On orthogonal systems, two-sided bases and regular subfactors

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Abstract. We prove that a regular subfactor of type $II_1$ with finite Jones index always admits a two-sided Pimsner-Popa basis. This is preceded by a pragmatic revisit of Popa’s notion of orthogonal systems.

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1. Introduction

Let $N \subset M$ be a unital inclusion of von Neumann algebras equipped with a faithful normal conditional expectation $E$ from $M$ onto $N$. Then, a finite set $B := \{\lambda_1, \ldots, \lambda_n\} \subset M$ is called a left Pimsner-Popa basis for $M$ over $N$ via $E$ if every $x \in M$ can be expressed as $x = \sum_{i=1}^n E(x\lambda_i^*)\lambda_i$ - see [14, 17, 16, 9, 20] and the references therein. Similarly, $B$ is called a right Pimsner-Popa basis for $M$ over $N$ via $E$ if every $x \in M$ can be expressed as $x = \sum_{j=1}^n \lambda_j E(\lambda_j^*x)$. And, $B$ is said to be a two-sided basis if it is simultaneously a left and a right Pimsner-Popa basis. It is readily seen that a type $II_1$ subfactor that admits a two-sided basis is always extremal (Proposition 3.1).

An extensively exploited result of Pimsner and Popa (from [14]) states that if $N \subset M$ is a subfactor of type $II_1$ with finite Jones index ([7]), then there always exists a left (equivalently, a right) Pimsner-Popa basis for $M$ over $N$ via the unique trace preserving conditional expectation $E_N : M \to N$. As noted above, non-extremal subfactors do not admit two-sided bases. So, it is natural to ask whether there always exists a two-sided basis for every finite index extremal subfactor or not. In fact, it has also been asked publicly

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by Vaughan Jones at various places - see, for instance, the second talk by M. Izumi in the workshop organized in honour of V. S. Sunder’s 60th birthday in Chennai during March-April 2012. Given the fact that every irreducible regular subfactor of finite index is a group subfactor, it is not surprising that such a subfactor always admits a two-sided orthonormal basis, as was illustrated in [6] (also see [2]) . However, it seems to be a difficult question to answer in general. In this article, we answer this question in affirmative for all regular subfactors of type $II_1$ with finite Jones index (without assuming extremality) in:

**Theorem 3.10.** Let $N \subset M$ be a regular subfactor of type $II_1$ with finite Jones index. Then, $M$ admits a two-sided basis over $N$.

As a consequence, we deduce that every finite index regular subfactor of type $II_1$ is extremal.

Recall that an inclusion $Q \subset P$ of von Neumann algebras is said to be regular if its group of normalizers $N_P(Q) := \{u \in U(P) : uQu^* = Q\}$ generates $P$ as von Neumann algebra, i.e., $N_P(Q)'' = P$. Our proof is essentially self contained and does not depend on any structure theorem for regular subfactors.

An effort has been made to keep this article as self-contained as possible. The reader is assumed only to have some basic knowledge of subfactor theory, for instance, as discussed in the first few chapters of [9].

Here is a brief outline of the content of this article.

As mentioned in the abstract, we first revisit, in Section 2, Popa’s ([17]) notion of an orthogonal system for an inclusion of von Neumann algebras $N \subset M$ with a faithful normal conditional expectation from $M$ onto $N$. This generalizes the notion of an orthonormal basis for a subfactor $N \subset M$ of type $II_1$ introduced by Pimsner and Popa in [14]. Dropping orthogonality, Jones and Sunder, in [9], generalized the notion of orthonormal basis and gave another formulation of basis for $M$ over $N$ (as recalled in the first paragraph of Introduction). Very much on the lines of [9], we introduce and discuss the notion of a Pimsner-Popa system, which generalizes Popa’s notion of an orthogonal system.

If $N \subset M$ is an inclusion of finite von Neumann algebras with a fixed faithful normal tracial state $\text{tr}$ on $M$, then for any Pimsner-Popa system $\{\lambda_1, \cdots, \lambda_k\}$ for $N \subset M$ with respect to the unique $\text{tr}$-preserving conditional expectation from $M$ onto $N$, it turns out that the positive operator $f := \sum_{i=1}^k \lambda_i e_1 \lambda_i^*$ is a projection in $M_1$ (Lemma 2.3), which we call the support of the system, where as usual $e_1$ denotes the Jones projection for the canonical basic construction $N \subset M \subset M_1$. An astute reader must have already noticed that, if the support of $\{\lambda_i\}$ equals 1, then it is in fact a Pimsner-Popa basis (in the sense of [9]) for $M$ over $N$.

On the other hand, for a finite index subfactor $N \subset M$ of type $II_1$, we observe that for every projection $f \in M_1$ there exists a Pimsner-Popa system
with support \( f \) (Proposition 2.8). An useful consequence of this observation yields:

**Theorem 2.10** Let \( N \subset M \) be a subfactor of type \( II_1 \) with finite index. Then, any Pimsner-Popa system \( \{\lambda_1, \ldots, \lambda_k\} \) for \( M \) over \( N \) can be extended to a Pimsner-Popa basis for \( M \) over \( N \).

One application being that we deduce in Corollary 2.14 that every subfactor of finite index admits a Pimsner-Popa basis (not necessarily orthonormal) containing at least \( |G| \) many unitaries, where \( G \) is the general Weyl group of the subfactor (as defined in the next paragraph).

