Evolution of the first eigenvalue of weighted \(p\)-Laplacian along the Ricci-Bourguignon flow

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Abstract. Let \(M\) be an \(n\)-dimensional closed Riemannian manifold with metric \(g\), \(d\mu = e^{-\phi(x)}d\nu\) be the weighted measure and \(\Delta_{p,\phi}\) be the weighted \(p\)-Laplacian. In this article we will investigate monotonicity for the first eigenvalue problem of the weighted \(p\)-Laplace operator acting on the space of functions along the Ricci-Bourguignon flow on closed Riemannian manifolds. We find the first variation formula for the eigenvalues of the weighted \(p\)-Laplacian on a closed Riemannian manifold evolving by the Ricci-Bourguignon flow and we obtain various monotonic quantities. At the end we find some applications in 2-dimensional and 3-dimensional manifolds and give an example.

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1. Introduction

A smooth metric measure space is a triple \((M, g, d\mu)\), where \(g\) is a metric, \(d\mu = e^{-\phi(x)}d\nu\) is the weighted volume measure on \((M, g)\) related to function \(\phi \in C^\infty(M)\) and \(d\nu\) is the Riemannian volume measure. Such spaces have been used more widely in the work of mathematicians, for instance, Perelman used it in [13]. Let \(M\) be an \(n\)-dimensional closed Riemannian manifold with metric \(g\).

Over the last few years the geometric flows as the Ricci-Bourguignon flow have been a topic of active research interest in both mathematics and physics. A geometric flow is an evolution of a geometric structure under a differential equation related to a functional on a manifold, usually associated
with some curvature. The family \( g(t) \) of Riemannian metrics on \( M \) is called a Ricci-Bourguignon flow when it satisfies the equations

\[
\frac{d}{dt}g(t) = -2Ric(g(t)) + 2\rho R(g(t))g(t) = -2(Ric - \rho R)g(t),
\]

with the initial condition

\[ g(0) = g_0 \]

where \( Ric \) is the Ricci tensor of \( g(t) \), \( R \) is the scalar curvature and \( \rho \) is a real constant. When \( \rho = 0 \), \( \rho = \frac{1}{2} \), \( \rho = \frac{1}{n} \) and \( \rho = \frac{1}{2(n-1)} \), the tensor \( Ric - \rho Rg \) corresponds to the Ricci tensor, Einstein tensor, the traceless Ricci tensor and Schouten tensor respectively. In fact the Ricci-Bourguignon flow is a system of partial differential equations which was introduced by Bourguignon for the first time in 1981 (see [3]). Short time existence and uniqueness for solution to the Ricci-Bourguignon flow on \([0, T)\) have been shown by Catino et al. in [6] for \( \rho < \frac{1}{2(n-1)} \). When \( \rho = 0 \), the Ricci-Bourguignon flow is the Ricci flow.

Let \( f : M \to \mathbb{R} \), \( f \in W^{1,p}(M) \) where \( W^{1,p}(M) \) is the Sobolev space. For \( p \in [1, +\infty) \), the \( p \)-Laplacian of \( f \) defined as

\[
\Delta_p f = div(|\nabla f|^{p-2}\nabla f) = |\nabla f|^{p-2}\Delta f + (p-2)|\nabla f|^{p-4}(Hess f)(\nabla f, \nabla f).
\]

The Witten-Laplacian is defined by \( \Delta_\phi = \Delta - \nabla \phi \cdot \nabla \), which is a symmetric diffusion operator on \( L^2(M, \mu) \) and is self-adjoint. Now, for \( p \in [1, +\infty) \) and any smooth function \( f \) on \( M \), we define the weighted \( p \)-Laplacian on \( M \) by

\[
\Delta_{p,\phi} f = e^\phi div \left( e^{-\phi}|\nabla f|^{p-2}\nabla f \right) = \Delta_p f - |\nabla f|^{p-2}\nabla \phi \cdot \nabla f.
\]

In the weighted \( p \)-Laplacian when \( \phi \) is a constant function, the weighted \( p \)-Laplace operator is just the \( p \)-Laplace operator and when \( p = 2 \), the weighted \( p \)-Laplace operator is the Witten-Laplace operator.

Let \( \Lambda \) satisfies in \(-\Delta_{p,\phi} f = \Lambda|f|^{p-2}f\), for some \( f \in W^{1,p}(M) \), in this case we say \( \Lambda \) is an eigenvalue of the weighted \( p \)-Laplacian \( \Delta_{p,\phi} \) at time \( t \in [0, T) \). Notice that \( \Lambda \) equivalently satisfies in

\[
-\int_M f \Delta_{p,\phi} f d\mu = \Lambda \int_M |f|^p d\mu,
\]

where \( d\mu = e^{-\phi(x)}d\nu \) and \( d\nu \) is the Riemannian volume measure Using integration by parts, it results that

\[
\int_M |\nabla f|^p d\mu = \Lambda \int_M |f|^p d\mu,
\]

in above equation, \( f(x, t) \) called eigenfunction corresponding to eigenvalue \( \Lambda(t) \). The first non-zero eigenvalue \( \lambda(t) = \lambda(M, g(t), d\mu) \) is defined as follows

\[
\lambda(t) = \inf_{0 \neq f \in W^{1,p}_0(M)} \left\{ \int_M |\nabla f|^p d\mu : \int_M |f|^p d\mu = 1 \right\},
\]
where $W_{0}^{1,p}(M)$ is the completion of $C_{0}^{\infty}(M)$ with respect to the Sobolev norm

$$
\|f\|_{W^{1,p}} = \left( \int_{M} |f|^{p} d\mu + \int_{M} |\nabla f|^{p} d\mu \right)^{\frac{1}{p}}.
$$

The eigenvalue problem for weighted $p$-Laplacian has been extensively studied in the literature [14, 15].

The problem of monotonicity of the eigenvalue of geometric operator is a known and an intrinsic problem. Recently many mathematicians study properties of evolution of the eigenvalue of geometric operators (for instance, Laplace, $p$-Laplace, Witten-Laplace) along various geometric flows (for example, Yamabe flow, Ricci flow, Ricci-Bourguignon flow, Ricci-harmonic flow and mean curvature flow). The main study of evolution of the eigenvalue of geometric operator along the geometric flow began when Perelman in [13] showed that the first eigenvalue of the geometric operator $-4\Delta + R$ is nondecreasing along the Ricci flow, where $R$ is scalar curvature.

Then Cao [5] and Chen et al. [7] extended the geometric operator $-4\Delta + R$ to the operator $-\Delta + cR$ on closed Riemannian manifolds, and investigated the monotonicity of eigenvalues of the operator $-\Delta + cR$ under the Ricci flow and the Ricci-Bourguignon flow, respectively.

Author in [2] studied the monotonicity of the first eigenvalue of Witten-Laplace operator $-\Delta_{\phi}$ along the Ricci-Bourguignon flow with some assumptions and in [1] investigated the evolution for the first eigenvalue of the $p$-Laplacian along the Yamabe flow.

In [11] and [10] have been studied the evolution for the first eigenvalue of geometric operator $-\Delta_{\phi} + \frac{R}{2}$ along the Yamabe flow and the Ricci flow, respectively. For the other recent research in this subject, see [9, 8, 17].

Motivated by the described above works, in this paper, we will study the evolution of the first eigenvalue of the weighted $p$-Laplace operator whose metric satisfying the Ricci-Bourguignon flow (1.1) and $\phi$ evolves by $\frac{\partial \phi}{\partial t} = \Delta \phi$ that is $(M^{n}, g(t), \phi(t))$ satisfying in following system

$$
\begin{aligned}
\frac{d}{dt} g(t) &= -2Ric(g(t)) + 2\rho R(g(t))g(t) = -2(Ric - \rho Rg), \\
\frac{\partial \phi}{\partial t} &= \Delta \phi, \\
g(0) &= g_{0}, \\
\phi(0) &= \phi_{0},
\end{aligned}
$$

where $\Delta$ is Laplace operator of metric $g(t)$.

