^0$, we have that*

(1) *In $\widetilde{C}_0^1(k)$*

$$\delta_0\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}}{p^{l+1}}\right) = t\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{t}_2$$

(2) *In $\widetilde{C}_0^2(k)$*

$$\delta_0\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}}{p^{n+1}}\right) = s\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_1.$$

(3) *In $\widetilde{C}_0^3(k)$, $\delta_0 = 0$.*

**Proof.** If $q_1 \leqslant k$, it is a direct computation from

$$\begin{aligned} d(\widehat{v}_1^{sp^n}) &= sp^{n+1}\widehat{v}_1^{sp^n-1}\widehat{t}_1 + \cdots \\ d(\widehat{v}_2^{tp^l}) &= tp^{l+1}\widehat{v}_2^{tp^l-1}\widehat{t}_2 + \cdots \qquad \bmod (v_1). \end{aligned}$$

If $q_1 > k$, in $H^1M_2^0$, we obtain that

$$\delta_0\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}}{p^k}\right) = p^{q_1-k}\delta_0\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}}{p^{q_1}}\right) = 0$$

□

From Lemma 4.1, It is concluded that the cokernel of $\delta_0$ is $C_1(k) \oplus \zeta_1 C_0(k) \oplus \zeta_2 C_0(k)$.

**Lemma 4.2.** *The connecting homomorphism $\delta_1 : H^1L(k,1) \longrightarrow H^2M_2^0$ acts on the sub-modules $\widetilde{C}_1(k) \oplus \widetilde{\zeta_2 C}_0(k)$ as:*

(1) *In $\widetilde{C}_1^1(k)$*

$$\delta_1\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{t}_2}{p^{n+1}}\right) = s\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{t}_1\widehat{t}_2.$$

*In $\widetilde{C}_1^2(k)$*

$$\delta_1\left(\frac{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_1}{p^{l+1}}\right) = t\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{t}_1\widehat{t}_2.$$

(2) *In $\widetilde{C}_1^5(k)$*

$$\delta_1\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2}{p^{[\frac{m}{2}]+1}}\right) = \widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2.$$

(3) *In $\widetilde{C}_1^6(k)$*

$$\delta_1\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2}{p^{n+1}}\right) = s\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_1\zeta_2.$$

(4) *In $\widetilde{C}_1^7(k)$*

$$\delta_1\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2}{p^{l+1}}\right) = t\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{t}_2\zeta_2.$$

(5) *In $\widetilde{C}_1^3(k)$, $\widetilde{C}_1^4(k)$ and $\widetilde{C}_1^8(k)$, $\delta_1 = 0$.*

**Proof.** From

$$\begin{aligned} d(\widehat{v}_1^{sp^n}) &\equiv sp^{n+1}\widehat{v}_1^{sp^n-1}\widehat{t}_1 + \cdots, \\ d(\widehat{v}_2^{tp^l}) &\equiv tp^{l+1}\widehat{v}_2^{tp^l-1}\widehat{t}_2 + \cdots \qquad \bmod (v_1) \end{aligned} \tag{4.1}$$

we conclude that for $s \neq 0$, $t \neq 0$

$$\begin{aligned} d(\widehat{v}_1^{sp^n-1}\widehat{t}_1 + \cdots) &\equiv 0, \\ d(\widehat{v}_2^{tp^l-1}\widehat{t}_2 + \cdots) &\equiv 0 \qquad \bmod (v_1). \end{aligned}$$

This implies the equations in (1). Equations in (2)-(4) are concluded from Equation (4.1) and $\bmod (p^{[\frac{m}{2}]+2}, v_1)$

$$d\zeta_2 = p^{[\frac{m}{2}]+1}\widehat{t}_2^p \otimes \widehat{t}_1^p \quad (m \text{ is even})$$

$$d\zeta_2 = p^{[\frac{m}{2}]+1}\widehat{t}_1^p \otimes \widehat{t}_2^p \quad (m \text{ is odd}).$$

If $q_1 > k$

$$\delta_1\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{t}_2}{p^k}\right) = p^{q_1-k}\delta_1\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{t}_2}{p^{q_1}}\right) = 0$$

$$\delta_1\left(\frac{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_2}{p^k}\right) = p^{q_1-k}\delta_1\left(\frac{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_2}{p^{q_1}}\right) = 0$$

If $q_2 > k$

$$\delta_1\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2}{p^k}\right) = p^{q_2-k}\delta_1\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_2}{p^{q_2}}\right) = 0$$

□

**Lemma 4.3.** *The connecting homomorphism $\delta_2 : H^2L(k,1) \longrightarrow H^3M_2^0$ is:*

(1) *In $\widetilde{C}_2^1(k)$*

$$\delta_2\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2}{p^{n+1}}\right) = s\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_1\zeta_1\zeta_2$$

*In $\widetilde{C}_2^2(k)$*

$$\delta_2\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}\zeta_1\zeta_2}{p^{l+1}}\right) = t\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{t}_2\zeta_1\zeta_2$$

