Analyticity and kernel stabilization of unbounded derivations on $C^*$-algebras

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Abstract. We first show that a derivation studied recently by E. Christensen has a set of analytic elements which is strong operator topology-dense in the algebra of bounded operators on a Hilbert space, which strengthens a result of Christensen. Our second main result shows that this derivation has kernel stabilization, that is, no elements have derivative eventually equal to 0 unless their first derivative is 0. As applications, we (1) show that a family of derivations on $C^*$-algebras studied by Bratteli and Robinson has kernel stabilization, and (2) we provide sufficient conditions for when two operators which satisfy the Heisenberg Commutation Relation must both be unbounded.

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1. Introduction

Given an algebra $\mathcal{A}$ with involution and a fixed element $a \in \mathcal{A}$ such that $a = a^*$, the map $\delta_a : \mathcal{A} \to \mathcal{A}$ by $\delta_a(b) = [ia, b]$ (where $[x, y] = xy - yx$) is a $*$-derivation, that is, $\delta_a(b^*) = \delta_a(b)^*$ for all $b \in \mathcal{A}$. Conversely, for an arbitrary $*$-derivation $\delta : \mathcal{A} \to \mathcal{A}$, certain conditions on the algebra can imply $\delta = \delta_a$ for some $a \in \mathcal{A}$. The correspondence between derivations on algebras and their representation as commutators has a rich history and is deeply connected to the mathematical formulation of quantum mechanics.
To illustrate, a quantum system can be modeled by a Hilbert space $H$ and the associated Hamiltonian of that quantum system is given by a self-adjoint operator $D$ whose domain is a dense subspace of $H$. Despite the potential for $D$ to be unbounded, we wish to consider commutators of $D$ with elements of $B(H)$. As not every $x \in B(H)$ will result in the commutator $[D, x]$ being defined and bounded on a dense subspace of $H$, the definition of the derivation “$\delta_D$” is ambiguous. A plethora of literature is dedicated to exploring the various definitions of $\delta_D$ and their corresponding domains, and in each situation, if $D$ is unbounded then the domain of $\delta_D$ is a proper subspace of $B(H)$. In turn, further research has been dedicated to the study of unbounded derivations on an abstract $C^*$-algebra. The unboundedness of such a derivation creates complexities that are not found with derivations defined on the entire $C^*$-algebra. In [6], Kadison summarizes three of the many significant results pertaining to bounded derivations:

1. Every such derivation on a commutative $C^*$-algebra is 0. (This follows from the Singer-Wermer Theorem from 1955 in [12].)
2. Sakai (1959) showed in [10] that every derivation on a $C^*$-algebra is automatically bounded, thus affirmatively settling a 1953 conjecture of Kaplansky.
3. In [7], Kaplansky showed every bounded derivation $\delta$ of a type I von Neumann algebra $M$ is inner, i.e., there exists $a \in M$ such that $\delta = \delta_a$.

We turn our attention to densely-defined derivations on $C^*$-algebras. Our primary setting of interest is a $*$-derivation $\delta_w^D$ on $B(H)$ defined by commutation with a fixed (possibly unbounded) self-adjoint operator $D$. In Section 2 we give a formal definition of $\delta_w^D$, its domain, domains of its higher powers, and state its desirable properties. All of these can be found in [3]. In particular, Christensen shows that the domain of $\delta_w^D$ is strong operator topology (SOT)-dense in $B(H)$. We strengthen this property in Theorem 3.15, stated as the following theorem.

**Theorem.** The set of analytic elements for $\delta_w^D$ is SOT-dense in $B(H)$.

Our second main result, Theorem 4.6, shows $\delta_w^D$ has a property called kernel stabilization.

**Theorem.** If $H$ is a Hilbert space and $D$ is a (possibly unbounded) self-adjoint operator on $H$, then $\ker(\delta_w^D)^n = \ker \delta_w^D$ for all $n \in \mathbb{N}$.

The proof requires use of Christensen’s work in [4] and [3]. Let $D$ be an unbounded self-adjoint operator on $H$. Seeking to formalize the connection between commutators and unbounded derivations on $B(H)$ of the form $\delta_D$, Christensen showed in [3] that $x \in B(H)$ makes $[D, x]$ defined and bounded on a core for $D$ if and only if for every $h, k \in H$, the map $t \mapsto \langle e^{itD}xe^{-itD}h, k \rangle$ is continuously differentiable. If $x$ satisfies this, we say $x$ is weakly $D$-differentiable, denoted $x \in \text{dom } \delta_w^D$. Define $\delta_w^D(x)$ to be the
bounded extension of \([iD, x]\) to all of \(H\). Christensen defines higher weak \(D\)-differentiability in \([4]\) and extends the aforementioned equivalence.

In Section 4, we prove Theorem 4.6, and in Section 5, we give two applications. The first extends the property of kernel stabilization to a class of unbounded \(*\)-derivations on \(C^*\)-algebras described in the following theorem.

**Theorem 1.1** (Bratteli-Robinson, Theorem 4 \([1]\)). Let \(\delta\) be a derivation of a \(C^*\)-algebra \(A\), and assume there exists a state \(\omega\) on \(A\) which generates a faithful cyclic representation \((\pi, H, f)\) satisfying

\[
\omega(\delta(a)) = 0, \quad \forall a \in \text{dom } \delta.
\]

Then \(\delta\) is closable and there exists a symmetric operator \(S\) on \(H\) such that

\[
\text{dom } S = \{h \in H : h = \pi(a)f \text{ for some } a \in A\}
\]

and \(\pi(\delta(a))h = [S, \pi(a)]h\), for all \(a \in \text{dom } \delta\) and all \(h \in \text{dom } S\). Moreover, if the set \(A(\delta)\) of analytic elements for \(\delta\) is dense in \(A\), then \(S\) is essentially self-adjoint on \(\text{dom } S\). For \(x \in B(H)\) and \(t \in \mathbb{R}\), define

\[
\alpha_t(x) := e^{\text{St}}xe^{-\text{St}}
\]

where \(\overline{S}\) denotes the self-adjoint closure of \(S\). It follows that \(\alpha_t(\pi(A)) = \pi(A)\) for all \(t \in \mathbb{R}\), and \(\{\alpha_t\}_{t \in \mathbb{R}}\) is a strongly continuous group of automorphisms with closed infinitesimal generator \(\tilde{\delta}\) equaling the closure of \(\pi \circ \delta|_{\text{dom } \delta}\).

Physically, we interpret \(\omega\) as an invariant state of the quantum system whose observables lie in \(A\). Also, we interpret the condition \(\omega(\delta(x)) = 0\) for all \(x \in \text{dom } \delta\) as saying \(\omega\) is an equilibrium state for the system. For more details, see the introduction of \([2]\). We state our application formally below.

**Application 1.** Let \(A\) be a \(C^*\)-algebra, \(\delta\) a derivation on \(A\), and \(\omega\) a state on \(A\) which satisfy the hypotheses of Theorem 1.1. For every \(n \in \mathbb{N}\), \(\ker \delta^n = \ker \delta\).

As a second application of Theorem 4.6, we provide sufficient conditions for when two operators satisfying the Heisenberg Commutation Relation must both be unbounded.

**Definition 1.2.** Let \(A\) and \(B\) be two (possibly unbounded) self-adjoint operators on a Hilbert space \(H\), with domains \(\text{dom } A\) and \(\text{dom } B\), respectively. We say \(A\) and \(B\) satisfy the Heisenberg Commutation Relation if there is a dense subspace \(K\) of \(H\) satisfying

\[
K \subseteq \text{dom } [A, B] := \{h \in \text{dom } A \cap \text{dom } B : Ah \in \text{dom } B, Bh \in \text{dom } A\}
\]

and \([A, B]k = ik\) for all \(k \in K\).

The classical example of such a pair is the Schrödinger pair, which we define in Example 5.8. Note both operators in this pair are unbounded. A large body of research has been committed to finding sufficient conditions for when two operators satisfying the Heisenberg Commutation Relation must be
unitarily equivalent to a direct sum of copies of the Schrödinger pair, thus implying that the two operators are unbounded. We provide a sufficient condition for when two operators satisfying the HCR must be unbounded without proving they are unitarily equivalent to a direct sum of copies of the Schrödinger pair.

**Application 2.** Let $A$ and $B$ be self-adjoint operators on a Hilbert space $H$ which satisfy the Heisenberg Commutation Relation on a dense subspace $K \subseteq H$. If $K$ is a core for both $A$ and $B$, then $A$ and $B$ must be unbounded.