2. Preliminaries

In this section, we will discuss the differentiable (of first nonzero eigenvalue and its corresponding eigenfunction of the weighted $p$-Laplacian $\Delta_{p,\phi}$ along the flow (1.8)). Let $M$ be a closed oriented Riemannian $n$-manifold and $(M, g(t), \phi(t))$ be a smooth solution of the evolution equations system
In what follows, we assume that $\lambda(t)$ exists and is $C^1$-differentiable under the flow \eqref{flow} in the given interval $t \in [0, T)$. The first nonzero eigenvalue of weighted $p$-Laplacian and its corresponding eigenfunction are not known to be $C^1$-differentiable. For this reason, we apply techniques of Cao \cite{4} and Wu \cite{17} to study the evolution and monotonicity of $\lambda(t) = \lambda(t, f(t))$, where $\lambda(t, f(t))$ and $f(t)$ are assumed to be smooth. For this end, we assume that at time $t_0$, $f_0 = f(t_0)$ is the eigenfunction for the first eigenvalue $\lambda(t_0)$ of $\Delta_{p, \phi}$. Then we have

$$\int_M |f(t_0)|^p \, d\mu_{g(t_0)} = 1. \tag{2.1}$$

Suppose that

$$h(t) := f_0 \left( \frac{\det(g_{ij}(t_0))}{\det(g_{ij}(t))} \right)^{\frac{1}{p(p-1)}}, \tag{2.2}$$

along the Ricci-Bourguignon flow $g(t)$. We assume that

$$f(t) = \frac{h(t)}{\left( \int_M |h(t)|^p \, d\mu \right)^{\frac{1}{p}}}, \tag{2.3}$$

which $f(t)$ is smooth function along the Ricci-Bourguignon flow, satisfied in $\int_M |f|^p \, d\mu = 1$ and at time $t_0$, $f$ is the eigenfunction for $\lambda$ of $\Delta_{p, \phi}$. Therefore, if $\int_M |f|^p \, d\mu = 1$ and

$$\lambda(t, f(t)) = -\int_M f \Delta_{p, \phi} f \, d\mu, \tag{2.4}$$

then $\lambda(t_0, f(t_0)) = \lambda(t_0)$.

3. Variation of $\lambda(t)$

In this section, we will find some useful evolution formulas for $\lambda(t)$ along the flow \eqref{flow}. We first recall some evolution of geometric structure along the Ricci-Bourguignon flow and then give a useful proposition about the variation of eigenvalues of the weighted $p$-Laplacian under the flow \eqref{flow}. From \cite{6}, we have:

**Lemma 3.1.** Under the Ricci-Bourguignon flow equation \eqref{flow}, we get

\begin{enumerate}
\item $\frac{\partial}{\partial t} g^{ij} = 2(R^{ij} - \rho R g^{ij})$,
\item $\frac{\partial}{\partial t} (d\nu) = (n\rho - 1)R d\nu$,
\item $\frac{\partial}{\partial t} (d\mu) = (-\phi_t + (n\rho - 1)R) d\mu$,
\item $\frac{\partial}{\partial t} (\Gamma^k_{ij}) = -\nabla_j R^k_i - \nabla_i R^k_j + \nabla^k R_{ij} + \rho(\nabla_j R \delta^k_i + \nabla_i R \delta^k_j - \nabla^k R g_{ij})$,
\item $\frac{\partial}{\partial t} R = [1 - 2(n-1)\rho] \Delta R + 2 |Ric|^2 - 2\rho R^2$,
\end{enumerate}
where $R$ is scalar curvature.

**Lemma 3.2.** Let $(M, g(t), \phi(t))$, $t \in [0, T)$ be a solution to the flow (1.8) on a closed oriented Riemannian manifold for $\rho < \frac{1}{2(n-1)}$. Let $f \in C^\infty(M)$ be a smooth function on $(M, g(t))$. Then we have the following evolutions:

\[
\frac{\partial}{\partial t} |\nabla f|^2 = 2R^{ij} \nabla_i f \nabla_j f - 2\rho R |\nabla f|^2 + 2g^{ij} \nabla_i f \nabla_j f_t, \tag{3.1}
\]

\[
\frac{\partial}{\partial t} |\nabla f|^{p-2} = (p-2)|\nabla f|^{p-4}\{R^{ij} \nabla_i f \nabla_j f - \rho R |\nabla f|^2 + g^{ij} \nabla_i f \nabla_j f_t\}, \tag{3.2}
\]

\[
\frac{\partial}{\partial t} (\Delta f) = 2R^{ij} \nabla_i \nabla_j f + \Delta f_t - 2\rho R \Delta f - (2-n)\rho \nabla^k R \nabla_k f, \tag{3.3}
\]

\[
\frac{\partial}{\partial t} (\Delta_p f) = 2R^{ij} \nabla_i (Z \nabla_j f) - 2\rho R \Delta_p f + g^{ij} \nabla_i (Z \nabla_j f) \tag{3.4}
\]

\[
+ g^{ij} \nabla_i (Z \nabla_j f_t) + \rho(n-2)Zg^{ij} \nabla_i R \nabla_j f,
\]

\[
\frac{\partial}{\partial t} (\Delta_{p, \phi} f) = 2R^{ij} \nabla_i (Z \nabla_j f) + g^{ij} \nabla_i (Z \nabla_j f) + g^{ij} \nabla_i (Z \nabla_j f_t) \tag{3.5}
\]

\[
- 2\rho R \Delta_{p, \phi} f + \rho(n-2)Zg^{ij} \nabla_i R \nabla_j f - Z \nabla \phi \nabla f - 2Zg^{ij} \nabla_i \phi \nabla_j f - Z \nabla \phi \nabla f_t,
\]

where $Z := |\nabla f|^{p-2}$ and $f_t = \frac{\partial f}{\partial t}$.

**Proof.** By direct computation in local coordinates we have

\[
\frac{\partial}{\partial t} |\nabla f|^2 = \frac{\partial}{\partial t} (g^{ij} \nabla_i f \nabla_j f) \tag{3.1}
\]

\[
= \frac{\partial g^{ij}}{\partial t} \nabla_i f \nabla_j f + 2g^{ij} \nabla_i f \nabla_j f_t \tag{3.1}
\]

\[
= 2R^{ij} \nabla_i f \nabla_j f - 2\rho R |\nabla f|^2 + 2g^{ij} \nabla_i f \nabla_j f_t, \tag{3.1}
\]

which exactly (3.1). We prove (3.2) by using (3.1) as follows

\[
\frac{\partial}{\partial t} |\nabla f|^{p-2} = \frac{\partial}{\partial t} (|\nabla f|^2)^{\frac{p-2}{2}} \tag{3.2}
\]

\[
= \frac{p-2}{2} |\nabla f|^2 \frac{\partial}{\partial t} (|\nabla f|^2) \tag{3.2}
\]

\[
= \frac{p-2}{2} |\nabla f|^{p-4}\{2R^{ij} \nabla_i f \nabla_j f - 2\rho R |\nabla f|^2 + 2g^{ij} \nabla_i f \nabla_j f_t\} \tag{3.2}
\]

\[
= (p-2)|\nabla f|^{p-4}\{R^{ij} \nabla_i f \nabla_j f - \rho R |\nabla f|^2 + g^{ij} \nabla_i f \nabla_j f_t\}, \tag{3.2}
\]
which is (3.2). Now previous lemma and $2\nabla^i R_{ij} = \nabla_j R$ result that
\[
\frac{\partial}{\partial t}(\Delta f) = \frac{\partial}{\partial t}[g^{ij}(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k})] \\
= \frac{\partial g^{ij}}{\partial t}(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k}) + g^{ij}(\frac{\partial^2 f_t}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f_t}{\partial x^k}) - g^{ij} \frac{\partial}{\partial t}(\Gamma^k_{ij} \frac{\partial f}{\partial x^k}) \\
= 2R^{ij}\nabla_i \nabla_j f - 2\rho R \Delta f + \Delta f_t - g^{ij}\left\{-\nabla_j R^k_i - \nabla_i R^k_j + \nabla^k R_{ij}\right\}\nabla_k f \\
- g^{ij}\rho(\nabla_j R \delta^k_i + \nabla_i R \delta^k_j - \nabla^k R g_{ij})\nabla_k f \\
= 2R^{ij}\nabla_i \nabla_j f + \Delta f_t - 2\rho R \Delta f - (2 - n)\rho \nabla^k R \nabla_k f.
\]