(2) *In $\widetilde{C}_2^3(k)$*

$$\delta_2\left(\frac{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_1\zeta_2}{p^{[\frac{m}{2}]+1}}\right) = \widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_1\zeta_1\zeta_2.$$

*In $\widetilde{C}_2^4(k)$*

$$\delta_2\left(\frac{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_1\zeta_2}{p^{l+1}}\right) = t\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{t}_1\widehat{t}_2\zeta_2.$$

(3) *In $\widetilde{C}_2^5(k)$*

$$\delta_2\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{t}_2\zeta_2}{p^{[\frac{m}{2}]+1}}\right) = \widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{t}_2\zeta_1\zeta_2$$

*In $\widetilde{C}_2^6(k)$*

$$\delta_2\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l-1}\widehat{t}_2\zeta_2}{p^{n+1}}\right) = s\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{t}_1\widehat{t}_2\zeta_2$$

(4) *In $\widetilde{C}_2^i(k)\,(7 \leqslant i \leqslant 10)$, $\delta_2 = 0$.*

**Proof.** Noted that $d(\zeta_1) \equiv 0 \mod (p^{[\frac{m}{2}]+2}, v_1)$. It is obvious by similar discussions in proofs of Lemma 4.1 and 4.2. □

**Lemma 4.4.** *The connecting homomorphism $\delta_3 : H^3L(k,1) \longrightarrow H^4M_2^0$ is:*

(1) *In* $\widetilde{C}_3^1(k)$

$$\delta_3\left(\frac{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l}\widehat{t}_1\zeta_1\zeta_2}{p^{l+1}}\right)=t\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{t}_1\widehat{t}_2\zeta_1\zeta_2$$

(2) *In* $\widetilde{C}_3^2(k)$

$$\delta_3\left(\frac{\widehat{v}_1^{sp^n}\widehat{v}_2^{tp^l}-1\widehat{t}_2\zeta_1\zeta_2}{p^{n+1}}\right)=s\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{t}_1\widehat{t}_2\zeta_1\zeta_2$$

(3) *In* $\widetilde{C}_3^3(k)$

$$\delta_3\left(\frac{\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{t}_1\widehat{t}_2\zeta_2}{p^{[\frac{m}{2}]+1}}\right)=\widehat{v}_1^{sp^n-1}\widehat{v}_2^{tp^l-1}\widehat{t}_1\widehat{t}_2\zeta_1\zeta_2$$

(4) *In* $\widetilde{C}_3^i(k)\,(4\leqslant i\leqslant 6)$, $\delta_3=0$.

**Proof.** It is obvious. □

**Lemma 4.5.** *The connecting homomorphism* $\delta_4: H^4L(k,1)\longrightarrow H^5M_2^0$ *is zero.*

**Proof.** Since $H^5M_2^0=0$, it is clear that $\delta_4=0$. □

**Proof of Theorem 2.1.** From the connecting homomorphisms $\delta_i\,(0\leqslant i\leqslant 4)$, The following exact sequences can be constructed.

$$0\to C_0(k)\to\widetilde{C}_0(k+1)\to\widetilde{C}_0(k)\xrightarrow{\delta_0}I_1(k)\to 0$$

$$0\to C_1(k)\to\widetilde{C}_1(k+1)\to\widetilde{C}_1(k)\xrightarrow{\delta_1}I_2^a(k)\to 0$$
$$0\to\zeta_1C_0(k)\to\zeta_1\widetilde{C}_0(k+1)\to\zeta_1\widetilde{C}_0(k)\xrightarrow{\delta_1}\zeta_1I_1(k)\to 0$$
$$0\to\zeta_2C_0(k)\to\widetilde{\zeta_2C}_0(k+1)\to\widetilde{\zeta_2C}_0(k)\xrightarrow{\delta_1}I_2^b(k)\to 0$$

$$0\to\zeta_1C_1(k)\to\zeta_1\widetilde{C}_1(k+1)\to\zeta_1\widetilde{C}_1(k)\xrightarrow{\delta_2}\zeta_1I_2^a(k)\to 0$$
$$0\to C_2(k)\to\widetilde{C}_2(k+1)\to\widetilde{C}_2(k)\xrightarrow{\delta_2}I_3(k)\to 0$$

$$0\to C_3(k)\to\widetilde{C}_3(k+1)\to\widetilde{C}_3(k)\xrightarrow{\delta_3}I_4(k)\to 0$$

$$0\to C_4(k)\to\widetilde{C}_4(k+1)\to\widetilde{C}_4(k)\xrightarrow{\delta_4}0$$