As an outline of the rest of the paper, Section 2 is devoted to providing background and summarizing some of Christensen's results from [4] and [3]. In Section 3, we prove SOT-density of the analytic elements in $B(H)$ for $\delta_w^D$, in Section 4 we prove kernel stabilization of $\delta_w^D$, and in Section 5 we provide applications of kernel stabilization.

### 2. Definition and properties of weak $D$-differentiability

Let $D$ be a self-adjoint operator with domain $\text{dom } D \subseteq H$. For any $t \in \mathbb{R}$, the operator $e^{itD}$ is unitary, and the one-parameter family $\{e^{itD}\}_{t \in \mathbb{R}}$ is strongly continuous. For $x \in B(H)$ and $t \in \mathbb{R}$, define $\alpha_t(x) := e^{itD}xe^{-itD}$. Then $\{\alpha_t\}_{t \in \mathbb{R}}$ defines a flow on $B(H)$, and more specifically, is a one-parameter automorphism group on $B(H)$. While the *infinitesimal generator* of this automorphism group in the norm topology of $B(H)$ is a natural derivation to consider, we focus instead on a related derivation with a larger domain.

**Definition 2.1.** An operator $x \in B(H)$ is *weakly $D$-differentiable* if there exists $y \in B(H)$ such that for every $h, k \in H$,

$$\lim_{t \to 0} \| \left( \frac{\alpha_t(x) - x}{t} - y \right) h, k \| = 0.$$  

Equivalently, for every $h, k \in H$ the function $t \mapsto \langle \alpha_t(x)h, k \rangle$ is continuously differentiable.

**Theorem 2.2** (Christensen, 3.8 [3]). Let $x$ be a bounded operator on $H$. The following properties are equivalent:

(i) $x$ is weakly $D$-differentiable.

(ii) There exists $y \in B(H)$ such that for every $h \in H$,

$$\lim_{t \to 0} \| \left( \frac{\alpha_t(x) - x}{t} - y \right) h \| = 0.$$  

(iii) There exists $c > 0$ such that for all $t \in \mathbb{R}$,

$$\| \alpha_t(x) - x \| \leq c |t|.$$  

(iv) The commutator $[iD, x]$ is defined and bounded on the domain of $D$.

(v) The commutator $[iD, x]$ is defined and bounded on a core for $D$.  

(vi) The sesquilinear form on \( \text{dom} \, D \times \text{dom} \, D \) given by
\[
(h, k) \mapsto i \langle xh, Dk \rangle - i \langle xDh, k \rangle
\]
is bounded.

(vii) The matrix \( m([iD, x])_{rc} = i(DP_r x P_c - P_r x P_c D) \) defines a bounded operator on \( H \), where \((P_n)_{n \in \mathbb{Z}}\) are the spectral projections of the intervals \((n - 1, n]\).

If any of the above conditions hold, then
\[
x(\text{dom} \, D) \subseteq \text{dom} \, D, \quad \delta_w^D(x) \big|_{\text{dom} \, D} = i[D, x].
\]

We write \( x \in \text{dom} \, \delta_w^D \) and the \( y \) in item (ii) satisfies \( y = \delta_w^D(x) \). Moreover, for any \( h, k \in H \),
\[
\frac{d^n}{dt^n} \langle \alpha_t(x)h, k \rangle = \langle \alpha_t((\delta_w^D)^n(x))h, k \rangle.
\]

Theorem 2.3 (Christensen, 3.9 [3]). The domain of definition \( \text{dom} \, \delta_w^D \) is a strongly dense \( * \)-subalgebra of \( B(H) \) and \( \delta_w^D \) is a \( * \)-derivation into \( B(H) \). The graph of \( \delta_w^D \) is weak operator topology closed.

In Theorem 3.15 we strengthen the first statement of Theorem 2.3 by proving that the analytic elements for \( \delta_w^D \) are SOT-dense in \( B(H) \).

Definition 2.4. An operator \( x \in B(H) \) is \( n \)-times weakly \( D \)-differentiable if for every \( k = 0, ..., n - 1 \), \( (\delta_w^D)^k(x) \in \text{dom} \, \delta_w^D \). We denote this by \( x \in \text{dom} \, (\delta_w^D)^n \).

Proposition 2.5 (Christensen, 2.6 [4]). A bounded operator \( x \) on \( H \) is \( n \)-times weakly \( D \)-differentiable if and only if for any pair \( h, k \in H \) the function \( t \mapsto \langle \alpha_t(x)h, k \rangle \) is \( n \)-times continuously differentiable. If \( x \) is \( n \)-times weakly \( D \)-differentiable, then
\[
\frac{d^n}{dt^n} \langle \alpha_t(x)h, k \rangle = \langle \alpha_t((\delta_w^D)^n(x))h, k \rangle.
\]

Analogous to Theorem 2.2, Christensen shows in [4] that higher order weak \( D \)-differentiability is directly tied to iterated commutators \([iD, ..., iD, x]]\).

Proposition 2.6 (Christensen, 3.3 [4]). Let \( x \in \text{dom} \, (\delta_w^D)^n \). Then for \( k = 1, ..., n \),
\[
(i) \quad (\delta_w^D)^{k-1}(x)(\text{dom} \, D) \subseteq \text{dom} \, D
(ii) \quad x(\text{dom} \, D^k) \subseteq \text{dom} \, D^k
(iii) \quad \text{dom} \, [iD, ..., [iD, x]] \text{\( k \)-times} = \text{dom} \, D^k
(iv) \quad (\delta_w^D)^k(x) \big|_{\text{dom} \, D^k} = [iD, ..., [iD, x]] \text{\( k \)-times}
(v) \quad (\delta_w^D)^k(x) \text{is the bounded extension of} \ [iD, ..., [iD, x]] \text{\( k \)-times} \text{from} \text{dom} \, D^k \text{to all of} \ H.
\]

Theorem 2.7 (Christensen, 4.1 [4]). Let \( x \in B(H) \) and \( n \) be a natural number. The following are equivalent:
(i) $x \in \text{dom } (\delta^D_w)^n$.
(ii) $x$ is $n$ times weakly $D$-differentiable.
(iii) For all $k = 1, \ldots, n$, $x(\text{dom } D^k) \subseteq \text{dom } D^k$ and $[iD, \ldots, [iD, x]]$ is defined and bounded on $\text{dom } D^k$ with closure $(\delta^D_w)^k(x)$.
(iv) There exists a core $\mathcal{X}$ for $D$ such that for any $k = 1, \ldots, n$, the operator $[iD, \ldots, [iD, x]]$ is defined and bounded on $\mathcal{X}$.

**Notation 2.8.** For notational convenience, we define

$$d^k(x) := [iD, \ldots, [iD, x]]$$

for each $k \in \mathbb{N}$.

### 3. Density of the analytic elements for $\delta^D_w$

**Definition 3.1.** Let $S$ be an operator on a Banach space $X$. An element $x \in X$ is an analytic element for $S$ if

1. $x \in \text{dom } S^n$ for all $n \in \mathbb{N}$ and
2. there exists $t_x > 0$ such that for all $0 \leq t < t_x$, the following series converges:

$$\sum_{n=0}^{\infty} \frac{\|S^n x\|}{n!} t^n.$$

**Notation 3.2.** Let $A(S)$ denote the set of analytic elements for $S$.

By Nelson’s Analytic Vector Theorem in [8], a symmetric operator $S$ on a Hilbert space $H$ is essentially self-adjoint if and only if $A(S)$ is dense in $H$. In particular, if $D$ is a self-adjoint operator, then the set $A(D)$ is dense in $H$. An analogous statement for $\delta^D_w$ spurs our investigation. To relate the analytic elements for $D$ and $\delta^D_w$, we exploit an equivalent notion of analyticity for the one-parameter families for which $D$ and $\delta^D_w$ are infinitesimal generators: $\{e^{itD}\}_{t \in \mathbb{R}}$ and $\{\alpha_t\}_{t \in \mathbb{R}}$, respectively. We first introduce the notion of analytic elements for a general one-parameter family on a Banach space, and then we specialize to our setting.