Let $Z = |\nabla f|^{p-2}$ we get
\[
\frac{\partial}{\partial t}(\Delta_p f) = \frac{\partial}{\partial t}(\text{div}(|\nabla f|^{p-2}\nabla f)) = \frac{\partial}{\partial t}(g^{ij}\nabla_i(Z\nabla_j f)) \\
= \frac{\partial g^{ij}}{\partial t}\nabla_i Z\nabla_j f + g^{ij}Z\nabla_i \nabla_j f \\
= \frac{\partial g^{ij}}{\partial t}\nabla_i Z\nabla_j f + g^{ij}\nabla_i Zf_t + g^{ij}\nabla_i Z\nabla_j f_t \\
+ Z_1\Delta f + Z\frac{\partial}{\partial t}(\Delta f) \\
= 2R^{ij}\nabla_i Z\nabla_j f - 2\rho R g^{ij}\nabla_i Z\nabla_j f + g^{ij}\nabla_i Zf_t + Z_1\Delta f \\
+ Z\{2R^{ij}\nabla_i \nabla_j f + \Delta f_t - 2\rho R \Delta f - (2 - n)\rho \nabla^k R \nabla_k f\} \\
= 2R^{ij}\nabla_i (Z\nabla_j f) - 2\rho R \Delta_p f + g^{ij}\nabla_i (Zf_t) \\
+ g^{ij}\nabla_i (Z\nabla_j f_t) + \rho(n-2)Zg^{ij}\nabla_i R \nabla_j f.
\]

We have $\Delta_{p,\phi} f = \Delta_p f - |\nabla f|^{p-2}\nabla \phi.\nabla f$. Taking derivative with respect to time of both sides of last equation and (3.4) imply that
\[
\frac{\partial}{\partial t}(\Delta_{p,\phi} f) = \frac{\partial}{\partial t}(\Delta_p f) - Z\frac{\partial g^{ij}}{\partial t}\nabla_i \phi \nabla_j f - Z_t g^{ij}\nabla_i \phi \nabla_j f - Zg^{ij}\nabla_i \phi_t \nabla_j f \\
- Zg^{ij}\nabla_i \phi \nabla_j f_t \\
= 2R^{ij}\nabla_i (Z\nabla_j f) - 2\rho R \Delta_p f + g^{ij}\nabla_i (Zf_t) + g^{ij}\nabla_i (Z\nabla_j f_t) \\
+ \rho(n-2)Zg^{ij}\nabla_i R \nabla_j f - 2ZR^{ij}\nabla_i \phi \nabla_j f + 2pZR g^{ij}\nabla_i \phi \nabla_j f \\
- Z_t g^{ij}\nabla_i \phi \nabla_j f - Zg^{ij}\nabla_i \phi_t \nabla_j f - Zg^{ij}\nabla_i \phi \nabla_j f_t,
\]
it results (3.5). □

**Proposition 3.3.** Let $(M, g(t), \phi(t)), \ t \in [0, T)$ be a solution of the flow (1.8) on the smooth closed oriented Riemannain manifold $(M^n, g_0, \phi_0)$ for $\rho < \frac{1}{2(1-n)}$. If $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted $p$-Laplacian $\Delta_{p,\phi}$ corresponding to the eigenfunction $f(t)$ under the
flow (1.8), then
\[
\frac{\partial}{\partial t} \lambda(t, f(t))_{t=t_0} = \lambda(t_0)(1 - np) \int_M R|f|^p \, d\mu - (1 + \rho p - \rho n) \int_M R|\nabla f|^p \, d\mu \\
+ p \int_M ZR^{ij}\nabla_i f \nabla_j f \, d\mu + \lambda(t_0) \int_M (\Delta \phi)|f|^p \, d\mu \\
- \int_M (\Delta \phi)|\nabla f|^p \, d\mu.
\]  
(3.6)

**Proof.** Let \( f(t) \) be a smooth function where \( f(t_0) \) is the corresponding eigenfunction to \( \lambda(t_0) = \lambda(t_0, f(t_0)) \). \( \lambda(t, f(t)) \) is a smooth function and taking derivative of both sides \( \lambda(t, f(t)) = -\int_M f \Delta_{\rho, \phi} f \, d\mu \) with respect to time, we get
\[
\frac{\partial}{\partial t} \lambda(t, f(t))_{t=t_0} = -\frac{\partial}{\partial t} \int_M f \Delta_{\rho, \phi} f \, d\mu.
\]  
(3.7)

Now by applying condition \( \int_M |f|^p \, d\mu = 1 \) and the time derivative, we can have
\[
\frac{\partial}{\partial t} \int_M |f|^p \, d\mu = 0 = \frac{\partial}{\partial t} \int_M |f|^{p-2} f^2 \, d\mu \\
= \int_M (p-1)|f|^{p-2} f f_t \, d\mu + \int_M |f|^{p-2} f \frac{\partial}{\partial t} (f \, d\mu),
\]  
(3.8)

hence
\[
\int_M |f|^{p-2} f \left[(p-1) f_t \, d\mu + \frac{\partial}{\partial t} (f \, d\mu)\right] = 0.
\]  
(3.9)

On the other hand, using (3.5), we obtain
\[
\frac{\partial}{\partial t} \int_M f \Delta_{\rho, \phi} f \, d\mu = \int_M \frac{\partial}{\partial t} (\Delta_{\rho, \phi} f) \, f \, d\mu + \int_M \Delta_{\rho, \phi} f \frac{\partial}{\partial t} (f \, d\mu) \\
= 2 \int_M R^{ij}\nabla_i (Z \nabla_j f) \, f \, d\mu - 2 \rho \int_M R \Delta_{\rho, \phi} f \, f \, d\mu \\
+ \int_M g^{ij}\nabla_i (Z_t \nabla_j f) \, f \, d\mu + \int_M g^{ij}\nabla_i (Z \nabla_j f_t) \, f \, d\mu \\
+ \rho(n-2) \int_M Z \nabla R, \nabla f \, f \, d\mu - \int_M Z \nabla \phi, \nabla f_t \, f \, d\mu \\
- \int_M Z \nabla \phi_t, \nabla f \, d\mu - \int_M Z \nabla \phi, \nabla f_t \, d\mu \\
- 2 \int_M R^{ij} Z \nabla_i \phi \nabla_j f \, f \, d\mu - \int_M \lambda |f|^{p-2} f \frac{\partial}{\partial t} (f \, d\mu).
\]  
(3.10)

By the application of integration by parts, we can conclude that
\[
\int_M g^{ij}\nabla_i (Z_t \nabla_j f) \, f \, d\mu = -\int_M Z_t |\nabla f|^2 \, d\mu + \int_M Z_t \nabla f, \nabla \phi f \, d\mu.
\]  
(3.11)
Similarly, integration by parts implies that

\[ \int_M g^{ij} \nabla_i(Z \nabla_j f_t) f \, d\mu = - \int_M Z \nabla f_t \cdot \nabla f \, d\mu + \int_M Z \nabla f_t \cdot \nabla \phi f \, d\mu, \tag{3.12} \]

and

\[ \int_M R^{ij} \nabla_i(Z \nabla_j f) f \, d\mu = - \int_M Z R^{ij} \nabla_i f \nabla_j f \, d\mu + \int_M Z R^{ij} \nabla_i f \nabla \phi f \, d\mu \]

\[ - \int_M Z \nabla_i R^{ij} \nabla_j f f \, d\mu. \tag{3.13} \]

But, we can write

\[ 2 \int_M Z \nabla_i R^{ij} \nabla_j f f \, d\mu = 2 \int_M Z g^{ik} g^{jl} \nabla_j f \nabla_i R \nabla f \, d\mu = \int_M Z g^{ik} \nabla j f \nabla_i R f \, d\mu \]

\[ = \int_M R \Delta_p f f \, d\mu - \int_M R |\nabla f|^p \, d\mu. \tag{3.14} \]

Putting (3.14) in (3.13), yields

\[ 2 \int_M R^{ij} \nabla i(Z \nabla_j f) f \, d\mu = -2 \int_M Z R^{ij} \nabla i f \nabla_j f \, d\mu + 2 \int_M Z R^{ij} \nabla i f \nabla \phi f \, d\mu \]

\[ - \int_M \lambda R |f|^p \, d\mu + \int_M R |\nabla f|^p \, d\mu. \tag{3.15} \]