From the structure of $H^*M_2^0$

$$H^0M_2^0=C_0(k)$$

$$H^1M_2^0 = I_1(k) \oplus C_1(k) \oplus \zeta_1 C_0(k) \oplus \zeta_2 C_0(k)$$
$$H^2M_2^0 = I_2^a(k) \oplus \zeta_1 I_1(k) \oplus \zeta_1 C_1(k) \oplus I_2^b(k) \oplus C_2(k)$$
$$H^3M_2^0 = I_3(k) \oplus C_3(k)$$
$$H^4M_2^0 = I_4(k) \oplus C_4(k)$$

It is easy to prove that

$$\operatorname{coker}\delta_0 = C_1(k) \oplus \zeta_1 C_0(k) \oplus \zeta_2 C_0(k)$$
$$\operatorname{coker}\delta_1 = \zeta_1 C_1(k) \oplus C_2(k)$$
$$\operatorname{coker}\delta_2 = C_3(k)$$
$$\operatorname{coker}\delta_3 = C_4(k)$$

Consequently, we can construct $p$-torsion submodules $B^*(k)$ of $H^*L(k,1)$

$$B^0(k) = \widetilde{C}_0(k)$$
$$B^1(k) = \widetilde{C}_1(k) \oplus \zeta_1\widetilde{C}_0(k) \oplus \widetilde{\zeta_2 C}_0(k)$$
$$B^2(k) = \widetilde{C}_2(k) \oplus \zeta_1\widetilde{C}_1(k)$$
$$B^3(k) = \widetilde{C}_3(k)$$
$$B^4(k) = \widetilde{C}_4(k)$$
$$B^i(k) = 0\,(i>4)$$

such that the following diagram is commutative

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^sM_2^0 & \xrightarrow{p^k} & B^s(k+1) & \longrightarrow & B^s(k) & \xrightarrow{\delta_s} & H^{s+1}M_2^0 & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & H^sM_2^0 & \xrightarrow{p^k} & H^sL(k+1,1) & \longrightarrow & H^sL(k,1) & \xrightarrow{\delta_s} & H^{s+1}M_2^0 & \longrightarrow \cdots
\end{array}$$

Thus from the Bockstein spectral sequence (see [MiRW77], Remark 3.11), we conclude that

$$H^*L(k,1) \cong \Big((\widetilde{C}_0(k) \oplus \widetilde{C}_1(k)) \otimes E[\zeta_1]\Big) \oplus \widetilde{\zeta_2 C_0}(k) \oplus \widetilde{C}_2(k) \oplus \widetilde{C}_3(k) \oplus \widetilde{C}_4(k)$$

□

## 5. Differentials of $\widehat{h}_1^1$ and $\widehat{h}_2^1$

In this section, we will prove Lemma 3.3 and Lemma 3.4. The constructions of elements $\zeta_1$ and $\zeta_2$ are based on a collection of elements $\widehat{c}_{i,j} = c_{m+i,j}\,(1 \leqslant i \leqslant m+3, j \geqslant 1)$.

$$\widehat{c}_{1,j} = \widehat{t}_1^{p^j}, \quad \widehat{c}_{2,j} = \widehat{t}_2^{p^j},$$

$$\widehat{c}_{k,j} = \sum_{i=2}^{k-1} v_i^{p^j} p^{g(k,k-i)-1} \widehat{c}_{k-i,i+j} + p^{g(k)-1} \widehat{t}_k^{p^j} \quad (3 \leqslant k \leqslant m+3).$$

where

$$g(k) = \left[\tfrac{k+1}{2}\right], \quad g(k,t) = \left[\tfrac{k+1}{2}\right] - \left[\tfrac{t+1}{2}\right].$$

The leading item of $\widehat{c}_{i,j}$ is related to $\widehat{t}_1, \widehat{t}_1^p, \widehat{t}_2, \widehat{t}_2^p$. Define $\bar{j}$ as following.

$$\bar{j} = \begin{cases} 0 & j \text{ is even,} \\ 1 & j \text{ is odd.} \end{cases}$$

**Lemma 5.1.** *For $k \geqslant 1, j \geqslant 1$, we have*

$$\widehat{c}_{k,j} \equiv \begin{cases} \widehat{t}_1^{p^{\bar{j}}} \mod (p, v_1, v_2 - 1) & k \textit{ is odd,} \\ \widehat{t}_2^{p^{\bar{j}}} \mod (p, v_1, v_2 - 1, \widehat{t}_1^p, \widehat{t}_1) & k \textit{ is even.} \end{cases}$$

**Proof.** From Equation (3.13), we have

$$\begin{aligned} \widehat{t}_1^{p^2} &= \widehat{t}_1 \mod (p, v_1, v_2 - 1) \\ \widehat{t}_2^{p^2} &= \widehat{t}_2 - v_3 \widehat{t}_1^p + v_3^{p\omega} \widehat{t}_1 \mod (p, v_1, v_2 - 1) \\ &\equiv \widehat{t}_2 \mod (p, v_1, v_2 - 1, \widehat{t}_1^p, \widehat{t}_1) \end{aligned} \tag{5.1}$$