Given its importance, an important example of an orthogonal system for a finite index subfactor \( N \subset M \) that we illustrate (in Corollary 2.13) consists of a set containing coset representatives of, what we call, the generalized Weyl group of the subfactor \( N \subset M \), namely, the quotient group

\[
G := N_M(N)/U(N)U(N' \cap M).
\]

This group was first considered by Loi in \([12]\). Clearly, this group agrees with the Weyl group of the subfactor if the subfactor is irreducible, i.e., \( N' \cap M = \mathbb{C} \). Such coset representatives were also considered in \([4, 8, 14, 15, 11, 6]\) in the irreducible setup and used effectively.

Our second important class of examples of Pimsner-Popa systems comes from unital inclusions of finite dimensional \( C^* \)-algebras - see Section 2.2.2. This is done by employing the formalism of path algebras introduced independently by Sunder (\([19]\)) and Ocneanu (\([13]\)). Apart from these, Section 2 is also devoted to a detailed discussion of certain other useful properties related to Pimsner-Popa systems.

Finally, in Section 3, we settle the question of existence of two-sided basis for any finite index regular subfactor \( N \subset M \). This is achieved through a twofold strategy, namely, we first appeal to the formalism of path algebras to get hold of a two-sided basis for \( N' \cap M \) over \( \mathbb{C} \) with respect to the restriction of \( \text{tr}_M \) (in Proposition 3.3), which also turns out to be a two-sided basis for \( \mathcal{R} := N \vee (N' \cap M) \) over \( N \) (Lemma 3.4), and then, thanks to the regularity of \( N \subset M \), every set of coset representatives of the generalized Weyl group of \( N \subset M \) turns out to be a two-sided orthonormal basis consisting of normalizing unitaries for \( M \) over \( \mathcal{R} \) (Proposition 3.7). Ultimately, with an appropriate patching technique (Proposition 3.9), we deduce (in Theorem 3.10) that the product of these two two-sided bases forms a two-sided Pimsner-Popa basis for \( M \) over \( N \). And finally, employing the two-sided bases mentioned above and Watatani's notion of index of a conditional expectation, we derive (in Theorem 3.12) that

\[
[M : N] = |G| \dim_{\mathbb{C}}(N' \cap M),
\]

where \( G \) again denotes the generalized Weyl group of the subfactor \( N \subset M \).
2. Pimsner-Popa bases and systems

Recall, from [17], that given a unital inclusion of von Neumann algebras \( \mathcal{N} \subset \mathcal{M} \) with a faithful normal conditional expectation \( E \) from \( \mathcal{M} \) onto \( \mathcal{N} \), a family \( \{ m_j \} \) in \( \mathcal{M} \) is called a right orthogonal system for \( \mathcal{M} \) over \( \mathcal{N} \) with respect to \( E \) if \( E(m_i^* m_j) = \delta_{ij} f_j \) for some projections \( \{ f_j \} \) in \( \mathcal{N} \). In this article, we will be dealing only with finite right orthogonal systems.

2.1. Pimsner-Popa systems. On the lines of [9, §4.3], Popa’s notion of orthogonal systems generalizes naturally to the following:

**Definition 2.1.** Let \( \mathcal{N} \subset \mathcal{M} \) be a unital inclusion of von Neumann algebras with a faithful normal conditional expectation \( E \) from \( \mathcal{M} \) onto \( \mathcal{N} \). A finite subset \( \{ \lambda_j : j \in J \} \) in \( \mathcal{M} \) will be called a right Pimsner-Popa system for \( \mathcal{M} \) over \( \mathcal{N} \) with respect to \( E \) if the matrix \( Q = [q_{ij}] \) with entries \( q_{ij} := E(\lambda_i^* \lambda_j) \) is a projection in \( \mathcal{M}_J(\mathcal{N}) \).

Such a Pimsner-Popa system will be called a right orthogonal system if \( q_{ij} = \delta_{ij} q_j \) for some projections \( \{ q_j : j \in J \} \subset \mathcal{N} \). If each \( q_j \) is the identity operator, then such an orthogonal system will be called a right orthonormal system.

**Remark 2.2.** (1) Similarly, one defines left systems by considering the matrix \([E(\lambda_i^* \lambda_j)]\) in \( \mathcal{M}_J(\mathcal{N}) \). A collection which is both a left system and a right system will be called a two-sided system.

(2) Hereafter, by a Pimsner-Popa (resp., an orthogonal) system we will always mean a right Pimsner-Popa system for \( \mathcal{M} \) over \( \mathcal{N} \) with respect to \( E \) if the matrix \( Q = [q_{ij}] \) with entries \( q_{ij} := E(\lambda_i^* \lambda_j) \) is a projection in \( \mathcal{M}_J(\mathcal{N}) \).

In this subsection, we systematically study these objects and their generalities in the spirit of Pimsner-Popa basis.

Let \( \mathcal{N} \subset \mathcal{M} \) be a unital inclusion of finite von Neumann algebras with a fixed faithful normal tracial state \( \text{tr} \) on \( \mathcal{M} \) and let \( E_N \) denote the unique trace preserving normal conditional expectation from \( \mathcal{M} \) onto \( \mathcal{N} \). As is standard, \( e_1 \) will denote the Jones projection that implements the basic construction \( \mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \).