Now, replacing (3.11), (3.12) and (3.15) in (3.10), we obtain

\[ \frac{\partial}{\partial t} \int_M f \Delta_{p,\phi} f \, d\mu = -2 \int_M Z R^{ij} \nabla i f \nabla_j f \, d\mu - \int_M \lambda R |f|^p \, d\mu + \int_M R |\nabla f|^p \, d\mu \]

\[ + 2 \rho \int_M \lambda R |f|^p \, d\mu + \rho (n-2) \int_M Z \nabla R \nabla f f \, d\mu \]

\[ - \int_M Z_t |\nabla f|^2 \, d\mu - \int_M Z \nabla f_t \cdot \nabla f \, d\mu - \int_M Z \nabla \phi_t \cdot \nabla f f \, d\mu \]

\[ - \int_M \lambda |f|^{p-2} f f \frac{\partial}{\partial t} (f \, d\mu). \tag{3.16} \]

On the other hand of Lemma 3.2, we have

\[ Z_t = \frac{\partial}{\partial t} (|\nabla f|^{p-2}) = (p-2) |\nabla f|^{p-4} \{ R^{ij} \nabla i f \nabla_j f - \rho R |\nabla f|^2 + g^{ij} \nabla i f \nabla_j f_t \}. \tag{3.17} \]
Therefore, putting this into (3.16), we get

\[-\frac{\partial}{\partial t}\lambda(t, f(t))_{t=t_0} = -p \int_M Z R^{ij} \nabla_i f \nabla_j f \, d\mu + \lambda(t_0)(2\rho - 1) \int_M R|f|^p \, d\mu + (1 + \rho p - 2\rho) \int_M R|\nabla f|^p \, d\mu + \rho(n - 2) \int_M Z \nabla R. \nabla f \, d\mu - \lambda(t_0) \int_M |f|^{p-2} f \frac{\partial}{\partial t}(f \, d\mu). \]  

(3.18)

Also,

\[-(p - 1) \int_M Z \nabla f_t. \nabla f \, d\mu = (p - 1) \int_M \nabla(Z \nabla f) f_t \, d\mu - (p - 1) \int_M Z \nabla f. \nabla f_t \, d\mu = (p - 1) \int_M f_t \Delta_p \phi f \, d\mu = -(p - 1) \int_M \lambda|f|^{p-2} f_t f \, d\mu. \]  

(3.19)

Then we arrive at

\[-\frac{\partial}{\partial t}\lambda(t, f(t))_{t=t_0} = -p \int_M Z R^{ij} \nabla_i f \nabla_j f \, d\mu + \lambda(t_0)(2\rho - 1) \int_M R|f|^p \, d\mu + (1 + \rho p - 2\rho) \int_M R|\nabla f|^p \, d\mu + \rho(n - 2) \int_M Z \nabla R. \nabla f \, d\mu - \lambda(t_0) \int_M |f|^{p-2} f \left((p - 1) f_t f \, d\mu + \frac{\partial}{\partial t}(f \, d\mu)\right). \]  

(3.20)

Hence, (3.9) yields

\[-\frac{\partial}{\partial t}\lambda(t, f(t))_{t=t_0} = -p \int_M Z R^{ij} \nabla_i f \nabla_j f \, d\mu + \lambda(t_0)(2\rho - 1) \int_M R|f|^p \, d\mu + (1 + \rho p - 2\rho) \int_M R|\nabla f|^p \, d\mu + \rho(n - 2) \int_M Z \nabla R. \nabla f \, d\mu - \lambda(t_0) \int_M |f|^{p-2} f \left((p - 1) f_t f \, d\mu + \frac{\partial}{\partial t}(f \, d\mu)\right). \]  

(3.21)

By integration by parts, we get

\[\int_M Z \nabla \phi_t. \nabla f \, d\mu = \int_M \lambda|f|^p(\Delta \phi) \, d\mu - \int_M (\Delta \phi)|\nabla f|^p \, d\mu \]  

(3.22)

and

\[\int_M Z \nabla R. \nabla f \, d\mu = \int_M \lambda R|f|^p \, d\mu - \int_M R|\nabla f|^p \, d\mu. \]  

(3.23)

Plug in (3.22) and (3.23) into (3.21) imply that (3.6).
Corollary 3.4. Let \((M, g(t)), \ t \in [0, T),\) be a solution of the flow (1.1) on the smooth closed oriented Riemannain manifold \((M^n, g_0)\) for \(\rho < \frac{1}{2(n-1)}\). If \(\lambda(t)\) denotes the evolution the first non-zero eigenvalue of the weighted \(p\)-Laplacian \(\Delta_{p, \phi}\) corresponding to the eigenfunction \(f(x, t)\) under the Ricci-Bourguignon flow where \(\phi\) is independent of \(t\), then

\[
\frac{\partial}{\partial t} \lambda(t, f(t))|_{t=t_0} = \lambda(t_0)(1 - n\rho) \int_M R|f|^p \, d\mu - (1 + \rho\rho - \rho) \int_M R|\nabla f|^p \, d\mu 
+ p \int_M ZR^{ij} \nabla_i f \nabla_j f \, d\mu. \tag{3.24}
\]

We can get the evolution for the first eigenvalue of the geometric operator \(\Delta_p\) under the Ricci-Bourguignon flow (1.1) and along the Ricci flow, which was studied in [17]. Also, in Corollary 3.4, if \(p = 2\) then we can obtain the evolution for the first eigenvalue of the Witten-Laplace operator along the the Ricci-Bourguignon flow (1.1), which was investigated in [2].

Theorem 3.5. Let \((M, g(t), \phi(t)), \ t \in [0, T)\) be a solution of the flow (1.8) on the smooth closed oriented Riemannain manifold \((M^n, g_0)\) for \(\rho < \frac{1}{2(n-1)}\). Let \(R_{ij} - (\beta R + \gamma \Delta \phi) g_{ij} \geq 0\), \(\beta \geq \frac{1 + (n - 2)\rho}{\rho}\) and \(\gamma \geq \frac{1}{3}\) along the flow (1.8) and \(R < \Delta \phi\) in \(M \times [0, T)\). Suppose that \(\lambda(t)\) denotes the evolution the first non-zero eigenvalue of the weighted \(p\)-Laplacian \(\Delta_{p, \phi}\) then

1. If \(R_{\min}(0) \geq 0\), \(\lambda(t)\) is nondecreasing along the Ricci-Bourguignon flow for any \(t \in [0, T)\).
2. If \(R_{\min}(0) > 0\), then the quantity \(\lambda(t)(n - 2R_{\min}(0)T)^{\frac{1}{n}}\) is nondecreasing along the Ricci-Bourguignon flow for \(T \leq \frac{n}{R_{\min}(0)}\).
3. If \(R_{\min}(0) < 0\), then the quantity \(\lambda(t)(n - 2R_{\min}(0)T)^{\frac{1}{n}}\) is nondecreasing along the Ricci-Bourguignon flow for any \(t \in [0, T)\).

Proof. According to (3.6) of Proposition 3.3, we have

\[
\frac{\partial}{\partial t} \lambda(t, f(t))|_{t=t_0} \geq \lambda(t_0)(1 - n\rho) \int_M R|f|^p \, d\mu - (1 + \rho\rho - \rho) \int_M R|\nabla f|^p \, d\mu 
+ p\beta \int_M R|\nabla f|^p \, d\mu + p\gamma \int_M (\Delta \phi)|\nabla f|^p \, d\mu 
+ \lambda(t_0) \int_M R|f|^p \, d\mu - \int_M (\Delta \phi)|\nabla f|^p \, d\mu 
= \lambda(t_0)(2 - n\rho) \int_M R|f|^p \, d\mu + (p\gamma - 1) \int_M R|\nabla f|^p \, d\mu 
+ [p\beta - (1 + \rho\rho - \rho)] \int_M R|\nabla f|^p \, d\mu.
\]

On the other hand, the scalar curvature along the Ricci-Bourguignon flow evolves by

\[
\frac{\partial R}{\partial t} = (1 - 2(n - 1)\rho) \Delta R + 2|\text{Ric}|^2 - 2\rho R^2. \tag{3.26}
\]
The inequality $|Ric|^2 \geq \frac{R^2}{n}$ yields

$$\frac{\partial R}{\partial t} \geq (1 - 2(n - 1)\rho)\Delta R + 2\left(\frac{1}{n} - \rho\right)R^2. \quad (3.27)$$