Thus, if $k = 1$, then

$$\widehat{c}_{1,j} = \widehat{t}_1^{p^j} \equiv \widehat{t}_1^{p^{\bar{j}}} \mod (p, v_1, v_2 - 1)$$

If $k$ is odd and $3 \leqslant k \leqslant m+3$, then from the definition of $\widehat{c}_{k,j}$,

$$\widehat{c}_{k,j} \equiv \widehat{c}_{k-2,2+j} \mod (p, v_2 - 1)$$

By induction, $\mod (p, v_1, v_2 - 1)$

$$\widehat{c}_{k,j} \equiv \widehat{c}_{1,j+k-1} \equiv \widehat{t}_1^{p^{\overline{j+k-1}}} \equiv \widehat{t}_1^{p^{\bar{j}}}$$

If $k = 2$, then

$$\widehat{c}_{2,j} = \widehat{t}_2^{p^j} \equiv \widehat{t}_2^{p^{\bar{j}}} \mod (p, v_1, v_2 - 1, \widehat{t}_1^p, \widehat{t}_1)$$

If $k$ is even and $3 \leqslant k \leqslant m+3$, from the definition of $\widehat{c}_{k,j}$,

$$\begin{aligned} \widehat{c}_{k,j} &\equiv v_2^{p^j} \widehat{c}_{k-2,2+j} + v_3^{p^j} \widehat{c}_{k-3,3+j} \mod (p) \\ &\equiv \widehat{c}_{k-2,2+j} \mod (p, v_1, v_2 - 1, \widehat{t}_1^p, \widehat{t}_1) \quad (k-3 \text{ is odd}) \end{aligned}$$

By induction, $\mod (p, v_1, v_2 - 1, \widehat{t}_1^p, \widehat{t}_1)$

$$\widehat{c}_{k,j} \equiv \widehat{c}_{2,j+k-2} \equiv \widehat{t}_2^{p^{\overline{j+k-2}}} \equiv \widehat{t}_2^{p^{\bar{j}}}$$

This completes the proof. ∎

Apply Lemma 3.2, we can obtain the differentials of $\widehat{c}_{i,j} \mod (v_1)$.

$$
\begin{aligned}
d\widehat{c}_{k,j} &= p^{g(k)}\widehat{b}_k^{j-1} \quad (1\leqslant k\leqslant m+1),\\
d(\widehat{\widehat{c}}_{2,j}) &= p^{g(m+2)}\widehat{\widehat{b}}_2^{j-1} + p^{g(m+2)-2}d(\widehat{v}_1^{p^j})\widehat{c}_{1,m+1+j}\\
&\quad - p^{g(m+2)-1}\widehat{t}_1^{p^j}\otimes \widehat{t}_1^{p^{j+1}\omega},\\
d(\widehat{\widehat{c}}_{3,j}) &= p^{g(m+3)}\widehat{\widehat{b}}_3^{j-1}\\
&\quad + p^{g(m+3)-2}[d(\widehat{v}_1^{p^j})\widehat{c}_{2,m+1+j} + d(\widehat{v}_2^{p^j})\widehat{c}_{1,m+2+j}]\\
&\quad - p^{g(m+3)-1}(\widehat{t}_1^{p^j}\otimes \widehat{t}_2^{p^{j+1}\omega} + \widehat{t}_2^{p^j}\otimes \widehat{t}_1^{p^{j+2}\omega}).
\end{aligned}
\tag{5.2}
$$

Thus, if $m=2s$

$$
\begin{aligned}
d(\widehat{\widehat{c}}_{2,j}) &= -p^s\widehat{t}_1^{p^j}\otimes \widehat{t}_1^{p^{j+1}\omega} + p^{s+1}\widehat{\widehat{b}}_2^{j-1} + p^{s-1}d(\widehat{v}_1^{p^j})\widehat{c}_{1,2s+1+j},\\
d(\widehat{\widehat{c}}_{3,1}) &\equiv -p^{s+1}(\widehat{t}_1^{p}\otimes \widehat{t}_2^{p^2\omega} + \widehat{t}_2^{p}\otimes \widehat{t}_1^{p^3\omega}) \mod (p^{s+2},v_1),
\end{aligned}
$$

If $m=2s-1$

$$
\begin{aligned}
d(\widehat{\widehat{c}}_{2,j}) &= -\,p^s\widehat{t}_1^{p^j}\otimes \widehat{t}_1^{p^{j+1}\omega} + p^{s+1}\widehat{\widehat{b}}_2^{j-1} + p^{s-1}d(\widehat{v}_1^{p^j})\widehat{c}_{1,2s+j},\\
d(\widehat{\widehat{c}}_{3,1}) &\equiv -\,p^{s}(\widehat{t}_1^{p}\otimes \widehat{t}_2^{p^2\omega} + \widehat{t}_2^{p}\otimes \widehat{t}_1^{p^3\omega}) \mod (p^{s+1},v_1),
\end{aligned}
$$

We will construct $\zeta_1$ from $\widehat{\widehat{c}}_{2,1}$ and construct $\zeta_2$ from $\widehat{\widehat{c}}_{3,1}$, respectively. Before the construction, a technical lemma will be proved first.