**Definition 3.3.** Let $X$ be a Banach space and let $Y$ be a closed subspace of $X^*$. A one-parameter family $\{\tau_t\}_{t \in \mathbb{R}}$ of bounded linear maps of $X$ into itself is called a $\sigma(X, Y)$-continuous group of isometries of $X$ if

1. $\tau_0 = I$,
2. $\tau_{s+t} = \tau_s \tau_t$ for all $s, t \in \mathbb{R}$,
3. $\|\tau_t x\| = \|x\|$ for all $t \in \mathbb{R}$, $x \in X$,
4. $t \mapsto \tau_t(x)$ is $\sigma(X, Y)$-continuous for all $x \in X$, i.e.,

$$t \mapsto \psi(\tau_t(x))$$

is continuous for all $x \in X$ and $\psi \in Y$, and
(5) \( x \mapsto \tau_t(x) \) is \( \sigma(X,Y) \)-continuous for all \( t \in \mathbb{R} \).

**Definition 3.4.** Given a \( \sigma(X,Y) \)-continuous group of isometries \( \{ \tau_t \}_{t \in \mathbb{R}} \), an element \( x \in X \) is **analytic** for \( \{ \tau_t \}_{t \in \mathbb{R}} \) if there exist \( \lambda > 0 \), a strip \( I_\lambda := \{ z \in \mathbb{C} : \text{Im } z < \lambda \} \), and a function \( \varphi : I_\lambda \to X \) such that

1. \( \varphi(t) = \tau_t(x) \) for all \( t \in \mathbb{R} \) and
2. \( z \mapsto \varphi(z) \) is analytic on \( I_\lambda \) for all \( \psi \in Y \).

Proposition 3.6 states that Definition 3.1 and Definition 3.4 are equivalent when \( S \) is the **infinitesimal generator** of the family \( \{ \tau_t \}_{t \in \mathbb{R}} \).

**Definition 3.5.** Given a \( \sigma(X,Y) \)-continuous group of isometries \( \{ \tau_t \}_{t \in \mathbb{R}} \), the **infinitesimal generator** \( S \) for \( \{ \tau_t \}_{t \in \mathbb{R}} \) is the operator whose domain consists of all elements \( x \in X \) such that there exists \( x' \in X \) which satisfies

\[
\lim_{t \to 0} \psi \left( \frac{\tau_t(x) - x}{t} - x' \right) = 0 \quad \text{for all } \psi \in Y.
\]

If \( x \in \text{dom } S \) with corresponding difference quotient limit \( x' \), set \( Sx := x' \).

**Proposition 3.6** (Bratteli-Robinson, [2]). If \( \{ \tau_t \}_{t \in \mathbb{R}} \) is a \( \sigma(X,Y) \)-continuous group of isometries with infinitesimal generator \( S \), then \( x \) is analytic for \( \{ \tau_t \}_{t \in \mathbb{R}} \) if and only if \( x \in \mathcal{A}(S) \).

Consider \( X = B(H) \), the one-parameter group of \( * \)-automorphisms \( \{ \alpha_t \}_{t \in \mathbb{R}} \), and the closed subspace of \( B(H)^* \) defined by

\[
Y := \{ \psi_{f,g} : f, g \in H, \psi_{f,g}(x) = \langle xf, g \rangle \}.
\]

Note that \( \sigma(X,Y) \) is precisely the weak operator topology (WOT) on \( B(H) \).

**Proposition 3.7.** The family \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is a WOT-continuous group of automorphisms with infinitesimal generator \( \delta^D_w \).

The WOT-continuity of \( \{ \alpha_t \}_{t \in \mathbb{R}} \) is a simple computation and showing \( \delta^D_w \) is the corresponding infinitesimal generator is immediate by the definition of weak \( D \)-differentiability. As a corollary of Propositions 3.6 and 3.7, we have the following:

**Corollary 3.8.** An operator \( x \in B(H) \) is analytic for \( \{ \alpha_t \}_{t \in \mathbb{R}} \) if and only if \( x \in \mathcal{A}(\delta^D_w) \).

**Notation 3.9.** Given \( h, k \in H \), define the rank-one operator \( h \otimes k^* \in B(H) \) by

\[
(h \otimes k^*)(f) := \langle f, k \rangle h \quad \text{for all } f \in H.
\]

**Notation 3.10.** Given subsets \( S_1, S_2 \subseteq H \), let

\[
F(S_1, S_2) := \text{span}\{ h \otimes k^* : h \in S_1, k \in S_2 \}.
\]

We simply denote \( F(S_1, S_1) \) by \( F(S_1) \).

**Lemma 3.11.** If \( S_1, S_2 \subseteq H \) are dense subspaces, then \( F(S_1, S_2) \) is norm-dense in \( K(H) \).
The proof of Lemma 3.11 is just an $\varepsilon$-argument using norm-density of $F(H)$ in $K(H)$. Our initial method for proving SOT-density of the set of analytic elements for $\delta^D_w$ in $B(H)$ was to show that any rank-one operator belonging to the set $F(A(D))$ is analytic for $\delta^D_w$. We were successful in proving the inclusion

$$F(\text{dom } D^n) \subseteq \text{dom } (\delta^D_w)^n$$

but extending this argument to show $F(A(D)) \subseteq A(\delta^D_w)$ fails. To remediate this argument, we chose to consider the set of finite-rank operators $F(A(D), R^{-1}[A(D^\#)])$, where $D^\#$ is conjugate to $D$ via the antunitary Riesz map, $R : H \to H^*$ given by

$$[R h](f) := \langle f, h \rangle \text{ for all } f \in H.$$

**Lemma 3.12.** The map $D^\# := R D R^{-1}$ is self-adjoint.

**Proof.** To show $D^\# = (D^\#)^*$, we must show $\text{dom } (D^\#)^* = \text{dom } D^\#$ and $D^\# \xi = (D^\#)^* \xi$ for all $\xi \in \text{dom } D^\#$. We first show $D^\#$ is a linear symmetric operator and then relate its adjoint’s domain to the domain of $D$. By definition, $\text{dom } D^\# = R(\text{dom } D)$. Thus, given $h \in \text{dom } D$ and $\lambda \in \mathbb{C}$, observe

$$D^\# (\lambda R h) = [R D R^{-1}] (\lambda R h) = [R D] (\lambda h) = R (\overline{\lambda D h})$$

$$= \lambda [R D R^{-1}] R h = \lambda D^\# (R h).$$

As $h \in \text{dom } D$ was arbitrary and $\text{dom } D^\# = R(\text{dom } D)$, we have $D^\# (\lambda \xi) = \lambda D^\# \xi$ for all $\xi \in \text{dom } D^\#$ and $\lambda \in \mathbb{C}$. It’s easy to check additivity of $D^\#$, so $D^\#$ is linear. For $f, h \in \text{dom } D$,

$$\left\langle D^\# R h, R f \right\rangle = \left\langle R D R^{-1} R h, R f \right\rangle$$

$$= \left\langle R D h, R f \right\rangle$$

$$= \left\langle f, D h \right\rangle$$

$$= \left\langle D f, h \right\rangle$$

$$= \left\langle R h, R D f \right\rangle$$

$$= \left\langle R h, D^\# R f \right\rangle.$$

As $f, h \in \text{dom } D$ were arbitrary and $\text{dom } D^\# = R(\text{dom } D)$, we have

$$\left\langle D^\# \xi, \eta \right\rangle = \left\langle \xi, D^\# \eta \right\rangle \text{ for all } \xi, \eta \in \text{dom } D^\#.$$

Hence, $D^\#$ is symmetric. By symmetry of $D^\#$, we have

$$\text{dom } D^\# \subseteq \text{dom } (D^\#)^* \text{ and } D^\# \xi = (D^\#)^* \xi \text{ for all } \xi \in \text{dom } D^\#.$$
Thus, it suffices to prove $\text{dom } (D\#)^* \subseteq \text{dom } D\#$. The domain of the adjoint of $D\#$ is the set

$$\text{dom } (D\#)^* = \{ \eta \in H^* : \text{the map } \text{dom } D\# \to \mathbb{C}; \xi \mapsto \langle D\# \xi, \eta \rangle \text{ is bounded} \}$$

$$= \{ \eta \in H^* : \text{the map } R(\text{dom } D) \to \mathbb{C}; \Rh \mapsto \langle D\# (\Rh), \eta \rangle \text{ is bounded} \}.$$ 

$$= \{ \eta \in H^* : \text{the map } R(\text{dom } D) \to \mathbb{C}; \Rh \mapsto \langle R^{-1} \eta, R^{-1} D\# (\Rh) \rangle \text{ is bounded} \}.$$ 

Hence, given $\eta \in \text{dom } (D\#)^*$, the map $R(\text{dom } D) \to \mathbb{C}$ defined by $\Rh \mapsto \langle R^{-1} \eta, \Rh \rangle$ for all $h \in \text{dom } D$ is a bounded linear functional. Then, as $R$ is isometric, the composition $D \to R(\text{dom } D) \to \mathbb{C}$ given by $h \mapsto \Rh \mapsto \langle R^{-1} \eta, \Rh \rangle$ defines a bounded linear functional on the domain of $D$. By the definition of the domain of $D^*$, this implies $R^{-1} \eta$ belongs to $\text{dom } D^*$. Further, self-adjointness of $D$ implies $R^{-1} \eta \in \text{dom } D$. Since $R$ is bijective, we conclude $\eta \in R(\text{dom } D) = \text{dom } D\#$. Therefore, $D\#$ is self-adjoint.