Since the solution to the corresponding ODE $y' = 2(\frac{1}{n} - \rho)y^2$ with initial value $c = \min_{x \in M} R(0) = R_{\min}(0)$ is

$$\sigma(t) = \frac{nc}{n - 2(1 - \rho)c t}, \quad (3.28)$$

Notice that $\sigma(t)$ defined on $[0, T')$ where $T' = \min\{T, \frac{n}{2(1 - n)\rho c}\}$ when $c > 0$ and on $[0, T)$ when $c \leq 0$. Using the maximum principle to (3.27), we have $R_{g(t)} \geq \sigma(t)$. Therefore, (3.25) becomes

$$\frac{d}{dt}\lambda(t, f(t)) |_{t=t_0} \geq A\lambda(t_0)\sigma(t_0),$$

where $A = p(\beta + \gamma) - \rho(\rho + 2n)$ and this results that in any sufficiently small neighborhood of $t_0$ as $I_0$, we obtain

$$\frac{d}{dt}\lambda(t, f(t)) \geq A\lambda(f, t)\sigma(t).$$

Integrating both sides of the last inequality with respect to $t$ on $[t_1, t_0] \subset I_0$, we have

$$\ln \frac{\lambda(t_0, f(t_0))}{\lambda(f(t_1), t_1)} \geq \ln\left(\frac{n - 2(1 - \rho)c t_1}{n - 2(1 - \rho)c t_0}\right)^{\frac{nA}{2(1 - n)\rho n}}.$$

Since $\lambda(t_0, f(t_0)) = \lambda(t_0)$ and $\lambda(f(t_1), t_1) \geq \lambda(t_1)$, we conclude that

$$\ln \frac{\lambda(t_0)}{\lambda(t_1)} \geq \ln\left(\frac{n - 2(1 - \rho)c t_1}{n - 2(1 - \rho)c t_0}\right)^{\frac{nA}{2(1 - n)\rho n}},$$

that is, the quantity $\lambda(t)(n - 2(1 - \rho)c t)^{\frac{nA}{2(1 - n)\rho n} \sigma}$ is strictly increasing in any sufficiently small neighborhood of $t_0$. Since $t_0$ is arbitrary, then $\lambda(t)(n - 2(1 - \rho)c t)^{\frac{nA}{2(1 - n)\rho n} \sigma}$ is strictly increasing along the flow (1.8) on $[0, T)$. Now we have,

1. If $R_{\min}(0) \geq 0$, by the non-negatively of $R_{g(t)}$ preserved along the Ricci-Bourguignon flow hence $\frac{d}{dt}\lambda(t, f(t)) \geq 0$, consequently $\lambda(t)$ is strictly increasing along the flow (1.1) on $[0, T)$.
2. If $R_{\min}(0) > 0$ then $\sigma(t)$ defined on $[0, T')$, thus the quantity $\lambda(t)(n - 2(1 - \rho)c t)^{\frac{nA}{2(1 - n)\rho n} \sigma}$ is nondecreasing along the flow (1.1) on $[0, T')$.
3. If $R_{\min}(0) < 0$ then $\sigma(t)$ defined on $[0, T')$, thus the quantity $\lambda(t)(n - 2(1 - \rho)c t)^{\frac{nA}{2(1 - n)\rho n} \sigma}$ is nondecreasing along the flow (1.1) on $[0, T')$.

$\square$

**Theorem 3.6.** Let $(M^n, g(t), \phi(t))$, $t \in [0, T)$ be a solution of the flow (1.8) on a closed Riemannian manifold $(M^n, g_0)$ with $R(0) > 0$ for $\rho < \frac{1}{2(n-1)}$. Let $\lambda(t)$ be the first eigenvalue of the weighted $p$-Laplacian $\Delta_{p, \phi}$, then $\lambda(t) \to$
+∞ in finite time for \( p \geq 2 \) where \( \text{Ric} - \nabla \phi \otimes \nabla \phi \geq \beta \text{Rg} \) in \( M \times [0, T) \) and \( \beta \in [0, \frac{1}{n}] \) is a constant.

**Proof.** The weighted \( p \)-Reilly formula on closed Riemannian manifolds (see [16]) as follows

\[
\int_M \left[ (\Delta_{p, \phi} f)^2 - |\nabla f|^{2p-4} |\text{Hess} f|_A^2 \right] d\mu
\]

\[
= \int_M |\nabla f|^{2p-4} (\text{Ric} + \nabla^2 \phi)(\nabla f, \nabla f) d\mu, \quad (3.29)
\]

where \( f \in C^\infty(M) \) and

\[
|\text{Hess} f|^2_A = |\text{Hess} f|^2 + \frac{p-2}{2} \frac{\nabla |\nabla f|^2}{|\nabla f|^2} + \frac{(p-2)^2}{4} \frac{<\nabla f, \nabla |\nabla f|^2>}{|\nabla f|^4}.
\] (3.30)

By a straightforward computation, we have the following inequality:

\[
|\nabla f|^{2p-4} |\text{Hess} f|^2 \geq \frac{1}{n} (\Delta_{p, \phi} f + |\nabla f|^{p-2} <\nabla \phi, \nabla f>)^2 \geq \frac{1}{1 + n} (\Delta_{p, \phi} f)^2 - |\nabla f|^{2p-4} |\nabla \phi, \nabla f|^2. \quad (3.31)
\]

Recall that \( \Delta_{p, \phi} f = -\lambda |f|^{p-2} f \), which implies

\[
\int_M (\Delta_{p, \phi} f)^2 d\mu = \lambda^2 \int_M |f|^{2p-2} d\mu. \quad (3.32)
\]

Combining (3.31) and (3.32), we can write

\[
\int_M \left[ (\Delta_{p, \phi} f)^2 - |\nabla f|^{2p-4} |\text{Hess} f|_A^2 \right] d\mu
\]

\[
\leq (1 - \frac{1}{1 + n}) \lambda^2 \int_M |f|^{2p-2} d\mu + \int_M |\nabla f|^{2p-4} |\nabla \phi, \nabla f|^2 d\mu, \quad (3.33)
\]

putting (3.33) in (3.29) yields

\[
(1 - \frac{1}{1 + n}) \lambda^2 \int_M |f|^{2p-2} d\mu + \int_M |\nabla f|^{2p-4} |\nabla \phi, \nabla f|^2 d\mu \geq
\]

\[
\int_M |\nabla f|^{2p-4} \text{Ric}(\nabla f, \nabla f) d\mu + \int_M |\nabla f|^{2p-4} \nabla^2 \phi(\nabla f, \nabla f) d\mu. \quad (3.34)
\]

By identifying \( \nabla \phi \otimes \nabla \phi(\nabla f, \nabla f) \) with \( |\nabla \phi, \nabla f|^2 \) (see [12]), we obtain

\[
\int_M |\nabla f|^{2p-4} \nabla \phi \otimes \nabla \phi(\nabla f, \nabla f) d\mu = \int_M |\nabla f|^{2p-4} |\nabla \phi, \nabla f|^2 d\mu. \quad (3.35)
\]

Therefore, it and \( \text{Ric} - \nabla \phi \otimes \nabla \phi \geq \beta \text{Rg} \) yield that

\[
(1 - \frac{1}{1 + n}) \lambda^2 \int_M |f|^{2p-2} d\mu
\]

\[
\geq \beta \int_M R|\nabla f|^{2p-2} d\mu + \int_M |\nabla f|^{2p-4} \nabla^2 \phi(\nabla f, \nabla f) d\mu. \quad (3.36)
\]
Now, since $\phi$ satisfies in $\varphi_t = \Delta \varphi$, we get
\[ |\nabla^2 \varphi| \geq \frac{1}{\sqrt{n}} |\Delta \varphi| = \frac{1}{\sqrt{n}} |\varphi_t|. \] (3.37)

Hence,
\[ (1 - \frac{1}{1+n}) \lambda^2 \int_M |f|^{2p-2} d\mu \geq \beta \int_M R|\nabla f|^{2p-2} d\mu + \frac{1}{\sqrt{n}} \int_M |\varphi_t||\nabla f|^{2p-2} d\mu \]
\[ \geq (\beta R_{\min}(t) + \frac{1}{\sqrt{n}} \min_{x \in M} |\varphi_t|) \int_M |\nabla f|^{2p-2} d\mu. \] (3.38)

Multiplying $\Delta_{p,\varphi} f = -\lambda |f|^{p-2} f$ by $|f|^{p-2} f$ on both sides, we obtain
\[ |f|^{p-2} f \Delta_{p,\varphi} f = -\lambda |f|^{2p-2} f. \]

