Composition operators on the Dirichlet space of the upper half-plane

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Abstract. It is well known that Hardy and weighted Bergman spaces of the upper half-plane do not support compact composition operators (see [M99] and [SS03]). In this paper, we prove that unlike Hardy and Bergman spaces, the Dirichlet space of the upper half-plane does support compact composition operators. Furthermore, bounded analytic symbols, which in the case of Hardy and weighted Bergman spaces of the upper half-plane do not even induce bounded composition operators, can induce compact composition operators on the Dirichlet space of the upper half-plane.

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1. Introduction and preliminaries

Let $\Omega$ be a domain in the complex plane $\mathbb{C}$ and let $\varphi$ be a holomorphic self-map of $\Omega$. Then the equation $C_\varphi f = f \circ \varphi$, for $f$ analytic in $\Omega$, defines a composition operator $C_\varphi$ with inducing map $\varphi$. During the past few decades, composition operators have been studied extensively on spaces of functions analytic on the open unit disk $\mathbb{D}$. As a consequence of the Littlewood subordination principle it is known that every analytic self-map $\varphi$ of $\mathbb{D}$ induces a bounded composition operator on Hardy and weighted Bergman spaces of the open unit disk. However, a self-map $\varphi$ of $\mathbb{D}$ does not necessarily induce a bounded composition operator on the Dirichlet space of the open unit disk $\mathbb{D}$. An obvious necessary condition for it is that $\varphi$ be in the Dirichlet space of the open unit disk. But this condition is not sufficient. A necessary and sufficient condition for $\varphi$ to induce a bounded composition operator on...
the Dirichlet space of the open unit disk is given in terms of the counting function and Carleson measures (see [JM97] and [Z98] and the references therein). For more about composition operators, we refer to [CoM95] and [S93].

If we move to Hardy and weighted Bergman spaces of the upper half-plane

\[ \Pi^+ = \{ z \in \mathbb{C} : \Im z > 0 \} \]

the situation is entirely different. There do exist analytic self-maps of the upper half-plane which do not induce bounded composition operators. Moreover, Hardy and Bergman spaces of the upper half-plane do not support compact composition operators (see [M99] and [SS03]). Recent work on composition operators on Hardy and weighted Bergman spaces of the upper half-plane can be found in [BT12],[CKS17],[EW11],[EJ12],[M89],[M99],[SS03], and [SiSh80].

Composition operators on the Dirichlet space \( D_{\Pi^+} \) of the upper half-plane remain untouched so far. In this paper, we characterize compact composition operators on the Dirichlet space of the upper half-plane. Recall that a function \( f \) that is analytic in the upper half-plane \( \Pi^+ \) belongs to the Dirichlet space \( D_{\Pi^+} \) if and only if

\[ \int_{\Pi^+} |f'(z)|^2 dA(z) < \infty, \]

where \( dA(z) = dx dy \) is ordinary area measure. The norm on \( D_{\Pi^+} \) is defined as

\[ ||f||_{D_{\Pi^+}}^2 = |f(i)|^2 + \int_{\Pi^+} |f'(z)|^2 dA(z), \]

For \( z \in \Pi^+ \) and \( 0 < r < 1 \) we define

\[ S(z, r \Im z) = \{ w \in \Pi^+ : |w - z| < r \Im z \}. \]

Then for \( 0 < r < 1/3 \), there exists a positive integer \( M \) and a sequence \( \{ z_n \} \) in \( \Pi^+ \) such that

\[ \bigcup_{n=1}^{\infty} S(z_n, 3r y_n) = \Pi^+ \]

and every point in \( \Pi^+ \) belongs to at most \( M \) sets in \( \{ S(z_n, 3r y_n) \}_{n \in \mathbb{N}} \); see [KK01].

2. Boundedness and compactness of \( C_\varphi \) on \( D_{\Pi^+} \).

In this section we characterize bounded and compact composition operators on the Dirichlet space of the upper half plane in terms of the counting function and Carleson measures.