Another application of Nelson’s Analytic Vector Theorem in [8] implies that the set of analytic elements for $D\#$, denoted $\mathcal{A}(D\#)$, are dense in $H^*$. As $R^{-1} : H^* \to H$ is antiunitary, it follows that $R^{-1}[\mathcal{A}(D\#)]$ is dense in $H$. By Lemma 3.11, we obtain norm-density of $F(\mathcal{A}(D), R^{-1}[\mathcal{A}(D\#)])$ in the compact operators.

**Proposition 3.13.** If $h \in \mathcal{A}(D)$ and $k \in R^{-1}[\mathcal{A}(D\#)]$, then $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t \in \mathbb{R}}$.

**Proof.** Let $h \in \mathcal{A}(D)$ and $k \in R^{-1}[\mathcal{A}(D\#)]$. To prove $h \otimes k^*$ is analytic for $\{\alpha_t\}_{t \in \mathbb{R}}$ in the WOT, we must find $\lambda > 0$ and a function $\varphi : I_\lambda \to B(H)$ such that

1. $\varphi(t) = \alpha_t(h \otimes k^*)$ for all $t \in \mathbb{R}$ and
2. $z \mapsto \langle \varphi(z)f, g \rangle$ is analytic on $I_\lambda$ for all $f, g \in H$.

We shall construct $\varphi$ using functions obtained from analytic properties of $h$ and $k$. As $h \in \mathcal{A}(D)$, Proposition 3.6 implies $h$ is analytic for $\{e^{itD}\}_{t \in \mathbb{R}}$. Thus, there exist $\lambda_h > 0$ and a function $\varphi_h : I_{\lambda_h} \to H$ such that

1. $\varphi_h(t) = e^{itD}h$ for all $t \in \mathbb{R}$ and
2. $z \mapsto \langle \varphi_h(z), g \rangle$ is analytic on $I_{\lambda_h}$ for all $g \in H$.

As $k \in R^{-1}[\mathcal{A}(D\#)]$, there exists a unique $\eta \in \mathcal{A}(D\#)$ such that $k = R^{-1} \eta$. Since $\eta$ is analytic for $D\#$, it is analytic for $\{e^{itD\#}\}_{t \in \mathbb{R}}$ by Proposition 3.6. Thus, there exist $\lambda_\eta > 0$ and a function $\varphi_\eta : I_{\lambda_\eta} \to H^*$ such that

1. $\varphi_\eta(t) = e^{itD\#} \eta$ for all $t \in \mathbb{R}$ and
Note that in (2) for \( \eta \), we simply identified \( H^* \) with \( R(H) \).

Set \( \lambda := \min\{\lambda_h, \lambda_\eta\} \), and fix \( z \in I_\lambda \). Define a map
\[
[n, \cdot] : H \times H \to \mathbb{C} \text{ by } [f, g] := \langle \varphi_h(z), g \rangle \langle \varphi_\eta(z), Rf \rangle \text{ for all } f, g \in H.
\]

Sesquis-linearity of the inner products on \( H \) and \( H^* \) and antilin-earity of \( R \) establishes that \([n, \cdot]\) is a sesquilinear form. Moreover, for any \( f, g \in H \),
\[
||f, g|| = ||\langle \varphi_h(z), g \rangle \langle \varphi_\eta(z), Rf \rangle|| \leq ||\varphi_h(z)|| \|g\| \|\varphi_\eta(z)|| \|f\|.
\]

As \( h, \eta \), and \( z \) are all fixed, \([n, \cdot]\) defines a bounded sesquilinear form on \( H \). Hence, for each \( z \in I_\lambda \), the Riesz Representation Theorem provides an operator \( \varphi(z) \in B(H) \) such that
\[
\langle \varphi(z)f, g \rangle = [f, g] = \langle \varphi_h(z), g \rangle \langle \varphi_\eta(z), Rf \rangle \text{ for all } f, g \in H.
\]
As the two maps \( z \mapsto \langle \varphi_h(z), g \rangle \) and \( z \mapsto \langle \varphi_\eta(z), Rf \rangle \) are analytic on \( I_\lambda \) for all \( f, g \in H \), their product \( z \mapsto \langle \varphi(z)f, g \rangle \) is analytic on \( I_\lambda \) for all \( f, g \in H \). Furthermore, for each \( t \in \mathbb{R} \),
\[
\langle \varphi(t)f, g \rangle = \langle e^{itD}h, g \rangle \langle e^{itD}^* \eta, Rf \rangle = \langle e^{itD}h, g \rangle \langle f, e^{itD}k \rangle = \langle \alpha_t(h \otimes k^*)f, g \rangle.
\]
As \( f, g \in H \) were arbitrary, we have \( \varphi(t) = \alpha_t(h \otimes k^*) \) for all \( t \in \mathbb{R} \). Therefore, \( h \otimes k^* \) is analytic for \( \{\alpha_t\}_{t \in \mathbb{R}} \) in the WOT.

Lemma 3.14. If \( S \) is a subspace of \( B(H) \) such that \( S \cap F(H) \) is norm-dense in \( K(H) \), then \( S \) is SOT-dense in \( B(H) \).

Theorem 3.15. The set of analytic elements for \( \delta_w^D \) are SOT-dense in \( B(H) \).

Proof. By Proposition 3.6, the set of analytic elements for \( \delta_w^D \) is precisely the set of analytic elements for \( \{\alpha_t\}_{t \in \mathbb{R}} \). Since the set of analytic elements for \( \{\alpha_t\}_{t \in \mathbb{R}} \) is a linear space, Proposition 3.13 implies \( F(A(D), R^{-1}[A(D^\#)]) \) is contained in \( A(\delta_w^D) \). In particular,
\[
F(A(D), R^{-1}[A(D^\#)]) \subseteq A(\delta_w^D) \cap F(H).
\]

By Lemma 3.11 and Nelson’s Analytic Vector Theorem, we know that \( F(A(D), R^{-1}[A(D^\#)]) \) is norm-dense in \( K(H) \). Thus, by the above inclusion, we then have that \( A(\delta_w^D) \cap F(H) \) is norm-dense in \( K(H) \). From Lemma 3.14 we obtain SOT-dense of \( A(\delta_w^D) \) in \( B(H) \). \( \square \)

4. Kernel stabilization of \( \delta_w^D \)

In this section, we show for any self-adjoint operator \( D \) on a Hilbert space, \( \ker(\delta_w^D)^n = \ker \delta_w^D \) for all \( n \in \mathbb{N} \). We call this property kernel stabilization.

We now present the motivating example for Theorem 4.6. Given a \( \sigma \)-finite measure space \( (X, \mu) \), define
\[
\text{diag} : L^\infty(X, \mu) \to B(L^2(X, \mu))
\]
diag(f) := M_f,

where \( M_f g = fg \) for each \( g \in L^2(X, \mu) \). Throughout, we denote the standard orthonormal basis for \( \ell^2(\mathbb{Z}) \) by \( \{e_j : j \in \mathbb{Z}\} \), and we denote the matrix representation of an operator \( x \in B(\ell^2(\mathbb{Z})) \) with respect to the standard orthonormal basis by \( [x_{rc}] \) where

\[
x_{rc} := \langle x\epsilon_c, \epsilon_r \rangle.
\]

**Example 4.1.** Define \( (Df)(j) := jf(j) \) for \( f \in \text{dom } D \), where

\[ \text{dom } D := \{ f \in \ell^2(\mathbb{Z}) : \sum_{j \in \mathbb{Z}} j^2 |f(j)|^2 < \infty \} . \]

Then,

(a) the operator \( D \) is self-adjoint.