Then integrating by parts and using the Hölder inequality for $p > 2$, we obtain
\[ \lambda \int_M |\nabla f|^{2p-2} d\mu = -\int_M |f|^{p-2} f \Delta_{p,\varphi} f d\mu = (p-1) \int_M |\nabla f|^{p} |f|^{p-2} d\mu \]
\[ \leq (p-1) \left[ \int_M (|\nabla f|^{p})^{\frac{2p-2}{p}} d\mu \right]^{\frac{p}{2p-2}} \left[ \int_M (|f|^{p-2})^{\frac{2p-2}{p-2}} d\mu \right]^{\frac{p-2}{2p-2}} \]
\[ = (p-1) \left[ \int_M |\nabla f|^{2p-2} d\mu \right]^{\frac{p}{2p-2}} \left[ \int_M |f|^{2p-2} d\mu \right]^{\frac{p-2}{2p-2}}. \]

So, we can conclude that
\[ \int_M |\nabla f|^{2p-2} d\mu \geq \left( \frac{\lambda}{p-1} \right)^{\frac{2p-2}{p}} \int_M |f|^{2p-2} d\mu, \]
which implies
\[ \left(1 - \frac{1}{1+n}\right) \lambda^2 \int_M |f|^{2p-2} d\mu \]
\[ \geq \left( \beta R_{\min}(t) + \frac{1}{\sqrt{n}} \min_{x \in M} |\varphi_t| \right) \left( \frac{\lambda}{p-1} \right)^{\frac{2p-2}{p}} \int_M |f|^{2p-2} d\mu, \]
or, more precisely,
\[ \left[ (1 - \frac{1}{1+n}) \lambda^2 - (\beta R_{\min}(t) + \frac{1}{\sqrt{n}} \min_{x \in M} |\varphi_t|)(\frac{\lambda}{p-1})^{\frac{2p-2}{p}} \right] \int_M |f|^{2p-2} d\mu \geq 0. \]

Since $\int_M |f|^{2p-2} d\mu \geq 0$, for $p > 2$ we get
\[ \lambda(t) \geq \left( \beta R_{\min}(t) + \frac{1}{\sqrt{n}} \min_{x \in M} |\varphi_t| \right) \frac{1 + n\alpha}{1 + n\alpha - \alpha} \frac{1}{(p-1)(p-1) \lambda^{2p-2}}. \]
Since $R_{\min}(t) \to +\infty$ (see [6]) and $\min_{x \in M} |\phi_t|$ is finite, then $\lambda(t) \to +\infty$. For $p = 2$, (3.38) yields that
\[
(1 - \frac{1}{1 + n})\lambda^2 \int_M |f|^2 d\mu \geq (\beta R_{\min}(t) + \frac{1}{\sqrt{n}} \min_{x \in M} |\phi_t|) \lambda \int_M |f|^2 d\mu,
\]
hence,
\[
\lambda(t) \geq (\beta R_{\min}(t) + \frac{1}{\sqrt{n}} \min_{x \in M} |\phi_t|) \frac{1 + n\alpha}{1 + \alpha}.
\]
This implies that $\lambda(t) \to +\infty$. \hfill \Box

**Corollary 3.7.** Let $(M, g(t))$, $t \in [0, T)$, be a solution of the flow (1.1) on the smooth closed Riemannian manifold $(M^n, g_0)$, $\phi$ is independent of $t$, $\frac{1}{6} < \rho < \frac{1}{4}$ and $\lambda(t)$ be the first eigenvalue of the weighted $p$-Laplacian $\Delta_{p, \phi}$. If $R_{ij} > \frac{1 + pp - 3\rho}{p} R_{gij}$ on $M^n \times \{0\}$ and $c = R_{\min}(0) \geq 0$ then the quantity $\lambda(t)(3 - 2(1 - 3\rho)ct)^{\frac{3}{2}}$ is nondecreasing along the flow (1.1) for $p \geq 3$.

**Proof.** The pinching inequality $R_{ij} > \frac{1 + pp - 3\rho}{p} R_{gij}$ for $\frac{1}{6} < \rho < \frac{1}{4}$ and $p \geq 3$ is preserved along the Ricci-Bourguignon flow. Therefore, we have
\[
R_{ij} > \frac{1 + pp - 3\rho}{p} R_{gij}, \quad \text{on} \ [0, T) \times M.
\]
Now according to Corollary 3.4, we get
\[
\frac{\partial}{\partial t} \lambda(t, f(t))|_{t=t_0} \geq \lambda(t_0)(1 - n\rho) \int_M R|f|^2 d\mu,
\]
hence, similar to the proof of Theorem 3.5, we have $R_{g(t)} \geq \sigma(t)$ on $[0, T)$ and then
\[
\frac{\partial}{\partial t} \lambda(t, f(t))|_{t=t_0} \geq \lambda(t_0)(1 - n\rho)\sigma(t_0)
\]
thus we arrive at the quantity $\lambda(t)(3 - 2(1 - 3\rho)ct)^{\frac{3}{2}}$ is nondecreasing. \hfill \Box

**Theorem 3.8.** Let $(M, g(t), \phi(t))$, $t \in [0, T)$ be a solution of the flow (1.8) on the smooth closed oriented Riemannian manifold $(M^n, g_0)$ for $\rho < \frac{1}{2(n - 1)}$

Let $0 < R_{ij} < \frac{1 + pp - np}{p} R_{gij}$ on $M^n \times [0, T)$ and $R < \Delta\phi$ in $M \times [0, T)$.

Suppose that $\lambda(t)$ denotes the evolution the first non-zero eigenvalue of the weighted $p$-Laplacian $\Delta_{p, \phi}$ and $C = R_{\max}(0)$ then the quantity $\lambda(t)(1 - CA)^{-\frac{n}{n-1}}$ is strictly decreasing along the flow (1.8) on $[0, T')$ where $T' = \min\{T, \frac{1}{CA}\}$ and $A = 2(n(\frac{1}{p} - 1)^2 - \rho)$.

**Proof.** The proof is similar to proof of Theorem 3.5 with the difference that we need to estimate the upper bound of the right hand (3.6). Notice that $R_{ij} < \frac{1 + pp - np}{p} R_{gij}$ implies that $|\text{Ric}|^2 < n(\frac{1 + pp - np}{p})^2 R^2$. So, the evolution of the scalar curvature under the Ricci-Bourguignon flow evolve by (3.26) and it yields
\[
\frac{\partial R}{\partial t} \leq (1 - 2(n - 1)\rho)\Delta R + 2(n(1 + pp - np)^2 - \rho)R^2.
\] (3.39)
Applying the maximum principle to (3.39), we have \( 0 \leq R_g(t) \leq \gamma(t) \) where
\[
\gamma(t) = \left[ C^{-1} - 2\left(n\left(\frac{1 + p\rho - n\rho}{p}\right)^2 - \rho\right)t \right]^{-1} = \frac{C}{1 - CA}\text{ on } [0, T').
\]
Replacing \( 0 \leq R_g(t) \leq \gamma(t) \) and \( R_{ij} < 1 - \left(\frac{n - p}{n}\right)\rho R_{g_{ij}} \) into equation (3.6), we can write
\[
\frac{d}{dt} \lambda(t, f(t)) \leq \left(1 - \frac{n - p}{n}\rho\right)\lambda(t, f(t)) \text{ in } M^n \times [0, T)
\]
and
\[
\int_M R f^2 d\mu - ap \int_M |\nabla f|^p d\mu \geq 0
\]
combining (3.40), (3.41) and (3.42), we arrive at \( \frac{d}{dt} \lambda(f(t), t) > 0 \) in any sufficiently small neighborhood of \( t_0 \). Since \( t_0 \) is arbitrary, then \( \lambda(t) \) is strictly increasing along the Ricci-Bourguignon flow on \([0, T)\). \( \square \)

**Theorem 3.9.** Let \((M, g(t)), t \in [0, T)\) be a solution of the Ricci-Bourguignon flow (1.1) on a closed manifold \( M^n \) and \( \rho < \frac{1}{2(n-1)} \). Let \( \lambda(t) \) be the first nonzero eigenvalue of the weighted \( p \)-Laplacian of the metric \( g(t) \) and \( \phi \) be independent of \( t \). If there is a non-negative constant \( a \) such that
\[
R_{ij} - \frac{1 - (n - p)\rho}{p} R_{g_{ij}} \geq -ag_{ij} \text{ in } M^n \times [0, T) \tag{3.40}
\]
and
\[
R \geq \frac{pa}{1 - n\rho} \text{ in } M^n \times \{0\} \tag{3.41}
\]
then \( \lambda(t) \) is strictly monotone increasing along the Ricci-Bourguignon flow.