Let \( \varphi : \Pi^+ \rightarrow \Pi^+ \) be an analytic map and \( w \) be a point in \( \Pi^+ \). Let \( \{ z_k \} \) be the points in the upper half plane for which \( \varphi(z_k) = w \) counting multiplicities. Define the counting function

\[ n_\varphi : \Pi^+ \rightarrow \mathbb{N} \cup \{ \infty \} \]
by
\[ n_\varphi(w) = \text{Card}\{z \in \Pi^+ : \varphi(z_k) = w\} \]
when the set \(\{z \in \Pi^+ : \varphi(z_k) = w\}\) is finite, and \(n_\varphi(w) = \infty\) otherwise. Also we set \(n_\varphi(w) = 0\) if \(w \not\in \varphi(\Pi^+)\). Define
\[ \mathfrak{N}_\varphi(w) = \begin{cases} n_\varphi(w) & (w \in \varphi(\Pi^+)) \\ 0 & \text{otherwise}. \end{cases} \]
The counting function \(\mathfrak{N}_\varphi\) produces the following non-univalent change of variables formula. The proof is standard, but we include it for completeness.

**Proposition 2.1.** Let \(0 < r < 1/3\) be fixed. Let \(f \in \mathcal{D}_{\Pi^+}\) and \(\varphi\) be a non-constant analytic self-map of \(\Pi^+\). Then
\[ \int_{\Pi^+} |(f' \circ \varphi)(w)|^2|\varphi'(w)|^2dA(z) = \int_{\Pi^+} |f'(w)|^2\mathfrak{N}_\varphi(w)dA(w). \]  

**Proof.** Let
\[ \Pi_0^+ = \{z \in \Pi^+ : \varphi'(z) = 0\}. \]
Then the set \(\Pi^+ \setminus \Pi_0^+\) is at most countable. If \(z \in \Pi_0^+\), then \(\varphi\) is one-one on \(S(z, r\Im(z))\) for some \(r \in (0, 1/3)\). Thus we have
\[ \int_{S(z, r\Im(z))} |(f' \circ \varphi)(w)|^2|\varphi'(w)|^2dA(w) = \int_{\varphi(S(z, r\Im(z)))} |f'(w)|^2dA(w). \]
The disks \(S(z, r\Im(z))\) form a cover for \(\Pi_0^+\) and we can pick a countable sub-cover \(\{S(z_n, r\Im(z_n)) : n \in \mathbb{N}\}\). Let \(B_1 = S(z_1, ry_1)\) and
\[ B_n = S(z_n, r\Im(z_n)) \setminus \bigcup_{k=1}^{n-1} B_k \]
for all \(n \geq 2\). Then \(\{B_n : n \in \mathbb{N}\}\) is a pairwise disjoint cover of \(\Pi^+_0\). Using (2.1), we have
\[ \int_{\Pi^+} |(f' \circ \varphi)(w)|^2|\varphi'(w)|^2dA(w) = \sum_{n=1}^{\infty} \int_{B_n} |(f' \circ \varphi)(w)|^2|\varphi'(w)|^2dA(w) \]
\[ = \sum_{n=1}^{\infty} \int_{\varphi(B_n)} |f'(w)|^2dA(w) \]
\[ = \int_{\Pi^+} |f'(w)|^2\sum_{n=1}^{\infty} \chi_{\varphi(B_n)}(w)dA(w) \]
\[ = \int_{\Pi^+} |f'(w)|^2\mathfrak{N}_\varphi(w)dA(w). \]
This completes the proof of the non-univalent change of variables formula. \(\Box\)

From now onwards, constants are denoted by \(C\). They are positive and not necessarily the same at each occurrence.
Theorem 2.2. Let $0 < r < 1/3$ be fixed and $\varphi$ be a non-constant analytic self-map of $\Pi^+$. Then $C_\varphi$ is bounded on $D_{\Pi^+}$ if and only if

$$\sup_{z \in \Pi^+} \frac{1}{(3z)^2} \int_{S(z, r \Im z)} N_\varphi(w) dA(w) < \infty. \quad (2.2)$$