**Lemma 5.2.** *Let $i,j,k$ be integers in $\{0,1\}$. There exists a non-negative integer $N$ depending on $i,j,k$ such that*

$$p^N\widehat{t}_1^{p^j}\otimes \widehat{t}_{1+i}^{p^k}$$

*is a coboundary in the cobar complex $\Omega_{\Gamma(2,m)}(E_m(2)_*/(p^{N+1},v_1))$.*

**Proof.** In the following, the underlined elements with the same subscripts amount to zero.

(1) $\widehat{t}_1\otimes\widehat{t}_1$, $\widehat{t}_1^p\otimes\widehat{t}_1^p$, $\widehat{t}_2\otimes\widehat{t}_2$, $\widehat{t}_2^p\otimes\widehat{t}_2^p$

$$
\begin{aligned}
d(-\tfrac{1}{2}\widehat{t}_1^2) &= \widehat{t}_1\otimes\widehat{t}_1,\\
d(-\tfrac{1}{2}\widehat{t}_1^{2p}) &= \widehat{t}_1^p\otimes\widehat{t}_1^p \mod (p)\\
d(-\tfrac{1}{2}\widehat{t}_2^2) &= \widehat{t}_2\otimes\widehat{t}_2 \mod (v_1)\\
d(-\tfrac{1}{2}\widehat{t}_2^{2p}) &= \widehat{t}_2^p\otimes\widehat{t}_2^p \mod (p,v_1)
\end{aligned}
$$

(2) $\widehat{t}_1 \otimes \widehat{t}_2$

$$d(\widehat{v}_1\widehat{t}_2) = p\widehat{t}_1 \otimes \widehat{t}_2$$

(3) $\widehat{t}_1 \otimes \widehat{t}_1^p$

$$\begin{aligned} d(\widehat{v}_1\widehat{t}_1^{p^3}) &= p\widehat{t}_1 \otimes \widehat{t}_1^{p^3} + \underline{\widehat{v}_1 \cdot p\widehat{b}_1^2}_1, \\ d(pv_2^{-p}\widehat{v}_1\widehat{t}_3^p) &= p^2 v_2^{-p}\widehat{t}_1\widehat{t}_3^p - \underline{p\widehat{v}_1\widehat{b}_1^2}_1 \\ &\quad + p^2 v_2^{-p}\widehat{v}_1\widehat{b}_3^0, \end{aligned}$$

Then

$$\begin{aligned} d(\widehat{v}_1\widehat{t}_1^{p^3} + pv_2^{-p}\widehat{v}_1\widehat{t}_3^p) &\equiv p\widehat{t}_1 \otimes \widehat{t}_1^{p^3} \mod (p^2) \\ &\equiv pv_2^{p^2\omega - p}\widehat{t}_1 \otimes \widehat{t}_1^p \mod (p^2, v_1) \end{aligned}$$

Let $a = v_2^{p-p^2\omega}(\widehat{v}_1\widehat{t}_1^{p^3} + pv_2^{-p}\widehat{v}_1\widehat{t}_3^p)$. Thus

$$d(a) \equiv p\widehat{t}_1 \otimes \widehat{t}_1^p \mod (p^2, v_1).$$

(4) $\widehat{t}_1 \otimes \widehat{t}_2^p$

$$\begin{aligned} d(\widehat{v}_1\widehat{t}_2^{p^3}) &= p\widehat{t}_1 \otimes \widehat{t}_2^{p^3} + \underline{p\widehat{v}_1\widehat{b}_2^2}_1, \\ d(pv_2^{-p}\widehat{v}_1\widehat{t}_4^p) &\equiv -\underline{p\widehat{v}_1\widehat{b}_2^2}_1 - \underline{pv_2^{-p}v_3^p\widehat{v}_1\widehat{b}_1^3}_2 \mod (p^2) \\ d(-pv_2^{-p^2-p}v_3^p\widehat{v}_1\widehat{t}_3^{p^2}) &\equiv \underline{pv_2^{-p}v_3^p\widehat{v}_1\widehat{b}_1^3}_2 \mod (p^2). \end{aligned}$$

These imply that $p\widehat{t}_1 \otimes \widehat{t}_2^{p^3}$ is a coboundary modulo $(p^2)$. Furthermore,

$$\begin{aligned} p\widehat{t}_1 \otimes \widehat{t}_2^{p^3} \equiv p\widehat{t}_1 \otimes (&v_2^{p^2\omega-1}\widehat{t}_2 - v_3v_2^{p^2\omega-p-1}\widehat{t}_1^p \\ &+ v_2^{-1}v_3^{p\omega}\widehat{t}_1)^p \mod (p^2, v_1) \end{aligned}$$