**Lemma 2.3.** Let \( \mathcal{N} \subset \mathcal{M} \), \( E_N \) be as in the preceding paragraph and let \( \{ \lambda_1, \ldots, \lambda_k \} \) be a Pimsner-Popa system for \( \mathcal{M}/\mathcal{N} \). Then, the positive operator \( \sum_i \lambda_i e_1 \lambda_i^* \) is a projection in \( \mathcal{M}_1 \).

**Proof.** The idea of the proof is essentially borrowed from [14] and [9]. Consider the projection \( Q = [q_{ij}] := [E_N(\lambda_i^* \lambda_j)] \) in \( M_k(\mathcal{N}) \). Let \( v_i := \lambda_i e_1 \) for
1 ≤ i ≤ k and V ∈ M_k(M_1) be the matrix given by

\[ V = \begin{bmatrix}
  v_1 & v_2 & \cdots & v_n \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{bmatrix}. \]

Now, since \( v_i^*v_j = e_1\lambda^*_i\lambda_j e_1 = q_{ij}e_1 \), we see that \( V^*V = QE = EQ \), where \( E \) is the diagonal matrix \( \text{diag}(e_1, \ldots, e_1) \) in \( M_k(M_1) \). So, \( V \) is a partial isometry in \( M_k(M_1) \). In particular, \( VV^* \) is a projection in \( M_k(M_1) \), thereby implying that \( \sum_i v_i v_i^* = \sum_i \lambda_i e_1\lambda^*_i \) is a projection in \( M_1 \).

\[ \square \]

**Definition 2.4.** Let \( \mathcal{N} \subset \mathcal{M} \) and \( E_N \) be as in Lemma 2.3. For any Pimsner-Popa system \( \{ \lambda_i : 1 \leq i \leq n \} \) for \( \mathcal{M} \) over \( \mathcal{N} \), the projection \( \sum_{i=1}^n \lambda_i e_1\lambda_i^* \in \mathcal{M}_1 \) will be called the support of the system \( \{ \lambda_i : 1 \leq i \leq n \} \).

**Remark 2.5.** (1) A subcollection of an orthogonal (resp., orthonormal) system is also an orthogonal (resp., orthonormal) system.

(2) A Pimsner-Popa system with support equal to 1 turns out to be a Pimsner-Popa basis for \( \mathcal{M} \) over \( \mathcal{N} \) (as mentioned in Section 1). For such a basis, the sum \( \sum_{i=1}^n \lambda_i e_1\lambda_i^* \) is independent of the basis (see [20]) and is called the Watatani index of \( \mathcal{N} \subset \mathcal{M} \). This quantity is denoted by \( \text{Index}_w(\mathcal{N} \subset \mathcal{M}) \).

If \( \mathcal{N} \subset \mathcal{M} \) is a finite index subfactor of type \( II_1 \), then it is known that \( \text{Index}_w(\mathcal{N} \subset \mathcal{M}) = [\mathcal{M} : \mathcal{N}] \) - see [20].

The following useful equivalence is folklore and will be used on few occasions.

**Lemma 2.6.** Let \( \mathcal{N} \subset \mathcal{M} \) and \( E_N \) be as in Lemma 2.3. Then, for any finite set \( \{ \lambda_i : 1 \leq i \leq n \} \) in \( \mathcal{M} \), \( \{ \lambda_i : 1 \leq i \leq n \} \) is a Pimsner-Popa basis for \( \mathcal{M}/\mathcal{N} \) if and only if \( \sum_{i=1}^n \lambda_i e_1\lambda_i^* = 1 \).

Unlike above characterization of a Pimsner-Popa basis (Lemma 2.6), the converse of Lemma 2.3 may not be true; that is, if for some projection \( f \neq 1 \) in \( \mathcal{M}_1 \) there is a finite set \( \{ \lambda_i \} \subset \mathcal{M} \) satisfying \( \sum_i \lambda_i e_1\lambda_i^* = f \), then there is no obvious reason why \( \{ \lambda_i \} \) should be a Pimsner-Popa system for \( \mathcal{M}/\mathcal{N} \). However, in some specific cases the situation is better.

**Proposition 2.7.** Let \( \mathcal{N} \subset \mathcal{M} \) be a subfactor of type \( II_1 \) with \( [\mathcal{M} : \mathcal{N}] < \infty \), \( \{ \lambda_i : 1 \leq i \leq n \} \) be a finite subset of \( \mathcal{M} \) and \( f \) be a projection in \( \mathcal{M}_1 \) satisfying the following three conditions:

1. \( f \geq e_1 \),
2. \( \sum_i \lambda_i e_1\lambda_i^* = f \) and
3. \( \{ \lambda_i : 1 \leq i \leq n \} \subseteq \{ f \}' \cap \mathcal{M} \).