Proof. Suppose that (2.2) holds. By the closed graph theorem, we need to show that $C_\varphi f \in D_{\Pi^+}$ whenever $f \in D_{\Pi^+}$. By Proposition 2.1, we have

$$\int_{\Pi^+} |(C_\varphi f)'(w)|^2 dA(w) = \int_{\Pi^+} |f'(w)|^2 N_\varphi(w) dA(w). \quad (2.3)$$

The right side of (2.3) is dominated by

$$\sum_{n=1}^{\infty} \int_{S(z_n, r \Im z_n)} |f'(w)|^2 N_\varphi(w) dA(w) \quad (2.4)$$

By the subharmonicity of $|f'(w)|^2$, (2.4) is further dominated by a constant multiple of

$$\sum_{n=1}^{\infty} \int_{S(z_n, 2r \Im z_n)} |f'(w)|^2 dA(w) \leq C \sum_{n=1}^{\infty} \int_{S(z_n, 2r \Im z_n)} dA(w) \leq CM \|f\|_{D_{\Pi^+}}^2.$$

Conversely, suppose that $C_\varphi$ is bounded, then

$$\|C_\varphi f\|_{D_{\Pi^+}}^2 \leq C \|f\|_{D_{\Pi^+}}^2.$$
Thus $f_z \in \mathcal{D}_{\Pi^+}$. Moreover,
\[
\sup_{z \in \Pi^+} \|f_z\|_{\mathcal{D}_{\Pi^+}} \leq C.
\]
So, from (2.3), we have
\[
\frac{1}{(3z)^2} \int_{S(z, r \Im z)} \Re \varphi(w) dA(w) \leq C \int_{S(z, r \Im z)} |f_z'(w)|^2 \Re \varphi(w) dA(w)
\]
\[
\leq \|C_\varphi f\|^2_{\mathcal{D}_{\Pi^+}} \leq C.
\]
Since $z \in \Pi^+$ is arbitrary, the desired result follows.

It is well known (see [M89], [M99] and [SS03]) that a linear fractional map
\[
\tau(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R} \text{ and } ad - bc > 0,
\]
induces a bounded composition operator on the Hardy or weighted Bergman spaces of the upper half plane if and only if $c = 0$. However, in view of Theorem 2.2, every linear fractional map $\tau$ defined in (2.5) induces a bounded composition operator on $\mathcal{D}_{\Pi^+}$.

As in [CKS17], let $\hat{\Pi^+}$ denote the set $\overline{\Pi^+} \cup \{\infty\}$. For any function $F(z)$,
\[
\lim_{z \to \partial \hat{\Pi^+}} F(z) = 0
\]
means that
\[
\sup_{z \in \Pi^+ \setminus K} |F(z)| \to 0
\]
as the compact set $K \subset \Pi^+$ expands to the whole of $\Pi^+$, or equivalently, that $F(z) \to 0$ as $\Im z \to 0^+$ and $F(z) \to 0$ as $|z| \to \infty$.

**Theorem 2.3.** Let $\varphi$ be a non-constant holomorphic self-map of $\Pi^+$ such that $C_\varphi$ is bounded on $\mathcal{D}_{\Pi^+}$. Then $C_\varphi$ is compact on $\mathcal{D}_{\Pi^+}$ if and only if there exists $r \in (0, 1)$ such that
\[
\lim_{z \to \partial \Pi^+} \frac{1}{(3z)^2} \int_{S(z, r \Im z)} \Re \varphi(w) dA(w) = 0. \tag{2.6}
\]