From (1) and (3), $p\widehat{t}_1 \otimes \widehat{t}_1$, $p\widehat{t}_1 \otimes \widehat{t}_1^p \mod (p^2, v_1)$ are coboundaries. Hence we can find an element $a_2$ such that

$$d(a_2) \equiv p\widehat{t}_1 \otimes \widehat{t}_2^p \mod (p^2, v_1)$$

(5) $\widehat{t}_2 \otimes \widehat{t}_1^p$, $\widehat{t}_2 \otimes \widehat{t}_2^p$

By similar discussions in (3) and (4), we can obtain $a_3$, $a_4$ such that

$$d(a_3) \equiv p\widehat{t}_2 \otimes \widehat{t}_1^p \mod (p^2, v_1),$$

$$d(a_4) \equiv p\widehat{t}_2 \otimes \widehat{t}_2^p \mod (p^2, v_1).$$

(6) $\widehat{t}_1^p \otimes \widehat{t}_2^p$
It will be proved in Lemma 3.4.

□

Based on this lemma, we will finish the constructions of $\zeta_1, \zeta_2$ and prove Lemma 3.3 and Lemma 3.4.

**Proof of Lemma 3.4.** If $m = 2s$, $[\frac{m}{2}] + 2 = s + 2$. Equation (5.2) implies that $\mod (p^{s+2}, v_1, v_2 - 1)$

$$\begin{aligned} d(\widehat{\widehat{c}}_{3,1}) &\equiv -p^{s+1}(\widehat{t}_1^p \otimes \widehat{t}_2^{p^2\omega} + \widehat{t}_2^p \otimes \widehat{t}_1^{p^3\omega}) \\ &\equiv -p^{s+1}(\widehat{t}_1^p \otimes \widehat{t}_2^{p^2\omega} + \widehat{t}_2^p \otimes \widehat{t}_1^p) \end{aligned}$$

From Equation (3.13),

$$v_2\widehat{t}_2^{p^2} = v_2^{p^2\omega}\widehat{t}_2 - v_3v_2^{p^2\omega-p}\widehat{t}_1^p + v_3^{p\omega}\widehat{t}_1 \mod (p, v_1)$$

Thus, by induction, $\widehat{t}_1^p \otimes \widehat{t}_2^{p^2\omega}$ is a linear combination of $\widehat{t}_1^p \otimes \widehat{t}_2, \widehat{t}_1^p \otimes \widehat{t}_1^p, \widehat{t}_1^p \otimes \widehat{t}_1$ with coefficients in $\mathbb{Z}/p[v_2, v_3]$. This implies that there exists an element $\zeta_2$ such that

$$d(\zeta_2) = p^{s+1}\widehat{t}_2^p \otimes \widehat{t}_1^p \mod (p^{s+2}, v_1)$$

Furthermore, $\zeta_2$'s leading item is same as the one of $\widehat{\widehat{c}}_{3,1}$, which is $\widehat{t}_1^p$.

If $m = 2s-1$, $[\frac{m}{2}]+2 = s+1$. Equation (5.2) shows that $\mod (p^{s+1}, v_1, v_2-1)$

$$\begin{aligned} d(\widehat{\widehat{c}}_{3,1}) &\equiv -p^{s}(\widehat{t}_1^p \otimes \widehat{t}_2^{p^2\omega} + \widehat{t}_2^p \otimes \widehat{t}_1^{p^3\omega}) \\ &\equiv -p^{s+1}(\widehat{t}_1^p \otimes \widehat{t}_2^{p^2\omega} + \widehat{t}_2^p \otimes \widehat{t}_1^p) \end{aligned}$$

By similar discussions as the case $m = 2s$, we have an element $\zeta_2$ with a leading item $\widehat{t}_1^p$ and

$$d(\zeta_2) = p^s\widehat{t}_1^p \otimes \widehat{t}_2^p \mod (p^{s+1}, v_1)$$

This completes the proof. □

**Proof of Lemma 3.3.** If $m = 2s$, then $[\frac{m}{2}] + 2 = s + 2$. Equation (5.2) implies the differential of $c_{2m+2,3}$ is as follows

$$d(\widehat{\widehat{c}}_{2,3}) = -p^s\widehat{t}_1^{p^3} \otimes \widehat{t}_1^{p^4\omega} + p^{s+1}\widehat{\widehat{b}}_2^2 + p^{s-1}d(\widehat{v}_1^{p^3})\widehat{c}_{1,2s+4}$$

Since $d(\widehat{v}_1^{p^3}) \equiv 0 \mod (p^3)$, we can obtain the following equivalence relation modulo $(p^{s+2}, v_1)$. The underlined elements with the same subscripts amount to zero.