(b) an operator \( x \in B(\ell^2(\mathbb{Z})) \) is \( n \)-times weakly \( D \)-differentiable if and only if for every \( k \leq n \), \( x(\text{dom } D^k) \subseteq \text{dom } D^k \) and the matrix \( [i^k(r - c)^kx_{rc}] \) with dense domain \( \text{dom } D^k \) extends to a bounded operator on \( \ell^2(\mathbb{Z}) \). When either condition is satisfied,

\[ [\delta^n_w(x)_{rc}]_{\text{dom } D^n} = [i^n(r - c)^nx_{rc}] . \]

(c) for any \( g \in \ell^\infty(\mathbb{Z}) \), \( \delta^n_w(M_g) = 0 \).

(d) for all \( n \in \mathbb{N} \), \( \ker(\delta^n_w) = \text{diag}(\ell^\infty(\mathbb{Z})) \).

**Proof.** (a) See Example 7.1.5 of [11].

(b) Matrix multiplication shows for any \( r, c \in \mathbb{Z} \),

\[ d^k(x)_{rc} = i^k(r - c)^kx_{rc} . \]

Given \( x \in B(\ell^2(\mathbb{Z})) \) such that \( x(\text{dom } D^k) \subseteq \text{dom } D^k \) for each \( k \leq n \), the domain of \( d^k(x) \) is \( \text{dom } D^k \). Theorem 2.7 states \( x \) is \( n \)-times weakly \( D \)-differentiable if and only if for every \( k \leq n \), \( x(\text{dom } D^k) \subseteq \text{dom } D^k \) and \( d^k(x) \) is bounded on \( \text{dom } D^k \). It follows that \( x \) is \( n \)-times weakly \( D \)-differentiable if and only if \( x(\text{dom } D^k) \subseteq \text{dom } D^k \) and \([d^k(x)_{rc}] = [i^k(r - c)^kx_{rc}] \) is bounded on \( \text{dom } D^k \). As \( D \) is self-adjoint, \( \text{dom } D^k \) is dense in \( \ell^2(\mathbb{Z}) \) for all \( k \in \mathbb{N} \). Therefore, \([d^k(x)_{rc}] \) extends to a bounded matrix on all of \( \ell^2(\mathbb{Z}) \). By Theorem 2.7, the closure \( (\delta^n_w)^n(x) \) is the extension of \([i^n(r - c)^nx_{rc}] \) to all of \( \ell^2(\mathbb{Z}) \).

(c) Fix \( g \in \ell^\infty(\mathbb{Z}) \), and let \( f \in \text{dom } D \). We show \( M_g f \in \text{dom } D \). Observe

\[
\sum_{j \in \mathbb{Z}}|j(M_g f)(j)|^2 = \sum_{j \in \mathbb{Z}}|jg(j)f(j)|^2 \leq ||g||_\infty^2 \left( \sum_{j \in \mathbb{Z}}|jf(j)|^2 \right) < \infty.
\]

As \( f \in \text{dom } D \) was arbitrary, \( M_g(\text{dom } D) \subseteq \text{dom } D \), and hence, the commutator \([iD, M_g] \) is a well-defined linear operator on \( \text{dom } D \). Furthermore, \( iD \) and \( M_g \) are diagonal matrices with complex entries (which commute), so the commutator \([iD, M_g] \) is simply a restriction of the 0 operator to \( \text{dom } D \). Theorem 2.2 implies \( M_g \in \text{dom } \delta^n_w \) and \( \delta^n_w(M_g) \) is
the extension of \([iD, M_g]\) to all of \(H\). In particular, \(\delta_w^D(M_g) = 0\). Hence, 
\(M_g \in \ker \delta_w^D\), and since \(g \in \ell^\infty(\mathbb{Z})\) was arbitrary, \(\text{diag}(\ell^\infty(\mathbb{Z})) \subseteq \ker \delta_w^D\).

(d) Part (c) quickly implies \(\text{diag}(\ell^\infty(\mathbb{Z})) \subseteq \ker(\delta_w^D)^n\) for all \(n \in \mathbb{N}\). We now show if \((\delta_w^D)^n(x) = 0\), then \(x \in \text{diag}(\ell^\infty(\mathbb{Z}))\). If \(x \in \text{dom} (\delta_w^D)^n\) and \((\delta_w^D)^n(x) = 0\), then \(x \in B(\ell^2(\mathbb{Z}))\) and \((\delta_w^D)^n(x)_{rc} = 0\) for every \(r, c \in \mathbb{Z}\).

By part (b),

\[
[(\delta_w^D)^n(x)_{rc}]_{\text{dom} D^n} = [(r - c)^n x_{rc}],
\]
thus, \((r - c)^n x_{rc} = 0\) for every \(r, c \in \mathbb{Z}\). If \(r \neq c\), it must be that \(x_{rc} = 0\), i.e., \(x\) must be zero off the diagonal. As \(x \in B(\ell^2(\mathbb{Z}))\), we conclude \(x \in \text{diag}(\ell^\infty(\mathbb{Z}))\). Therefore, \(\ker(\delta_w^D)^n = \text{diag}(\ell^\infty(\mathbb{Z}))\) for all \(n \in \mathbb{N}\).

This kernel stabilization phenomenon initially appears unique to the setting of Example 4.1; the self-adjoint operator is multiplicity-free (the von Neumann algebra generated by its spectral projections is a maximal abelian self-adjoint subalgebra of \(B(\ell^2(\mathbb{Z}))\)) and its eigenvectors constitute our choice of orthonormal basis. Below, we show our example is not unique; kernel stabilization holds for every self-adjoint operator on any Hilbert space.

**Proposition 4.2.** Let \(H\) be a Hilbert space and \(D\) a self-adjoint operator. Then \(\ker \delta_w^D\) is a von Neumann algebra.

**Proof.** The identity \(I\) of \(B(H)\) is easily shown to be in \(\ker \delta_w^D\). Let \(x \in \ker \delta_w^D\). As \(\text{dom} \delta_w^D\) is a *-algebra by Theorem 2.3, \(x^* \in \text{dom} \delta_w^D\). Since \(\delta_w^D\) is a *-derivation, \(\delta_w^D(x^*) = \delta_w^D(x)^* = 0\). Therefore, \(x^* \in \ker \delta_w^D\). Finally, if \(x, y \in \ker \delta_w^D\), then \(xy \in \text{dom} \delta_w^D\) and \(\delta_w^D(xy) = \delta_w^D(x)y + x\delta_w^D(y) = 0\), so \(xy \in \ker \delta_w^D\).

Let \((x_\lambda) \subset \ker \delta_w^D\) be a net converging in the weak operator topology to some \(x \in B(H)\). We show \(x \in \text{dom} \delta_w^D\) and \(\delta_w^D(x) = 0\). Because \(\delta_w^D(x_\lambda) = 0\) for all \(\lambda\), we trivially have \(\delta_w^D(x_\lambda) \stackrel{\text{WOT}}{\to} 0\). By Theorem 2.3, the graph of \(\delta_w^D\) is weak operator topology closed. Therefore, \(x \in \text{dom} \delta_w^D\) and \(\delta_w^D(x) = 0\). We conclude \(\ker \delta_w^D\) is a von Neumann algebra.

**Notation 4.3.** Let \(\mathcal{P}_D\) denote the collection of all spectral projections for \(D\) obtained through the spectral theorem for unbounded self-adjoint operators. Also, let

\[
\mathcal{M}_D := \mathcal{P}_D'.
\]

We give further description of the structure \(\ker \delta_w^D\) in terms of \(\mathcal{M}_D\) in the following lemma and proposition.