**Proof.** By Corollary 3.4, we write evolution of first eigenvalue as follows
\[
\frac{d}{dt} \lambda(t, f(t))_{t=t_0} = (1 - n\rho)\lambda(t_0) \int_M R f^2 d\mu + p \int_M (R_{ij} - \frac{1 - (n - p)\rho}{p} R_{g_{ij}}) |\nabla f|^p - 2\nabla_i f \nabla_j f d\mu \geq 0
\]
combining (3.40), (3.41) and (3.42), we arrive at \( \frac{d}{dt} \lambda(f(t), t) > 0 \) in any sufficiently small neighborhood of \( t_0 \). Since \( t_0 \) is arbitrary, then \( \lambda(t) \) is strictly increasing along the Ricci-Bourguignon flow on \([0, T)\). \( \square \)

3.1. Variation of \( \lambda(t) \) on a surface. Now, we rewrite Proposition 3.3 and Corollary 3.4 in some remarkable particular cases.

**Corollary 3.10.** Let \((M^2, g(t)), t \in [0, T)\) be a solution of the Ricci-Bourguignon flow on a closed Riemannian surface \((M^2, g_0)\) for \( \rho < \frac{1}{2} \). If \( \lambda(t) \) denotes the evolution of the first eigenvalue of the weighted \( p \)-Laplacian under the Ricci-Bourguignon flow, then:
If $\partial \phi / \partial t = \Delta \phi$ then
\[
\frac{d}{dt} \lambda(t, f(t)) \big|_{t=t_0} = (1 - 2 \rho) \lambda(t_0) \int_M R |f|^p d\mu + \lambda(t_0) \int_M (\Delta \phi) |f|^p d\mu
\]
\[
- (1 + \rho \phi - 2 \rho - \frac{p}{2}) \int_M R |\nabla f|^p d\mu - \int_M (\Delta \phi) |\nabla f|^p d\mu.
\]
(3.43)

If $\phi$ is independent of $t$ then
\[
\frac{d}{dt} \lambda(t, f(t)) \big|_{t=t_0} = (1 - 2 \rho) \lambda(t_0) \int_M R |f|^p d\mu - (1 + \rho \phi - 2 \rho - \frac{p}{2}) \int_M |\nabla f|^p d\mu.
\]
(3.44)

Proof. In dimension $n = 2$, we have $Ric = \frac{1}{2} Rg$, then (3.6) and (3.24) imply that (3.43) and (3.44) respectively. □

Lemma 3.11. Let $(M^2, g(t))$, $t \in [0, T)$, be a solution of the Ricci-Bourguignon flow on a closed surface $(M^2, g_0)$ with nonnegative scalar curvature for $\rho < \frac{1}{2}$, $\phi$ be independent of $t$ and $p \geq 2$. If $\lambda(t)$ denotes the evolution of the first eigenvalue of the weighted $p$-Laplacian under the Ricci-Bourguignon flow, then
\[
\frac{\lambda(0)}{(1 - c(1 - 2 \rho) t^2)} \leq \lambda(t)
\]
on $(0, T')$ where $c = \min_{x \in M} R(0)$ and $T' = \min \left\{ T, \frac{1}{c(1 - 2 \rho)} \right\}$.

Proof. On a surface, we have $Ric = \frac{1}{2} Rg$, and for the scalar curvature $R$ on a closed surface $M$ along the Ricci-Bourguignon flow, we get
\[
\frac{c}{1 - c(1 - 2 \rho) t} \leq R, \quad \text{on } [0, T')
\]
(3.45)
where $T' = \min\{ T, \frac{1}{c(1 - 2 \rho)} \}$. According to (3.44) and $\int_M |f|^p d\mu = 1$, we have
\[
\frac{p c(1 - 2 \rho) \lambda(t, f(t))}{2} \cdot \frac{1}{1 - c(1 - 2 \rho) t} \leq \frac{d}{dt} \lambda(t, f(t))
\]
in any small enough neighborhood of $t_0$. After integrating the above inequality with respect to time $t$, this becomes
\[
\frac{\lambda(0, f(0))}{(1 - c(1 - 2 \rho) t)^2} \leq \lambda(t_0).
\]
Now, $\lambda(0, f(0)) \geq \lambda(0)$ yields that $\frac{\lambda(0)}{(1 - c(1 - 2 \rho) t)^2} \leq \lambda(t_0)$. Since $t_0$ is arbitrary, then $\frac{\lambda(0)}{(1 - c(1 - 2 \rho) t)^2} \leq \lambda(t)$ on $(0, T')$. □

Lemma 3.12. Let $(M^2, g_0)$ be a closed surface with nonnegative scalar curvature and $\phi$ be independent of $t$, then the eigenvalues of the weighted $p$-Laplacian are increasing under the Ricci-Bourguignon flow for $\rho < \frac{1}{2}$.
Proof. Along the Ricci-Bourguignon flow on a surface, we have

\[
\frac{\partial R}{\partial t} = (1 - 2\rho)(\Delta R + R^2)
\]

by the scalar maximum principle, the nonnegativity of the scalar curvature is preserved along the Ricci-Bourguignon flow (see [6]). Then (3.44) implies that \( \frac{\partial}{\partial t} \lambda(t, f(t)) |_{t=t_0} > 0 \), this results that in any sufficiently small neighborhood of \( t_0 \) as \( I_0 \), we get \( \frac{d}{dt} \lambda(t, f(t)) > 0 \). Hence, by integrating on the interval \([t_1, t_0] \subset I_0\), we have \( \lambda(t_1, f(t_1)) \leq \lambda(t_0, f(t_0)) \). Since \( \lambda(t_0, f(t_0)) = \lambda(t_0) \) and \( \lambda(t_1, f(t_1)) \geq \lambda(t_1) \), we conclude that \( \lambda(t_1) \leq \lambda(t_0) \). Therefore, the quantity \( \lambda(t) \) is strictly increasing in any sufficiently small neighborhood of \( t_0 \), but \( t_0 \) is arbitrary, then \( \lambda(t) \) is strictly increasing along the Ricci-Bourguignon flow on \([0, T)\).

3.2. Variation of \( \lambda(t) \) on homogeneous manifolds. In this section, we consider the behavior of the first eigenvalue when we evolve an initial homogeneous metric along the flow (1.8).

Proposition 3.13. Let \((M^n, g(t))\) be a solution of the Ricci-Bourguignon flow on the smooth closed homogeneous manifold \((M^n, g_0)\) for \( \rho < \frac{1}{2(n-1)} \).

Let \( \lambda(t) \) denote the evaluation of an eigenvalue under the Ricci-Bourguignon flow, then

\begin{enumerate}
  \item If \( \frac{\partial \phi}{\partial t} = \Delta \phi \) then
    \[
    \frac{d}{dt} \lambda(t, f(t))|_{t=t_0} = -pR\lambda(t_0) + p \int_M Z R^{ij} \nabla_i f \nabla_j f d\mu + \lambda(t_0) \int_M (\Delta \phi) |f|^p d\mu - \int_M (\Delta \phi) |f|^p d\mu. \tag{3.47}
    \]
  \item If \( \phi \) is independent of \( t \) then
    \[
    \frac{d}{dt} \lambda(t, f(t))|_{t=t_0} = -pR\lambda(t_0) + p \int_M Z R^{ij} \nabla_i f \nabla_j f d\mu. \tag{3.48}
    \]
\end{enumerate}

Proof. Since the evolving metric remains homogeneous and a homogeneous manifold has constant scalar curvature. Therefore (3.6) implies that

\[
\frac{d}{dt} \lambda(t, f(t))|_{t=t_0} = (1 - n\rho)\lambda(t_0)R \int_M f^2 d\mu + ((n - p)\rho - 1)R \int_M |\nabla f|^2 d\mu
+ p \int_M Z R^{ij} \nabla_i f \nabla_j f d\mu + \lambda(t_0) \int_M (\Delta \phi) |f|^p d\mu
- \int_M (\Delta \phi) |f|^p d\mu.
\]

But \( \int_M f^2 d\mu = 1 \) and \( \int_M |\nabla f|^2 d\mu = 1 \) therefore last equation results that (3.47) and (3.48). \]
3.3. Variation of $\lambda(t)$ on 3-dimensional manifolds. In this section, we consider the behavior of $\lambda(t)$ on 3-dimensional manifolds.