Then, \( \{ \lambda_i : 1 \leq i \leq n \} \) is a Pimsner-Popa system for \( \mathcal{M}/\mathcal{N} \).
Proof. Let $q_{ij} := E_N(\lambda_i^* \lambda_j)$ for $1 \leq i, j \leq n$. Clearly, $q_{ij}^* = q_{ji}$ and we have

$$\left( \sum_k q_{ik} q_{kj} \right) e_1 = \left( \sum_k E_N(\lambda_i^* \lambda_k) E_N(\lambda_k^* \lambda_j) \right) e_1$$

$$= \left( \sum_k E_N(\lambda_i^* \lambda_k E_N(\lambda_k^* \lambda_j)) \right) e_1$$

$$= \sum_k e_1 \lambda_i^* \lambda_k e_1 \lambda_k^* \lambda_j$$

$$= \sum_k e_1 \lambda_i^* f \lambda_j e_1$$

$$= q_{ij} e_1$$

for all $1 \leq i, j \leq n$. So, by the uniqueness part of the Pushdown Lemma [14, Lemma 1.2], we deduce that $\sum_k q_{ik} q_{kl} = q_{ij}$ for all $1 \leq i, j \leq n$. Thus, the matrix $Q := [q_{ij}]$ is a projection in $M_n(N)$. This completes the proof. \hfill \Box

The following observation is the crux of this section.

**Proposition 2.8.** Let $N \subset M$ be as in Proposition 2.7. Then, for any projection $f \in M_1$, there exists a Pimsner-Popa system $\{\lambda_1, \ldots, \lambda_n\}$ for $M/N$ with support equal to $f$.

**Proof.** The proof that we give is inspired by [9, Proposition 4.3.3(a)]. Fix an $n \geq [M : N]$. Since $0 \leq \text{tr}(f) \leq 1$, we obtain $n \geq \text{tr}(f)[M : N]$.

Since $M_n(N)$ is a II$_1$-factor, we can choose a projection $Q \in M_n(N)$ with $
\text{tr}_{M_n(N)}(Q) = \frac{\text{tr}(f)[M : N]}{n}$, Consider the diagonal matrix $P_1 := \text{diag}(f, 0, \ldots, 0)$ in $M_n(M_1)$. Then, $P_1$ is a projection with $\text{tr}_{M_n(M_1)}(P_1) = \frac{\text{tr}(f)}{n}$.

On the other hand, consider the projection $P_0 := QE$ in $M_n(M_1)$, where $E := \text{diag}(e_1, \ldots, e_1)$. Clearly,

$$\text{tr}_{M_n(M_1)}(P_0) = \frac{\sum_i \text{tr}(q_{ii} e_1)}{n} = \frac{\sum_i \text{tr}(q_{ii})}{n [M : N]} = \frac{\text{tr}_{M_n(N)}(Q)}{[M : N]} = \frac{\text{tr}(f)}{n},$$

so that, $P_1 \sim P_0$ in $M_n(M_1)$. Hence, there exists a partial isometry $V \in M_n(M_1)$ such that $V^* V = P_0$ and $V V^* = P_1$. Note that, the condition $VV^* = P_1$ forces $V$ to be of the form

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

for some $v_i$'s in $M_1$. These $v_i$'s then satisfy $\sum_i v_i^* v_i = f$ and $v_i^* v_j = q_{ij} e_1$ for all $1 \leq i, j \leq n$. In particular, $v_i^* v_i = q_{ii} e_1 \leq e_1$ for all $1 \leq i \leq n$.
Thus, \(|v_i| \leq e_1 \leq 1\) and this implies that \(|v_i| = |v_i|e_1\); so that, by polar decomposition of \(v_i\), we obtain \(v_i = w_i|v_i| = w_i|v_i|e_1 = v_ie_1\) for every \(1 \leq i \leq n\), where each \(w_i\) is an appropriate partial isometry.

Therefore, by the Pushdown Lemma [14, Lemma 1.2], we obtain a set \(\{\lambda_1, \ldots, \lambda_n\} \subset M\) such that \(v_i = \lambda_ie_1\) for all \(1 \leq i \leq n\). In particular,

\[
q_{ij}e_1 = v_i^*v_j = e_1\lambda_i^*\lambda_je_1 = E_N(\lambda_i^*\lambda_j)e_1;
\]

so that, by the uniqueness component of Pushdown Lemma, \(q_{ij} = E_N(\lambda_i^*\lambda_j)\) for all \(1 \leq i, j \leq n\). So, \(\{\lambda_1, \ldots, \lambda_n\}\) is a Pimsner-Popa system for \(M/N\) and its support is given by \(\sum_i \lambda_ie_1\lambda_i^* = \sum_i v_iv_i^* = f\).

\begin{proof}

\begin{enumerate}
\item An appropriate customization of above proof actually guarantees the existence of an orthogonal system as well. Indeed, if we choose a projection \(q \in N\) such that \(\text{tr}(q) = \frac{\text{tr}(f|M:N)}{n}\) and let \(Q := \text{diag}(q, q, \ldots, q) \in M_n(N)\) then clearly \(Q\) is a projection with \(\text{tr}_{M_n(N)}(Q) = \frac{\text{tr}(f|M:N)}{n}\). Then, a Pimsner-Popa system \(\{\lambda_1, \ldots, \lambda_n\}\) for \(M/N\) provided by the proof of Theorem 2.8 is in fact an orthogonal system for \(M/N\) with support \(f\).
\item We could even take a projection \(Q = (1, \ldots, 1, q) \in M_n(N)\), where \(q\) is a projection in \(N\) with \(\text{tr}_N(q) = \frac{\text{tr}(f[M:N] - n + 1)}{n}\). This choice of \(Q\) yields an orthogonal system \(\{\lambda_i : 1 \leq i \leq n\}\) with support \(f\) such that \(E_N(\lambda_i^*\lambda_i) = 1\) for all \(1 \leq i \leq n - 1\) and \(E_N(\lambda_n^*\lambda_n) = q\). In particular, if \(f = 1\), then we obtain an orthonormal basis (in the sense of [14]) for \(M/N\).
\end{enumerate}
\end{proof}

As mentioned in the Introduction, the following consequence can be used to construct bases with some specific requirements as we shall see, for instance, in Corollary 2.14.