**Lemma 4.4.** Suppose \(x \in B(H)\) satisfies \(x(\text{dom} D) \subseteq \text{dom} D\). If \(P \in \mathcal{P}_D\), then

\[
[P, [D, x]]h = [D, [P, x]]h
\]
for all \(h \in \text{dom} D\).
Proof. Let $B(\mathbb{R})$ denote the bounded Borel functions on $\mathbb{R}$, and for each $R \in \mathbb{R}$, define $\text{id}_R : \mathbb{R} \to \mathbb{R}$ by $\text{id}_R(t) = t$ whenever $-R \leq t \leq R$ and $\text{id}_R(t) = 0$ otherwise. The spectral theorem, stated as in Theorem 7.2.8 [11], provides a bounded Borel functional calculus for $D$, that is, a $*$-homomorphism $\Phi_D : B(\mathbb{R}) \to B(H)$ satisfying $\Phi_D(1) = I$,
\[
\text{dom } D = \{ h \in H : \lim_{R \to \infty} \|\Phi_D(\text{id}_R)h\| < \infty \},
\]
and
\[
Dh = \lim_{R \to \infty} \Phi_D(\text{id}_R)h
\]
for all $h \in \text{dom } D$. We claim for each $P \in \mathcal{P}_D$, $P(\text{dom } D) \subseteq \text{dom } D$ and $PDh = DPh$ for all $h \in \text{dom } D$. Given $P \in \mathcal{P}_D$, $P = \Phi_D(\chi_E)$ for some Borel set $E \subseteq \mathbb{R}$. Note that $(\text{id}_R \cdot \chi_E)(t) = 0$ if $t \not\in E \cap [-R, R]$, and otherwise $(\text{id}_R \cdot \chi_E)(t) = t$. Thus, for any $h \in \text{dom } D$,
\[
\lim_{R \to \infty} \|\Phi_D(\text{id}_R)P h\| = \lim_{R \to \infty} \|\Phi_D(\text{id}_R \cdot \chi_E)h\| \leq \lim_{R \to \infty} \|\Phi_D(\text{id}_R)h\| < \infty.
\]
Therefore, $P h \in \text{dom } D$, and as $h \in \text{dom } D$ was arbitrary, $P(\text{dom } D) \subseteq \text{dom } D$. Furthermore,
\[
\|DP h - P D h\| = \lim_{R \to \infty} \|\Phi_D(\text{id}_R)\Phi_D(\chi_E)h - \Phi_D(\chi_E)\Phi_D(\text{id}_R)h\|
\]
\[
= \lim_{R \to \infty} \|\Phi_D(\text{id}_R \cdot \chi_E)h - \Phi_D(\chi_E \cdot \text{id}_R)h\|
\]
\[
= \lim_{R \to \infty} \|\Phi_D(\text{id}_R \cdot \chi_E)h - \Phi_D(\text{id}_R \cdot \chi_E)h\|
\]
\[
= 0.
\]
Let $x \in B(H)$ and suppose $x(\text{dom } D) \subseteq \text{dom } D$. For $h \in \text{dom } D$, observe
\[
[P, [D, x]]h = P(Dx - xD)h - (Dx - xD)Ph
\]
\[
= PDxh - PxDh - DPh + xDPh
\]
\[
= DPxh - PxDh - DPh + xPDh
\]
\[
= DPxh - DPh + xPDh - DPh
\]
\[
= D(Px - xP)h + (xP - Px)Dh
\]
\[
= D(Px - xP)h - (Px - xP)Dh
\]
\[
= [D, [P, x]]h.
\]
Hence, $[P, [D, x]]h = [D, [P, x]]h$ for all $h \in \text{dom } D$, and as $P \in \mathcal{P}_D$ was arbitrary, this equality holds for any spectral projection of $D$. 

Proposition 4.5. $\mathcal{M}_D \subseteq \ker \delta_w^D = \mathcal{M}_D'$.

Proof. Let $P \in \mathcal{P}_D$. By the previous lemma, $[D, P] = 0$ on $\text{dom } D$, so $P \in \text{dom } \delta_w^D$ by Theorem 2.2. Moreover, $\delta_w^D(P)$ is the bounded extension of $i(DP - PD)$ to all of $H$, which is 0. Therefore, $P \in \ker \delta_w^D$. Proposition 4.2 implies $\mathcal{M}_D \subseteq \ker \delta_w^D$.

Let $x \in \ker \delta_w^D$. By Theorem 2.7, $x(\text{dom } D) \subseteq \text{dom } D$ and $\delta_w^D(x)|_{\text{dom } D} = [D, x]|_{\text{dom } D} = 0$. Then, by Theorem X.4.11 [5], $xf(D) \subseteq f(D)x$ for any
Let \( f \in B(\mathbb{R}) \). In particular, when \( f = \chi_E \) for some Borel subset \( E \subseteq \mathbb{R} \) and \( P \) denotes the corresponding spectral projection for \( D \), \( xP = Px \). Hence, \( x \) commutes with all projections in \( \mathcal{P}_D \), and as \( \mathcal{M}_D \) is generated as a von Neumann algebra by these projections, it follows that \( x \in \mathcal{M}_D \).

Let \( x \in \mathcal{M}_D' \). For each \( t \in \mathbb{R} \), \( e^{itD} \in \mathcal{M}_D \). Thus, \( \alpha_t(x) = e^{itD}xe^{-itD} = x \) for all \( t \in \mathbb{R} \). In particular, for any \( h, k \in H \), the function \( t \mapsto \langle \alpha_t(x)h, k \rangle = \langle xh, k \rangle \) is constant, and thus is continuously differentiable with derivative 0. Therefore, \( x \in \ker \delta_w^D \).

We now present our kernel stabilization result.

**Theorem 4.6.** If \( D \) is any self-adjoint operator on a Hilbert space \( H \), then for every \( n \in \mathbb{N} \),

\[
\ker(\delta_w^D)^n = \ker \delta_w^D.
\]

**Proof.** We first show \( \ker(\delta_w^D)^2 = \ker \delta_w^D \). The inclusion \( \ker \delta_w^D \subseteq \ker(\delta_w^D)^2 \) is clear. Let \( x \in \ker(\delta_w^D)^2 \). Proposition 4.5 states \( \ker \delta_w^D = \mathcal{M}_D' \). Thus, it suffices to prove \( x \in \mathcal{M}_D' \), which holds if and only if \( [P, x] = 0 \) for every \( P \in \mathcal{P}_D \). By Proposition 2.6, if \( x \in \ker(\delta_w^D)^2 \), then \( x(\text{dom } D) \subseteq \text{dom } D, \delta_w^D(x)(\text{dom } D) \subseteq \text{dom } D, \) and \( (\delta_w^D)^2(x)|_{\text{dom } D} = [iD, \delta_w^D(x)] \). Since \( (\delta_w^D)^2(x) = 0 \), it must be that \( [iD, \delta_w^D(x)] = 0 \). Thus, Theorem X.4.11 of [5] implies \( \delta_w^D(x) \) commutes with the bounded Borel functional calculus for \( D \), so, in particular, \( [P, \delta_w^D(x)] = 0 \) for every \( P \in \mathcal{P}_D \). Because \( \delta_w^D(x) \) and \( P \) both preserve the domain of \( D \), so does the commutator \( [P, \delta_w^D(x)] \). Thus, Lemma 4.4 implies

\[
0 = [P, \delta_w^D(x)]|_{\text{dom } D} = [P, [iD, x]]|_{\text{dom } D} = [iD, [P, x]]|_{\text{dom } D}.
\]

As \( [P, x] \in B(H) \), \( [P, x]|_{\text{dom } D} \subseteq \text{dom } D, \) and \( [iD, [P, x]] \) is bounded on the domain of \( D \), Theorem 2.7 implies \( [P, x] \in \ker \delta_w^D \). Hence, by Proposition 4.5, \( [P, x] \in \mathcal{M}_D' \). Therefore,

\[
[P, x] = (P + P^\perp)[P, x](P + P^\perp)
= P[P, x]P + P[P, x]P^\perp + P^\perp[P, x]P + P^\perp[P, x]P^\perp
= P[P, x]P + PP^\perp[P, x] + P^\perp P[P, x] + P^\perp[P, x]P^\perp
= P(Px - xP)P + 0 + 0 + P^\perp(Px - xP)P^\perp
= PxP - PxP + 0 + 0 = 0.
\]

As \( P \in \mathcal{P}_D \) was arbitrary, \( x \in \mathcal{M}_D' \). By Proposition 4.5, \( x \in \ker \delta_w^D \).