$$
\begin{aligned}
d(\widehat{\widehat{c}}_{2,3}) &\equiv -p^s\widehat{t}_1^{p^3}\otimes\widehat{t}_1^{p^4\omega}+p^{s+1}\widehat{\widehat{b}}{}_2^2\\
&\equiv -p^s v_2^N\widehat{t}_1^{p^3}\otimes\widehat{t}_1^{p^2}+p^{s+1}\widehat{\widehat{b}}{}_2^2 \quad \text{by (3.13)}\\
&\equiv -\underline{p^s v_2^{N+p\omega-1}\widehat{t}_1^{p^3}\otimes\widehat{t}_1}_1\\
&\quad+\underline{p^{s+1}v_2^{N+p^2\omega-p-1}\widehat{t}_1^{p}\otimes\widehat{t}_3}_2+\underline{p^{s+1}\widehat{\widehat{b}}{}_2^2}_3 \quad \text{by (3.13)}
\end{aligned}
\tag{5.3}
$$

Here $N$ is the unique integer which satisfies

$$\widehat{t}_1^{p^4\omega}\equiv v_2^N\widehat{t}_1^{p^2} \mod (p,v_1).$$

Let $a_1=v_2^{N+p\omega-1}\widehat{t}_1^{p^3}\eta_R(\widehat{v}_1)$ and $a_2=v_2^{N+p\omega-p-1}\widehat{t}_3^{p}\eta_R(\widehat{v}_1)$.

$$
\begin{aligned}
d(-p^{s-1}a_1) &= p^s v_2^{N+p\omega-1}[\underline{\widehat{t}_1^{p^3}\otimes\widehat{t}_1}_1-\underline{\widehat{b}_1^2\eta_R(\widehat{v}_1)}_4]\\
d(-p^s a_2) &\equiv p^s v_2^{N+p\omega-p-1}[p\underline{\widehat{t}_3^{p}\otimes\widehat{t}_1}_5+\underline{v_2^p\widehat{b}_1^2\eta_R(\widehat{v}_1)}_4\\
&\quad -p\widehat{b}_3^0\eta_R(\widehat{v}_1)]
\end{aligned}
$$

$$
\begin{aligned}
d(p^{s+1}v_2^{-p}\widehat{\widehat{t}}{}_4^p) &\equiv -p^{s+1}v_2^{-p}(v_2^p\widehat{\widehat{b}}{}_2^2+v_3^p\widehat{\widehat{b}}{}_1^3+\cdots+\widehat{v}_2^p\widehat{b}_2^{2s+2})\\
&\quad -p^{s+1}v_2^{-p}(\widehat{t}_1^p\otimes\widehat{t}_3^{p^2\omega}+\widehat{t}_2^p\otimes\widehat{t}_2^{p^3\omega}\\
&\quad +\widehat{t}_3^p\otimes\widehat{t}_1^{p^4\omega})+p^{s+2}v_2^{-p}\widehat{\widehat{b}}{}_4^0\\
&\equiv -\underline{p^{s+1}\widehat{\widehat{b}}{}_2^2}_3-\underline{p^{s+1}v_2^{N+p^{2s+2}-p-1}\widehat{t}_1^p\otimes\widehat{t}_3}_2\\
&\quad -\underline{p^{s+1}v_2^{N+p^{2s+1}-p-1}\widehat{t}_3^p\otimes\widehat{t}_1}_5\\
&\quad -p^{s+1}(v_3^p\widehat{\widehat{b}}{}_1^3+\cdots+\widehat{v}_2^p\widehat{b}_2^{2s+2})+\cdots
\end{aligned}
\tag{5.4}
$$

Here $\cdots$ is a linear combination of items $\widehat{t}_1^p\otimes\widehat{t}_2^p$, $\widehat{t}_1^p\otimes\widehat{t}_2$, $\widehat{t}_1^p\otimes\widehat{t}_1^p$, $\widehat{t}_1^p\otimes\widehat{t}_1$. Thus it is a coboundary modulo $(p^{s+1},v_1)$. Apply Equation (5.2), it is not difficult to check that $\mod (p^{s+1},v_1)$

$$
\begin{gathered}
p^{s+1}v_2^{N+p^{2s+1}-p-1}\widehat{b}_3^0\eta_R(\widehat{v}_1)\\
p^{s+1}(v_3^p\widehat{\widehat{b}}{}_1^3+\cdots+\widehat{v}_2^p\widehat{b}_2^{2s+2})
\end{gathered}
$$

are coboundaries. Consequently, we have an element $\alpha$ such that

$$d(\alpha)\equiv 0 \mod (p^{s+2},v_1)$$

The leading term of $\alpha$ is same as the leading term of $\widehat{\widehat{c}}_{2,3}$, which is $\widehat{t}_2^p$.