**Proof.** Arguing by contradiction, first assume that $C_\varphi$ is compact on $\mathcal{D}_{\Pi^+}$ but (2.6) does not hold. Then there is a $\delta > 0$ and a sequence $\{z_n\}_{n \in \mathbb{N}}$ in $\Pi^+$ such that $\Im z_n \to 0$ or $|z_n| \to \infty$ and
\[
\frac{1}{(3z)^2} \int_{S(z, r \Im z_n)} \Re \varphi(w) dA(w) > \delta
\]
for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$ consider the function
\[
f_n(w) = \frac{3z_n}{w - \overline{z_n}}.
\]
It is clear that $f_n$ is norm bounded and $f_n \to 0$ uniformly on compact subsets of $\Pi^+$ as $\Im z_n \to 0$ or $|z_n| \to \infty$. Thus $\|C_\varphi f_n\|_{\mathcal{D}_{\Pi^+}} \to 0$ as $\Im z_n \to 0$ or $|z_n| \to \infty$. On the other hand, by (2.3), we have

\[
\|C_\varphi f_n\|_{\mathcal{D}_{\Pi^+}} \geq \int_{S(z_n, r \Im z_n)} |f_n'(w)|^2 \mathcal{N}_\varphi(w) dA(w) \geq \frac{C}{(\Im z_n)^2} \int_{S(z_n, r \Im z_n)} \mathcal{N}_\varphi(w) dA(w) > C \delta,
\]

which is a contradiction.

Next, assume that (2.6) holds. Then for each $\epsilon > 0$ there is a compact subset $K$ of $\Pi^+$ such that

\[
\int_{S(z, r \Im z)} \mathcal{N}_\varphi(w) dA(w) < \epsilon (\Im z)^2 \tag{2.7}
\]

whenever $z \in \Pi^+ \setminus K$. Let $\{f_m\}$ be a sequence in $\mathcal{D}_{\Pi^+}$ such that

\[
\sup_m \|f_m\|_{\mathcal{D}_{\Pi^+}} \leq M_1
\]

and $f_m \to 0$ uniformly on compact subsets of $\Pi^+$ as $m \to \infty$. Then

\[
\|C_\varphi f_m\|_{\mathcal{D}_{\Pi^+}}^2 = |f_m(\varphi(i))|^2 + \int_{\Pi^+} |f_m'(w)|^2 \mathcal{N}_\varphi(w) dA(w)
\]

\[
= |f_m(\varphi(i))|^2 + \int_K |f_m'(w)|^2 \mathcal{N}_\varphi(w) dA(w) + \int_{\Pi^+ \setminus K} |f_m'(w)|^2 \mathcal{N}_\varphi(w) dA(w).
\]

Note that $|f_m(\varphi(i))|^2 \to 0$ as $m \to \infty$ and

\[
\int_K |f_m'(w)|^2 \mathcal{N}_\varphi(w) dA(w) \to 0 \quad \text{as} \quad m \to \infty. \tag{2.8}
\]

Thus to prove that $C_\varphi$ is compact on $\mathcal{D}_{\Pi^+}$ we just need to show that

\[
\int_{\Pi^+ \setminus K} |f_m'(w)|^2 \mathcal{N}_\varphi(w) dA(w) \to 0 \quad \text{as} \quad m \to \infty.
\]

As in the proof of Theorem 2.2, the above term is dominated by

\[
\sum_{n=1}^\infty \left( \frac{1}{(\Im z_n)^2} \int_{S(z_n, 2r \Im z_n)} |f_m'(w)|^2 dA(w) \right) \int_{S(z_n, 2r \Im z_n) \cap (\Pi^+ \setminus K)} \mathcal{N}_\varphi(w) dA(w).
\]

By (2.7), we have

\[
\int_{\Pi^+ \setminus K} |f_m'(w)|^2 \mathcal{N}_\varphi(w) dA(w) < \epsilon \sum_{n=1}^\infty \int_{S(z_n, 2r \Im z_n)} |f_m'(w)|^2 dA(w) \leq \epsilon M M_1 \tag{2.9}
\]
Combining (2.8) and (2.9), we obtain
\[
\int_{\Pi^+} |f_m^\prime(w)|^2 N_\varphi(z) dA(w) \to 0 \quad \text{as} \quad m \to \infty.
\]
Hence \( C_\varphi \) is compact on \( D_{\Pi^+} \). □