We proceed by induction on \( n \). The case when \( n = 1 \) is vacuous. Suppose \( \ker(\delta_w^D)^k = \ker \delta_w^D \) for some \( k \in \mathbb{N} \). Let \( x \in \ker(\delta_w^D)^{k+1} \). Then \( \delta_w^D(x) \in \ker(\delta_w^D)^k \), which equals \( \ker \delta_w^D \) by the inductive hypothesis. Hence, \( x \in \ker(\delta_w^D)^2 \). Since we have already shown \( \ker(\delta_w^D)^2 = \ker \delta_w^D \), we have \( x \in \ker \delta_w^D \). Therefore, \( \ker(\delta_w^D)^n = \ker \delta_w^D \) for all \( n \in \mathbb{N} \).
Remark 4.7. Let \( n \in \mathbb{N} \) be arbitrary. Kernel stabilization of \( \delta_u^D \) is equivalent to the following statement: Suppose \( x \in B(H) \), the domains of the iterated commutators \( d^k(x) \) for \( k = 1, \ldots, n \) contain a common core \( \mathcal{X} \) for \( D \), and \( d^k(x) \) is bounded on \( \mathcal{X} \) for all \( k = 1, \ldots, n \). If the continuous bounded extension of \( d^n(x) \) to all of \( H \) belongs to \( M^D_{\mathcal{F}} \), then \( [iD, x]^{(n)} \). Less formally, if \( [iD, \ldots, [iD, x]] \) and all lower commutators are well-defined and bounded on a core for \( D \), then
\[
[iD, \ldots, [iD, x]]^{(n)} = 0 \implies [iD, x] = 0.
\]

We are grateful to the referee’s hunch that this rephrasing of Theorem 4.6 in the case when \( n = 2 \) is similar to a theorem of C.R. Putnam’s in [9]. Upon investigation, we found that when \( n = 2 \), this statement is in fact equivalent to Theorem 1.6.3 of [9] in the self-adjoint setting. Putnam’s proof relies on techniques in the proof of Fuglede’s Theorem, whereas our proof is direct. Establishing the equivalence of these statements requires use of Christensen’s work in [4].

5. Applications of kernel stabilization (Theorem 4.6)

The first application is in the context of Theorem 1.1, which we copy below for convenience.

**Theorem 5.1** (Bratteli-Robinson, Theorem 4 [1]). Let \( \delta \) be a derivation of a \( C^* \)-algebra \( A \), and assume there exists a state \( \omega \) on \( A \) which generates a faithful cyclic representation \( (\pi, H, f) \) satisfying
\[
\omega(\delta(a)) = 0, \quad \forall a \in \text{dom} \delta.
\]
Then \( \delta \) is closable and there exists a symmetric operator \( S \) on \( H \) such that
\[
\text{dom } S = \{ h \in H : h = \pi(a)f \text{ for some } a \in A \}
\]
and \( \pi(\delta(a))h = [S, \pi(a)]h \), for all \( a \in \text{dom} \delta \) and all \( h \in \text{dom } S \). Moreover, if the set \( \mathcal{A}(\delta) \) of analytic elements for \( \delta \) is dense in \( A \), then \( S \) is essentially self-adjoint on \( \text{dom } S \). For \( x \in B(H) \) and \( t \in \mathbb{R} \), define
\[
\alpha_t(x) := e^{St}xe^{-St}
\]
where \( \overline{S} \) denotes the self-adjoint closure of \( S \). It follows that \( \alpha_t(\pi(A)) = \pi(A) \) for all \( t \in \mathbb{R} \), and \( \{\alpha_t\}_{t \in \mathbb{R}} \) is a strongly continuous group of automorphisms with closed infinitesimal generator \( \overline{\delta} \) equaling the closure of \( \pi \circ \delta|_{\mathcal{A}(\delta)} \).

We relate the infinitesimal generator \( \overline{\delta} \) to a derivation \( \delta_u^D \) studied by Christensen. Since the one-parameter automorphism group in Bratteli and Robinson’s Theorem given by \( \alpha_t(x) := e^{itD}xe^{-itD} \) for each \( t \in \mathbb{R} \) is strongly continuous, \( \overline{\delta} \) and \( \delta_u^D \) are precisely the same derivations.
Definition 5.2. An operator \( x \in B(H) \) is uniformly \( D \)-differentiable if there exists \( y \in B(H) \) such that
\[
\lim_{t \to 0} \left\| \frac{\alpha_t(x) - x}{t} - y \right\| = 0.
\]
We denote this by \( x \in \text{dom} \, \delta^D_u \) and \( \delta^D_u(x) = y \).

Proposition 5.3. \( \ker \delta^D_u = \ker \delta^D_w \).

Proof. Theorem 4.1 \([3]\) states \( x \in \text{dom} \, \delta^D_u \) if and only if \( x \in \text{dom} \, \delta^D_w \) and \( t \mapsto \alpha_t(\delta^D_w(x)) \) is norm continuous. Moreover, \( \delta^D_w \) extends \( \delta^D_u \). Thus, \( \ker \delta^D_u \subseteq \ker \delta^D_w \).

Let \( x \in \ker \delta^D_w \). Then \( t \mapsto \alpha_t(\delta^D_w(x)) = 0 \) is norm continuous, and hence, \( x \in \text{dom} \, \delta^D_u \). Moreover, \( \delta^D_u(x) = \delta^D_w |_{\text{dom} \, \delta^D(x)} = 0 \). Therefore, \( x \in \ker \delta^D_u \).

Corollary 5.4. For all \( n \in \mathbb{N} \), \( \ker(\delta^D_u)^n = \ker \delta^D_u \).

Proof. Fix \( n > 1 \) and let \( x \in \ker(\delta^D_w)^n \). Then \( (\delta^D_u)^{n-1}(x) \in \text{dom} \, \delta^D_u \). Hence, \( (\delta^D_u)^{n-1}(x) \in \text{dom} \, \delta^D_u \). Further, as \( x \in \text{dom} \, \delta^D_u \), we have \( x \in \text{dom} \, \delta^D_u \) and \( \delta^D_w(x) = \delta^D_u(x) \). Hence, \( x \in \text{dom} \, \delta^D_w \) and \( (\delta^D_u)^n(x) = (\delta^D_u)^n(x) = 0 \). By Theorem 4.6, \( x \in \ker \delta^D_u \). By Proposition 5.3, \( x \in \ker \delta^D_u \).

Given a self-adjoint operator \( D \), our proof of kernel stabilization of \( \delta^D_w \) relied on the relationship between \( \delta^D_w \) and commutation with \( D \). Intuitively, then, kernel stabilization is likely to occur for a derivation \( \delta \) on an abstract \( C^* \)-algebra that can be implemented, under an appropriate representation, as commutation with a self-adjoint operator. Bratteli and Robinson provide sufficient conditions for when a derivation on a \( C^* \)-algebra has such a representation.

Under this representation \( \pi \), Bratteli and Robinson construct an essentially self-adjoint operator \( S \) which implements the derivation’s action as commutation with \( S \). Once this essentially self-adjoint operator is in place, we use its self-adjoint closure \( D = \overline{S} \) to generate the weak-\( D \) derivation \( \delta^D_w \). We show \( \delta^D_w \) extends \( \delta \circ \pi \) and apply Theorem 4.6 (kernel stabilization of \( \delta^D_w \)) to obtain kernel stabilization of \( \delta \).

Definition 5.5. Given a one-parameter group \( \{ \alpha_t \}_{t \in \mathbb{R}} \) of maps on \( B(H) \), let \( \text{dom} \, \delta \) be the set of all \( x \in B(H) \) so that there exists \( y \in B(H) \) satisfying
\[
\lim_{t \to 0} \left\| \frac{\alpha_t(x) - x}{t} - y \right\| = 0.
\]
For \( x \in \text{dom} \, \delta \), let \( \delta(x) = y \) where \( y \) is the uniform limit described above. We call \( \delta \) the infinitesimal generator for \( \{ \alpha_t \}_{t \in \mathbb{R}} \).

Remark. When \( \alpha_t(x) := e^{itD}xe^{-itD} \) for some self-adjoint operator \( D \), Definition 5.5 is identical to the derivation \( \delta^D_u \) in Definition 5.2.
Lemma 5.6. If \( \delta, A, \pi, \) and \( \widetilde{\delta} \) are as in Theorem 1.1, then
\[
\ker \widetilde{\delta}^n \cap \pi(A(\delta)) = \pi(\ker \delta^n)
\]
for all \( n \in \mathbb{N} \).

Proof. Recall if \( a \in A(\delta) \), then Theorem 1.1 provides \( \widetilde{\delta}(\pi(a)) = \pi(\delta(a)) \). It follows by analyticity of \( a \) that \( \widetilde{\delta}^n(\pi(a)) = \pi(\delta^n(a)) \) for every \( n \in \mathbb{N} \). Suppose \( \widetilde{\delta}^n(\pi(a)) = 0 \). Then \( \pi(\delta^n(a)) = \widetilde{\delta}^n(\pi(a)) = 0 \), and since \( \pi \) is faithful, \( \delta^n(a) = 0 \). Therefore, \( \pi(a) \in \pi(\ker \delta^n) \).