**Theorem 2.10.** Let \(N \subset M\) be as in Proposition 2.7. Then, any Pimsner-Popa system \(\{\lambda_1, \ldots, \lambda_k\}\) for \(M/N\) can be extended to a Pimsner-Popa basis for \(M/N\).

\begin{proof}

Let \(f\) denote the support of the given system \(\{\lambda_i : 1 \leq i \leq k\}\). By Proposition 2.8, there exists a Pimsner-Popa system \(\{\lambda_{k+1}, \ldots, \lambda_{k+l}\}\) for \(M/N\) with support \(1 - f\). Then,

\[
\sum_{i=1}^{k+l} \lambda_i e_1 \lambda_i^* = \sum_{i=1}^{k} \lambda_i e_1 \lambda_i^* + \sum_{i=1}^{l} \lambda_{k+i} e_1 \lambda_{k+i}^* = f + (1 - f) = 1.
\]

Thus, by Lemma 2.6, \(\{\lambda_1, \ldots, \lambda_k, \lambda_{k+1}, \ldots, \lambda_{k+l}\}\) is a Pimsner-Popa basis for \(M/N\).
\end{proof}
2.2. Examples of Pimsner-Popa systems.

2.2.1. Pimsner-Popa bases and intermediate subalgebras. Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of finite von Neumann algebras. Let $\mathcal{P}$ be an intermediate von Neumann subalgebra, i.e., $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$. Fix a faithful normal tracial state on $\mathcal{M}$ and let $e_\mathcal{P}$ denote the canonical Jones projection for the basic construction $\mathcal{P} \subset \mathcal{M} \subset \mathcal{P}_1$. Let $\{\lambda_i\}$ be a finite set in $\mathcal{P}$. If $\{\lambda_i\}$ is a Pimsner-Popa basis for $\mathcal{P}/\mathcal{N}$, then it is easy to see that $\{\lambda_i\}$ is a Pimsner-Popa system for $\mathcal{M}/\mathcal{N}$ with support $e_\mathcal{P}$. Indeed, for any $x \in \mathcal{M}$, we have

$$
(\sum_i \lambda_i e_1 \lambda_i^*) x \Omega = \sum_i \lambda_i E_{\mathcal{N}}^\mathcal{M}(\lambda_i^* x) \Omega = \sum_i \lambda_i E_{\mathcal{N}}^\mathcal{P}(\lambda_i^* E_\mathcal{P}(x)) \Omega = E_\mathcal{P}^\mathcal{M}(x) \Omega = e_\mathcal{P}(x \Omega),
$$

where the second last equality holds because $\{\lambda_i\}$ is a basis for $\mathcal{P}$ over $\mathcal{N}$.

2.2.2. Inclusion of finite dimensional $C^*$-algebras. Let $A_0 \subset A_1$ be a unital inclusion of finite dimensional $C^*$-algebras with dimension vectors $\overrightarrow{m} = [m_1, \ldots, m_k]$ and $\overrightarrow{n} = [n_1, \ldots, n_l]$, respectively; so that

$$
A_0 \cong M_{m_1}(\mathbb{C}) \oplus \cdots \oplus M_{m_k}(\mathbb{C}) \quad \text{and} \quad A_1 \cong M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_l}(\mathbb{C}).
$$

We briefly recall the formalism of path algebras associated to such an inclusion, introduced independently by Ocneanu ([13]) and Sunder ([19]). For details, we refer the reader to [9, §5.4].

Let $\hat{C}$ denote the set of minimal central projections of a finite dimensional $C^*$-algebra $C$. With this notation, let $\hat{A}_0 = \{p_1^{(0)}, \ldots, p_{k}^{(0)}\}$ and $\hat{A}_1 = \{p_1^{(1)}, \ldots, p_{l}^{(1)}\}$. Let $A_{-1} := \mathbb{C}$ and put $\hat{\Omega} = \{\ast\}$. Consider the Bratteli diagram for $\mathbb{C} \subset A_0$ and let $\Omega_{[0]}$ denote the set of all directed edges starting from $\ast$ and ending at $p_i^{(0)}$ for some $1 \leq i \leq k$. Similarly, let $\Omega_{[0,1]}$ denote the set of edges in the Bratteli diagram of $A_0 \subset A_1$, and $\Omega_{[1]}$ denote the set of all paths starting from $\ast$ and ending at $p_j^{(1)}$ for some $1 \leq j \leq l$. For any edge or path $\beta$, $s(\beta)$ and $r(\beta)$ denotes the source vertex and range vertex of $\beta$. Let $\mathcal{H}_{[0]}$, $\mathcal{H}_{[0,1]}$ and $\mathcal{H}_{[1]}$ denote the corresponding Hilbert spaces with orthonormal bases indexed by $\Omega_{[0]}$, $\Omega_{[0,1]}$ and $\Omega_{[1]}$, respectively. Then, from [19] (also see [9]), there exist $C^*$-subalgebras $B_0 \subset B_1 \subset \mathcal{L}(\mathcal{H}_{[1]})$ such that the inclusion $A_0 \subset A_1$ is isomorphic to the inclusion $B_0 \subset B_1$ - see [9, Proposition 5.4.1(v)]. The pair $B_0 \subset B_1$ is called the path algebra model of the pair $A_0 \subset A_1$.