In [M89], Matache proved that if \( \varphi \) is a bounded analytic self mapping of \( \Pi^+ \), then \( C_\varphi \) cannot be bounded on the Hardy space \( H^p(\Pi^+) \). Also, bounded analytic self-maps of \( \Pi^+ \) cannot induce bounded composition operators on weighted Bergman spaces of the upper half-plane; see [SS03]. As an application of Theorem 2.3, we prove that there are non-trivial analytic self-maps of the upper half-plane that induce compact composition operators on \( D_{\Pi^+} \).

**Corollary 2.4.** Let \( \varphi \) be a conformal mapping from \( \Pi^+ \) to a relatively compact subset of \( \Pi^+ \). Then \( \varphi \) induces a compact composition operator on \( D_{\Pi^+} \).

**Proof.** Suppose that \( K = \varphi(\Pi^+) \) is a relatively compact subset of \( \Pi^+ \). Then
\[
N_\varphi(w) = \begin{cases} 
1 & \text{if } w \in K \\
0 & \text{if } w \notin K.
\end{cases}
\]
Therefore,
\[
\sup_{z \in \Pi^+} \frac{1}{(3z)^2} \int_{S(z,r3z)} N_\varphi(w) dA(w) = \sup_{z \in \Pi^+} \left( \frac{1}{(3z)^2} \int_{S(z,r3z) \cap K} N_\varphi(w) dA(w) + \frac{1}{(3z)^2} \int_{S(z,r3z) \cap (\Pi^+ \setminus K)} N_\varphi(w) dA(w) \right) \leq C \sup_{z \in \Pi^+} \frac{1}{(3z)^2} A(S(z,r3z) \cap \Pi^+) \leq C.
\]
Thus by (2.2), \( \varphi \) induces a bounded composition operator on \( D_{\Pi^+} \).

Next, we prove that \( \varphi \) induces a compact composition operator on \( D_{\Pi^+} \). In view of Theorem 2.3, we need to show that
\[
\lim_{z \to \partial \Pi^+} \frac{1}{(3z)^2} \int_{S(z,r3z)} N_\varphi(w) dA(w) = 0. \quad (2.10)
\]
Since \( K = \varphi(\Pi^+) \) is a relatively compact subset of \( \Pi^+ \) and
\[
\sup_{z \in \Pi^+} \frac{1}{(3z)^2} \int_{S(z,r3z)} N_\varphi(w) dA(w)
\]
is finite, so (2.10) is vacuously true. Hence \( \varphi \) induces a compact composition operator on \( D_{\Pi^+} \). □

Recall that the valence of an analytic self-mapping \( \varphi \) of \( \Pi^+ \) is
\[
N = \sup_{w \in \Pi^+} n_\varphi(w).
\]
The function $\varphi$ is said to have bounded valence if $N < \infty$, that is, if there is a positive integer $N$ such that $\varphi$ takes every value at most $N$ times in $\Pi^+$.

**Corollary 2.5.** Let $\varphi$ be of bounded valence and $\varphi$ maps $\Pi^+$ to a relatively compact subset of $\Pi^+$. Then $\varphi$ induces a compact composition operator on $D_{\Pi^+}$.

The proof follows on the same lines as the proof of Corollary 2.4. We omit the details.

**Example 2.6.** Let

$$\varphi(z) = 2i + \frac{1}{(z + i)\log(z + ei)} \quad (z \in \Pi^+).$$

Then $\varphi(\Pi^+)$ is a relatively compact subset of $\Pi^+$ (see Example 4.4 in [CKS17]). Thus $\varphi$ induces a compact composition operator on $D_{\Pi^+}$.

**Acknowledgements.** The authors thank the anonymous referee for providing suggestions that improved the paper. The first author thanks NBHM(DAE) (India) for the project grant 02011/30/2017/R&D II/12565.

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This paper is available via http://nyjm.albany.edu/j/2019/25-11.html.