Next, if $m=2s-1\,(s>1)$, then $[\frac{m}{2}]+1=s+1$. Equation (5.2) implies that $\mod (p^{s+1},v_1)$

$$d(\widehat{\widehat{c}}_{2,3})=-p^s\widehat{t}_1^{p^3}\otimes\widehat{t}_1^{p^3\omega}+p^{s+1}\widehat{\widehat{b}}{}_2^2+p^{s-1}d(\widehat{v}_1^{p^3})\widehat{c}_{1,2s+3} \tag{5.5}$$

Since $d(\widehat{v}_1^{p^3}) \equiv 0 \mod (p^3)$, we can obtain

$$\begin{aligned} d(\widehat{\widehat{c}}_{2,3}) &\equiv -p^s \widehat{t}_1^{p^3} \otimes \widehat{t}_1^{p^3\omega} \mod (p^{s+1}, v_1) \\ &\equiv -p^s v_2^N \widehat{t}_1^p \otimes \widehat{t}_1^p \mod (p^{s+1}, v_1) \end{aligned} \tag{5.6}$$

Here $N$ is the unique integer which satisfies

$$\widehat{t}_1^{p^3\omega} \equiv v_2^N \widehat{t}_1^p \mod (p, v_1).$$

Since

$$d(-\frac{p^s}{2} v_2^N \widehat{t}_1^{2p}) = p^s v_2^N \widehat{t}_1^p \otimes \widehat{t}_1^p$$

Let $\beta = \widehat{\widehat{c}}_{2,3} - \frac{p^s}{2} v_2^N \widehat{t}_1^{2p}$. We have

$$\beta \equiv \widehat{\widehat{c}}_{2,3} \mod (p), \quad d(\beta) \equiv 0 \mod (p^{s+1}, v_1)$$

From Lemma 5.1, $\beta$ has a leading item $\widehat{t}_1^p$.

Let $\zeta_1 = \alpha$ if $m = 2s$ and $\zeta_1 = \beta$ if $m = 2s-1$. It is obvious that $\zeta_1$ satisfies all assumptions of Lemma 3.3. This completes the proof. □

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## References

[IS08] Ichigi, Ippei; Shimomura, Katsumi. The modulo two homotopy groups of the $L_2$ localization of the Ravenel spectrum. *Cubo* **10** (2008), no. 3, 43–55. MR2467909, Zbl 1156.55011. 1124

[IST10] Ichigi, Ippei; Shimomura, Katsumi; Takeda, Rinko. The homotopy groups of $L_2$-localization of the Ravenel spectra $T(m)/v_1$ at the prime two. *Bol. Soc. Mat. Mexicana* (3) **16** (2010), no. 1, 53–61. MR2932533, Zbl 1254.55010. 1124

[LWY10] Liu, Xiugui; Wang, Xiangjun; Yuan, Zihong. The homotopy groups of $L_2T(1)/(v_1)$ at an odd prime. *Proc. Amer. Math. Soc.* **138** (2010), no. 3, 1143–1152. MR2566579, Zbl 1200.55020, doi: 10.1090/S0002-9939-09-10138-7. 1124

[MiRW77] Miller, Haynes R.; Ravenel, Douglas C.; Wilson, W. Stephen. Periodic phenomena in the Adams–Novikov spectral sequence. *Ann. of Math.* (2) **106** (1977), no. 3, 469–516. MR0458423, Zbl 0374.55022, doi: 10.2307/1971064. 1139

[Mor78] Morava, Jack. Completions of complex cobordism. *Geometric applications of homotopy theory* (Proc. Conf., Evanston, Ill., 1977), II, 349–361, Lecture Notes in Math., 658. *Springer, Berlin*, 1978. MR513583, Zbl 0394.55007. 1125

[NS07] Nakai, Hirofumi; Shimomura, Katsumi. On the homotopy groups of $E(n)$-local spectra with unusual invariant ideals. *Proceedings of the Nishida Fest* (Kinosaki 2003), 319–332, Geom. Topol. Monogr., 10. *Geom. Topol. Publ., Coventry*, 2007. MR2402792, Zbl 1258.55009, arXiv:0903.4662, doi: 10.2140/gtm.2007.10.319. 1124

[Rav84] Ravenel, Douglas C. Localization with respect to certain periodic homology theories. *Amer. J. Math.* **106** (1984), no. 2, 351–414. MR737778, Zbl 0586.55003, doi: 10.2307/2374308. 1123

[Rav86] Ravenel, Douglas C. Complex cobordism and stable homotopy groups of spheres. Pure and Applied Mathematics, 121. *Academic Press, Inc., Orlando, FL*, 1986. xx+413 pp. ISBN: 0-12-583430-6; 0-12-583431-4. MR860042, Zbl 0608.55001, doi: 10.1090/chel/347. 1125, 1131

[Shi95] Shimomura, Katsumi. The homotopy groups of the $L_2$-localized Mahowald spectrum $X\langle 1\rangle$. *Forum Math.* **7** (1995), no. 6, 685–707. MR1359422, Zbl 0835.55009, doi: 10.1515/form.1995.7.685. 1124

[Wan07] Wang, Xiangjun. $\pi_*(L_2T(1)/(v_1))$ and its applications in computing $\pi_*(L_2T(1))$ at the prime two. *Forum Math.* **19** (2007), no. 1, 127–147. MR2296069, Zbl 1118.55010, doi: 10.1515/FORUM.2007.006. 1124

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