Fix $\lambda, \mu \in \Omega_{[1]}$ with same end points. Define $e_{\lambda,\mu} \in B_1$ by

$$
e_{\lambda,\mu}(\alpha, \beta) = \delta_{\lambda,\alpha} \delta_{\mu,\beta} \quad \text{for all} \quad \alpha, \beta \in \Omega_{[1]}.
$$

Then, the set $\{e_{\lambda,\mu} : \lambda, \mu \in \Omega_{[1]} \text{ with } r(\lambda) = r(\mu)\}$ forms a system of matrix units for $B_1$ - see [9, Proposition 5.4.1 (iv)].
Now, let us assume that $A_0 \subset A_1$ has a faithful tracial state $\text{tr}$ on $A_1$. Let $E_{A_0}^{A_1} : A_1 \rightarrow A_0$ denote the unique tr-preserving conditional expectation. Let $\bar{t}^{(1)}$ be the trace vector corresponding to $\text{tr}$ and $\bar{t}^{(0)}$ be the one corresponding to $\text{tr}|_{A_0}$. Then, by [19] (also see [9]), we have

$$E_{B_0}(e_{\lambda,\mu}) = \delta_{\lambda_{0,0},\mu_{0,0}} \frac{\bar{t}^{(1)}_{\lambda(\lambda)}}{\bar{t}^{(0)}_{\lambda(0)}} e_{\lambda_{0,0},\mu_{0,0}}. \tag{2.1}$$

Now, consider $I := \{(\kappa, \beta) : \kappa \in \Omega_{[0,1]}, \beta \in \Omega_1, r(\kappa) = r(\beta)\}$ and, for each $(\kappa, \beta) \in I$, let

$$a_{\kappa,\beta} := \sum_{\theta \in \Omega_0, r(\theta) = s(\kappa)} e_{\theta \kappa, \beta}.$$

Then, by [9, Proposition 5.4.3], we have

$$E_{B_0}(a_{\kappa,\beta}(a_{\kappa',\beta'})^*) = \delta_{(\kappa,\beta),(\kappa',\beta')} \frac{\bar{t}^{(1)}_{\lambda(\kappa)}}{\bar{t}^{(0)}_{\lambda(0)}} \sum_{\theta, \theta' \in \Omega_0, r(\theta) = r(\theta') = s(\kappa)} e_{\theta \kappa, \theta' \kappa}. \tag{2.2}$$

Further, for each $p \in \widehat{A_0}$, consider a projection $j_p \in B_0$ (as in [9, Lemma 5.7.3]) given by

$$j_p = \frac{1}{n_p^2} \sum_{\alpha,\alpha' \in \Omega_0, r(\alpha) = r(\alpha') = p} e_{\alpha \kappa, \alpha' \kappa'}$$

where $(n_p^0)^2 = \dim p A_0$, and let $\lambda_{\kappa,\beta} := \left( \frac{n_p^{(0)}_{r(\kappa)}}{n_p^{(1)}_{r(\kappa)}} \right)^{-1/2} a_{\kappa,\beta}$. Then, by Equation 2.2, we obtain

$$E_{B_0}(\lambda_{\kappa,\beta}(\lambda_{\kappa',\beta'})^*) = \delta_{(\kappa,\beta),(\kappa',\beta')} j_{s(\kappa)}.$$

Therefore, $\{\lambda_{\kappa,\beta} : (\kappa, \beta) \in I\}$ is a left orthogonal system for $A_1/A_0$. This example will have a significant role to play in Section 3.

We will discuss some further useful properties of Pimsner-Popa systems in Section 2.4. Before that, let us digress to an important class of examples of orthonormal systems consisting of unitaries.

### 2.3. Generalized Weyl group and orthonormal systems.

In this subsection, we illustrate an important example of an orthonormal system consisting of unitaries, which will attract a good share of limelight of this article. Let $N \subset M$ be a subfactor of type $II_1$ (which is not necessarily irreducible), let $\mathcal{U}(N)$ (resp., $\mathcal{U}(M)$) denote the group of unitaries of $N$ (resp., $M$) and $\mathcal{N}_M(N) := \{u \in \mathcal{U}(M) : uNu^* = N\}$ denote the group of unitary normalizers of $N$ in $M$. It is straightforward to see that $\mathcal{U}(N)\mathcal{U}(N' \cap M) = \mathcal{U}(N' \cap M)\mathcal{U}(N)$ is a normal subgroup of $\mathcal{N}_M(N)$.
Definition 2.11. [12] The generalized Weyl group of a subfactor $N \subset M$ is defined as the quotient group

$$G := \mathcal{N}_M(N)/\mathcal{U}(N)\mathcal{U}(N' \cap M).$$

This group first appeared in [12, Proposition 5.2]. Note that the generalized Weyl group of an irreducible subfactor agrees with its Weyl group, namely, the quotient group $\mathcal{N}_M(N)/\mathcal{U}(N)$.