Conversely, suppose \( a \in \ker \delta^n \). Then \( a \in A(\delta) \) because \( \delta^j(a) = 0 \) for all \( j \geq n \) and \( \sum_{k=0}^{\infty} t^k \| \delta^k(a) \| = \sum_{k=0}^{n-1} t^k \| \delta^k(a) \| < \infty \) for any choice of \( t > 0 \). Similar to above, \( \widetilde{\delta}^n(\pi(a)) = \pi(\delta^n(a)) = \pi(0) = 0 \). Therefore, \( \pi(a) \in \ker \widetilde{\delta}^n \cap \pi(A(\delta)) \). The desired equality holds for all \( n \in \mathbb{N} \). \( \Box \)

Theorem 5.7. If \( \delta, A, \pi, \widetilde{\delta}, \) and \( S \) are as in Theorem 1.1, then \( \ker \delta^n = \ker \delta \).

Proof. Fix \( n \in \mathbb{N} \), and let \( a \in \ker \delta^n \). Then, \( a \in A(\delta) \) and \( \pi(a) \in \ker \widetilde{\delta}^n \) by Lemma 5.6. Note \( \widetilde{\delta} = \delta^D \) where \( D = \widetilde{S} \), so Proposition 5.4 implies \( \ker \widetilde{\delta}^n = \ker \widetilde{\delta} \) for all \( n \in \mathbb{N} \). Hence, \( \pi(a) \in \ker \widetilde{\delta} \cap \pi(A(\delta)) \). By another application of Lemma 5.6, we get \( a \in \ker \delta \). Therefore, \( \ker \delta^n = \ker \delta \) for all \( n \in \mathbb{N} \). \( \Box \)

The second application of Theorem 4.6 is related to the Heisenberg Commutation Relation, defined in Definition 1.2.

Example 5.8. The classical example of a pair satisfying the Heisenberg Commutation Relation is the Schrödinger pair, the quantum mechanical position operator \( Q \) and momentum operator \( P \) on \( L^2(\mathbb{R}) \). Let
\[
\text{dom } Q = \{ f \in L^2(\mathbb{R}) : \int_{\mathbb{R}} |xf(x)|^2 \, dx < \infty \}
\]
and, for \( g \in \text{dom } Q \), define \( (Qg)(x) = xg(x) \) for a.e. \( x \in \mathbb{R} \). It is shown in Example 7.1.5 of [11] that \( Q \) defines a self-adjoint operator. If a function \( f \) is absolutely continuous, denote its almost-everywhere defined derivative by \( f' \). Now, let
\[
\text{dom } P = \{ f \in L^2(\mathbb{R}) : f \text{ is absolutely continuous and } f' \in L^2(\mathbb{R}) \},
\]
and for \( h \in \text{dom } P \), define \( Ph := ih' \). It is shown in Theorem 6.30 of [13] that \( P \) defines a self-adjoint operator. Let \( S(\mathbb{R}) \) denote the Schwartz space on \( \mathbb{R} \), that is,
\[
S(\mathbb{R}) = \{ f \in C^\infty(\mathbb{R}) : \forall m, n \in \mathbb{N}, \|Q^m P^n f\|_\infty < \infty \}.
\]
Proposition X.6.5 of [5] shows \( S(\mathbb{R}) \) is dense in \( L^2(\mathbb{R}) \), and it is clear from its definition that \( S(\mathbb{R}) \) is contained in \( \text{dom } Q \cap \text{dom } P \) and is invariant under both \( Q \) and \( P \). Hence, \( S(\mathbb{R}) \subseteq \text{dom } [P,Q] \). Furthermore, \( [P,Q]g = ig \)
for all $g \in S(\mathbb{R})$. Therefore, $P$ and $Q$ satisfy the Heisenberg Commutation Relation.

If two operators are unitarily equivalent to a direct sum of copies of the Schrödinger pair, then they are certainly both unbounded. There are, however, examples of operators satisfying the Heisenberg Commutation Relation where one operator is bounded.

**Example 5.9.** For $f \in L^2[0,1]$, define $(Bf)(x) = xf(x)$ for a.e. $x \in [0,1]$. In contrast to its unbounded analogue $Q$, the operator $B$ is contractive. Let $AC[0,1]$ denote the set of functions which are absolutely continuous on $[0,1]$, and let
$$\text{dom } A = \{ f \in AC[0,1] : f' \in L^2[0,1], f(0) = f(1) \}.$$ 
For $g \in \text{dom } A$, define $Ag = ig'$. Example X.1.12 of [5] shows the operator $A$ with this particular domain is self-adjoint. Due to boundedness of $B$,
$$\text{dom } [A,B] = \{ f \in \text{dom } A : Bf \in \text{dom } A \}.$$ 
Choose
$$K := \{ f \in AC[0,1] : f' \in L^2[0,1], f(0) = f(1) = 0 \}.$$ 
Example X.1.11 of [5] shows $K$ is dense in $L^2[0,1]$ as it contains all polynomials $p$ on $[0,1]$ satisfying $p(0) = p(1) = 0$. Furthermore, we claim $K$ is invariant for $B$. Indeed, products of absolutely continuous functions are again absolutely continuous, so $(Bf)(x) = xg(x)$ for a.e. $x \in [0,1]$ defines an absolutely continuous function. The a.e.-defined derivative of $Bg$ is equivalent to $Bg' + g$ by the product rule. Moreover, $Bg' + g$ belongs to $L^2(\mathbb{R})$ as $g' \in L^2(\mathbb{R})$ and $B \in B(L^2[0,1])$. Lastly,
$$(Bg)(0) = 0 \cdot g(0) = 0 = 1 \cdot 0 = 1 \cdot g(1) = (Bg)(1).$$ 
Thus, $BK \subseteq K$. As a result, $K \subseteq \text{dom } [A,B]$. For $k \in K$, observe
$$[A,B]k = i \left( \frac{d}{dx} (Bk) - B(k') \right) = i(Bk' + k - Bk') = ik.$$ 
Therefore, $A$ and $B$ satisfy the Heisenberg Commutation Relation.

We claim the boundedness of the operators in Examples 5.8 and 5.9 differs due to the relative size of $\text{dom } [P,Q]$ in $L^2(\mathbb{R})$ versus $\text{dom } [A,B]$ in $L^2[0,1]$. In particular, $\text{dom } [A,B]$ does not contain a core for $A$ or $B$, while $\text{dom } [P,Q]$ contains $S(\mathbb{R})$, which is a core for both $P$ and $Q$.

**Theorem 5.10.** Let $A : \text{dom } A \to H$ and $B : \text{dom } B \to H$ be self-adjoint operators which satisfy the Heisenberg Commutation Relation on a dense subspace $K \subseteq H$. If $K$ is a core for both $A$ and $B$, then $A$ and $B$ are both unbounded.

**Proof.** Suppose that $K$ is a core for both $A$ and $B$. It is well-known that $A$ and $B$ cannot both be bounded and satisfy the Heisenberg Relation. Thus, without loss of generality, the only possibilities are that $A$ is bounded and
$B$ is unbounded, or both $A$ and $B$ are unbounded. Suppose that $A \in B(H)$. By the Heisenberg Commutation Relation, $[A, B]k = ik$ for all $k \in K$, or, equivalently, $[iB, A]k = k$ for all $k \in K$.

As $K$ is a core for $B$ and $\|iB, A\|_K = 1$, we have that $A \in \text{dom } \delta^B_w$. Furthermore, $\delta^B_w(A)$ is the continuous extension of the bounded and densely-defined operator $[iB, A]|_K$ to all of $H$, and thus, $\delta^B_w(A) = I$. Trivially, $I \in \text{dom } \delta^B_w$ and $\delta^B_w(I) = 0$, so $A \in \text{dom } (\delta^B_w)^2$ and $(\delta^B_w)^2(A) = 0$. By Theorem 4.6, $A \in \ker(\delta^B_w)^2 = \ker \delta^B_w$. But then

$$0 = \delta^B_w(A)|_K = [iB, A]|_K = I|_K,$$

which is absurd. Therefore, $A$ cannot be bounded. We conclude that if $A$ and $B$ satisfy the Heisenberg Commutation Relation on a common core for $A$ and $B$, then $A$ and $B$ must both be unbounded. □

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References


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