**Proposition 3.14.** Let $(M^3, g(t))$ be a solution of the Ricci-Bourguignon flow (1.1) for $\rho < \frac{1}{4}$ on a closed Riemannian manifold $M^3$ whose Ricci curvature is initially positive and there exists $0 \leq \epsilon \leq \frac{1}{3}$ such that

$$\text{Ric} \geq \epsilon Rg.$$ 

If $\phi$ is independent of $t$ and $\lambda(t)$ denotes the evolution of the first eigenvalue of the weighted $p$-Laplacian under the Ricci-Bourguignon flow then the quantity

$$e^{-\int_0^t A(\tau) d\tau} \lambda(t)$$

is nondecreasing along the Ricci-Bourguignon flow (1.1) for $p \leq 3$, where

$$A(t) = \frac{3c(1 - 3\rho)}{3 - 2(1 - 3\rho)ct} + (3\rho + p\epsilon - 1 - \rho p) \left( -2(1 - \rho)t + \frac{1}{C} \right)^{-1},$$

$C = R_{\text{max}}(0)$ and $c = R_{\text{min}}(0)$.

**Proof.** In [6], it has been shown that the pinching inequality $\text{Ric} \geq \epsilon Rg$ and nonnegative scalar curvature are preserved along the Ricci-Bourguignon flow (1.1) on closed manifold $M^3$. Then using (3.24), we obtain

$$\frac{d}{dt} \lambda(f(t)|_{t=t_0}) \geq (1 - 3\rho)\lambda(t_0) \int_M R f^2 d\mu + (3\rho - 1 - \rho p) \int_M R |\nabla f|^2 d\mu$$

$$+ p\epsilon \int_M R |\nabla f|^2 d\mu$$

$$= (1 - 3\rho)\lambda(t_0) \int_M R f^2 d\mu + (3\rho + p\epsilon - 1 - \rho p) \int_M R |\nabla f|^2 d\mu.$$ 

On the other hand, the scalar curvature under the Ricci-Bourguignon flow evolves by (3.26) for $n = 3$. By $|\text{Ric}|^2 \leq R^2$ we have

$$\frac{\partial R}{\partial t} \leq (1 - 4\rho)\Delta R + 2(1 - \rho)R^2.$$ 

Let $\gamma(t)$ be the solution to the ODE $y' = 2(1 - \rho)y^2$ with initial value $C = R_{\text{max}}(0)$. By the maximum principle, we have

$$R(t) \leq \gamma(t) = \left( -2(1 - \rho)t + \frac{1}{C} \right)^{-1} \quad (3.49)$$

on $[0, T')$, where $T' = \min\{T, \frac{1}{2(1 - \rho)C}\}$. Also, similar to proof of Theorem 3.5, we have

$$R(t) \geq \sigma(t) = \frac{3c}{3 - 2(1 - 3\rho)ct} \quad \text{on} \quad [0, T). \quad (3.50)$$
Hence,
\[
\frac{d}{dt} \lambda(t, f(t)) |_{t=t_0} \geq (1 - 3\rho)\lambda(t_0) \frac{3c}{3 - 2(1 - 3\rho)ct_0} \\
\qquad + (\rho - 1 + 2\epsilon)\lambda(t_0) \left(-2(1 - \rho)t_0 + \frac{1}{C}\right)^{-1} \\
= \lambda(t_0)A(t_0).
\]
This yields that in any sufficiently small neighborhood of \( t_0 \) as \( I_0 \), we obtain
\[
\frac{d}{dt} \lambda(t, f(t)) \geq \lambda(f, t)A(t).
\]
Integrating both sides of the last inequality with respect to \( t \) on \([t_1, t_0] \subset I_0\), we can write
\[
\ln \frac{\lambda(t_0, f(t_0))}{\lambda(t_1, f(t_1))} > \int_{t_1}^{t_0} A(\tau)d\tau.
\]
Since \( \lambda(t_0, f(t_0)) = \lambda(t_0) \) and \( \lambda(t_1, f(t_1)) \geq \lambda(t_1) \), we conclude that
\[
\ln \frac{\lambda(t_0)}{\lambda(t_1)} > \int_{t_1}^{t_0} A(\tau)d\tau.
\]
That is, the quantity \( \lambda(t)e^{-\int_0^t A(\tau)d\tau} \) is strictly increasing in any sufficiently small neighborhood of \( t_0 \). Since \( t_0 \) is arbitrary, then \( \lambda(t)e^{-\int_0^t A(\tau)d\tau} \) is strictly increasing along the Ricci-Bourguignon flow on \([0, T]\). □

**Proposition 3.15.** Let \((M^3, g(t))\) be a solution to the Ricci-Bourguignon flow for \( \rho < 0 \) on a closed homogeneous 3-manifold whose Ricci curvature is initially nonnegative and \( \phi \) be independent of \( t \) then the first eigenvalues of the weighted \( p \)-Laplacian is increasing.

**Proof.** In dimension three, the Ricci-Bourguignon flow preserves the nonnegativity of the Ricci curvature is preserved. From (3.48), its implies that \( \lambda(t) \) is increasing. □

**4. Example**

In this section, we consider the initial Riemannian manifold \((M^n, g_0)\) is Einstein manifold and then find evolving first eigenvalue of the weighted \( p \)-Laplace operator along the Ricci-Bourguignon flow.

**Example 4.1.** Let \((M^n, g_0)\) be an Einstein manifold i.e. there exists a constant \( a \) such that \( Ric(g_0) = ag_0 \). Assume that a solution to the Ricci-Bourguignon flow is of the form
\[
g(t) = u(t)g_0, \quad u(0) = 1
\]
where \( u(t) \) is a positive function. By a straightforward computation, we have
\[
\frac{\partial g}{\partial t} = u'(t)g_0, \quad Ric(g(t)) = Ric(g_0) = ag_0 = \frac{a}{u(t)}g(t), \quad R_g(t) = \frac{an}{u(t)},
\]
for this to be a solution of the Ricci-Bourguignon flow, we require
\[ u'(t)g_0 = -2Ric(g(t)) + 2\rho R_{g(t)}g(t) = (-2a + 2\rho n)g_0. \]
This shows that
\[ u(t) = (-2a + 2\rho n)t + 1, \]
so \( g(t) \) is an Einstein metric. Using formula (3.24) for evolution of first eigenvalue along the Ricci-Bourguignon flow, we obtain the following relation
\[
\frac{d}{dt} \lambda(t, f(t)) \big|_{t=t_0} = (1 - n\rho) \frac{\partial}{\partial u(t_0)} \lambda(t_0) \int_M |f|^p d\mu + 2 \frac{a}{u(t_0)} \int_M |\nabla f|^p d\mu \\
- \left((p - n)\rho - 1\right) \frac{\partial}{\partial u(t_0)} \int_M |\nabla f|^p d\mu = \frac{pa(1 - n\rho)\lambda(t_0)}{u(t_0)},
\]
This yields that in any sufficiently small neighborhood of \( t_0 \) as \( t_0 \), we get
\[
\frac{d}{dt} \lambda(t, f(t)) = \frac{pa(1 - n\rho)\lambda(t, f(t))}{(-2a + 2\rho n)t + 1}.
\]
Integrating the last inequality with respect to \( t \) on \([t_1, t_0] \subset I_0\), we have
\[
\ln \frac{\lambda(t_0, f(t_0))}{\lambda(t_1, f(t_1))} = \int_{t_1}^{t_0} \frac{pa(1 - n\rho)}{(-2a + 2\rho n)t + 1} \, dt = \ln \left(\frac{-2a(1 - n\rho)t_1 + 1}{-2a(1 - n\rho)t_0 + 1}\right)^{\frac{1}{p}}.
\]
Since \( \lambda(t_0, f(t_0)) = \lambda(t_0) \) and \( \lambda(t_1, f(t_1)) \geq \lambda(t_1) \), we conclude that
\[
\ln \frac{\lambda(t_0)}{\lambda(t_1)} > \ln \left(\frac{-2a(1 - n\rho)t_1 + 1}{-2a(1 - n\rho)t_0 + 1}\right)^{\frac{1}{p}}.
\]
That is, the quantity \( \lambda(t)|-2a(1 - n\rho)t + 1|^{\frac{1}{p}} \) is strictly increasing along the Ricci-Bourguignon flow on \([0, T]\).

References


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