The following two useful observations are well known for irreducible subfactors - see, for instance, [6, 8, 14, 15, 11, 12]. For the non-irreducible case, their proofs can be extracted readily from [12, Proposition 5.2].

Lemma 2.12. [12] Let $w \in \mathcal{N}_M(N) \setminus \mathcal{U}(N)\mathcal{U}(N' \cap M)$. Then, $E_N(w) = 0$.

In particular, for any two elements $v, u \in \mathcal{N}_M(N)$, $E_N(vu^*) = 0 = E_N(v^*u)$ if $[u] \neq [v]$ in the generalized Weyl group $G$.

Corollary 2.13. [12] Suppose $[M : N] < \infty$ and $G$ denotes the generalized Weyl group of the subfactor $N \subset M$. Then, any set of coset representatives $\{u_g : g = [u_g] \in G\}$ of $G$ in $\mathcal{N}_M(N)$ forms a two-sided orthonormal system for $M/N$. Also, $G$ is a finite group with order $\leq [M : N]$.

Corollary 2.14. Every finite index subfactor of type $II_1$ admits a Pimsner-Popa basis containing at least $|G|$ many unitaries.

Proof. By Corollary 2.13, there exists an orthonormal system for $M/N$ consisting of unitaries. Then, by Theorem 2.10, this orthonormal system can be extended to a Pimsner-Popa basis for $M/N$. This completes the proof.

Remark 2.15. Corollary 2.14 could be related somewhat to a recent question asked by Popa in [18] about the maximum number of unitaries possible in an orthonormal basis (in the sense of [14]) of a given subfactor. It, at least, tells us that every finite index subfactor $N \subset M$ of type $II_1$ always admits a Pimsner-Popa basis (not necessarily orthonormal) containing at least $|G|$ many unitaries.

In view of Corollary 2.14, calculating cardinality of $G$ becomes quite relevant. However, in practice, we are yet to find a suitable way to calculate the cardinality of $G$. Since the generalized Weyl group is the same as the Weyl group of an irreducible subfactor, it is always non-trivial for such a subfactor.

2.4. Some useful properties related to Pimsner-Popa systems.

Let $(N, P, Q, M)$ be a quadruple of $II_1$-factors, i.e., $N \subset P, Q \subset M$, with $[M : N] < \infty$. Let $\{\lambda_i : i \in I\}$ and $\{\mu_j : j \in J\}$ be (right) Pimsner-Popa bases for $P/N$ and $Q/N$, respectively. Consider two auxiliary operators $p(P, Q)$ and $p(Q, P)$ (as in [1]) given by

$$p(P, Q) = \sum_{i,j} \lambda_i \mu_j e_1 \mu_j^* \lambda_i^*$$

and

$$p(Q, P) = \sum_{i,j} \mu_j \lambda_i e_1 \lambda_i^* \mu_j^*. $$
Proof. Let $\{\lambda_i : i \in I\}$ be a Pimsner-Popa system for $M/Q$ with support $\frac{1}{\lambda} p(P, Q)$. Then, the following hold:

1. $\{\frac{1}{\sqrt{\lambda}} \lambda_i : i \in I\}$ is a Pimsner-Popa system for $M/Q$ with support $\frac{1}{\lambda} p(P, Q)$.

2. If $(N, P, Q, M)$ is a commuting square, then $\{\lambda_i\}$ can be extended to a Pimsner-Popa basis for $M/Q$.

Proof. (1) From [1, Lemma 3.2], we know that $\frac{1}{\lambda} p(P, Q)$ is a projection and, by [1, Lemma 3.4], $e_Q$ is a subprojection of $\frac{1}{\lambda} p(P, Q)$. Further, by [1, Proposition 2.25], we know that $p(P, Q) \in P' \cap Q_1$; so, it follows that $\{\lambda_i : i \in I\} \subseteq \{\frac{1}{\lambda} p(P, Q)\}' \cap M$. Also, we have

$$\sum_i \frac{1}{\sqrt{\lambda}} \lambda_i e_Q \frac{1}{\sqrt{\lambda}} \lambda_i^* = \frac{1}{\lambda} p(P, Q).$$

Thus, in view of Proposition 2.7, $\{\frac{1}{\sqrt{\lambda}} \lambda_i : i \in I\}$ is a Pimsner-Popa system for $M/Q$ with support $\frac{1}{\lambda} p(P, Q)$.

(2) Suppose that $(N, P, Q, M)$ is a commuting square. Then, by [1, Propositions 2.14 & 2.20], we know that $p(P, Q)$ is a projection. Thus, $\lambda = \|p(P, Q)\| = 1$ and the conclusion follows from (1) and Theorem 2.10. \qed

Proposition 2.17. Let $N \subset M$ be an irreducible subfactor of type $II_1$ with finite index and $\{\lambda_i\}$ be a Pimsner-Popa system for $M/N$ with support lying in $N' \cap M_1$. Then, $1 \leq \sum_i \lambda_i \lambda_i^* \leq [M : N]$.

Proof. Let $f$ denote the support of $\{\lambda_i\}$, i.e., $f = \sum_i \lambda_i e_1 \lambda_i^*$. Then, we obtain $\sum_i \lambda_i \lambda_i^* = [M : N] E_M(f)$. Since $N' \cap M = \mathbb{C}$, we have $E_M(f) = \text{tr}(f) \in [0, 1]$. Therefore, $\sum_i \lambda_i \lambda_i^* \leq [M : N]$.

On the other hand, since $f \in N' \cap M_1$ and $N' \cap M = \mathbb{C}$, by [14, Proposition 1.9], we have $\text{tr}(f) \geq \tau$. Then, by irreducibility of $N \subset M$ again, we have $\text{tr}(f) = E_M(f) = \tau \sum_i \lambda_i \lambda_i^*$. Hence, $\sum_i \lambda_i \lambda_i^* \geq 1$. \qed

We conclude this section with a small observation on a kind of local behaviour of orthogonal systems. Recall, from [7], that for a subfactor $N \subset M$ and a projection $f \in N' \cap M$, the index of $N$ at $f$ is given by $[M_f : N_f] = [M : N]_f$. Also, a finite index subfactor $N \subset M$ is said to be extremal, if $\text{tr}_N$ and $\text{tr}_M$ agree on $N' \cap M$. Clearly, if $N \subset M$ is irreducible, then it is extremal.
Proposition 2.18. Let $N \subset M$ be an irreducible subfactor of type $\text{II}_1$ with $[M : N] < \infty$ and $f \in N' \cap M_1$ be a projection. Then, for any orthogonal system $\{\lambda_i\}$ with support $f$, we have $\sum_i \lambda_i \lambda_i^* = \sqrt{[M_1 : N]} f$.

**Proof.** Since $N \subset M$ is extremal, the following local index formula holds (see [7]):

$$[fM_1 f : Nf] = [M_1 : N](\text{tr}_{M_1}(f))^2 = ([M : N]\text{tr}_{M_1}(f))^2.$$ 

On the other hand, since $\{\lambda_i\}$ is an orthogonal system, we obtain $\sum_i \lambda_i \lambda_i^* = [M : N]\text{tr}_{M_1}(f)$. This completes the proof. \hfill $\square$

3. Regular subfactor and two-sided basis

Before we pursue our hunt for a two-sided basis in a regular subfactor, as asserted in the Introduction, we first show that every finite index subfactor with a two-sided basis is extremal, which, most likely, is folklore.

**Proposition 3.1.** Let $N \subset M$ be a type $\text{II}_1$ subfactor with finite index. If there exists a two-sided basis for $M$ over $N$, then $N \subset M$ is extremal.

**Proof.** Given any right basis $\{\lambda_i : 1 \leq i \leq n\}$ for $M/N$, it is known (see, for instance, [1, Lemma 2.23]) that the $\text{tr}_{N'}$ preserving conditional expectation $E_{M'} : N' \to M'$ is given by

$$E_{M'}(x) = [M : N]^{-1}\sum_i \lambda_i x \lambda_i^*, \ x \in N'.$$

Thus, if $x \in N' \cap M$, then

$$\text{tr}_{N'}(x) = E_{M'\cap M}(x) = [M : N]^{-1}\sum_i \lambda_i x \lambda_i^*.$$ 

Now, let $\{\lambda_i : 1 \leq i \leq n\}$ be any two-sided basis for $M/N$. Then, we have $\sum_i \lambda_i^* e_1 \lambda_i = 1 = \sum_i \lambda_i e_1 \lambda_i^*$ so that $\sum_i \lambda_i^* \lambda_i = [M : N]1_M$ (after applying $E_{M'}$ on both sides of first equality). Thus, for any $x \in N' \cap M$, we have

$$\text{tr}_M(x) = [M : N]^{-1}\text{tr}_M(\sum_i \lambda_i^* \lambda_i)$$

$$= [M : N]^{-1}\text{tr}_M(\sum_i \lambda_i x \lambda_i^*)$$

$$= \text{tr}_M(\text{tr}_{N'}(x)1_M)$$

$$= \text{tr}_{N'}(x).$$

Hence, $N \subset M$ is extremal. \hfill $\square$

As the header suggests, this section is devoted to proving the existence of two-sided basis for a finite index regular subfactor. Keeping this in mind, from now onward, throughout this section, $N \subset M$ will denote a finite index subfactor of type $\text{II}_1$, which is not necessarily irreducible, and $\mathcal{R}$ will denote the intermediate von Neumann subalgebra generated by $N$ and $N' \cap M$, i.e., $\mathcal{R} = N \vee (N' \cap M)$. We first present some preparatory results that we require to deduce the main theorem.
Lemma 3.2. With notations as in the preceding paragraph, we have
\[ N_M(N) \subseteq N_M(R). \]

Proof. Let \( u \in N_M(N) \). Then, \( uNu^* = N \), and for \( x \in N' \cap M \), we have
\[ (uxu^*)n = uxu^*nuu^* = uu^*nuu^* = n(uxu^*) \]
for all \( n \in N \), i.e., \( u(N' \cap M)u^* = N' \cap M \). So, \( u(nxu^*) = (unu^*)(uxu^*) \in N \lor (N' \cap M) \)
for all \( n \in N \) and \( x \in N' \cap M \). Thus, we readily deduce that \( uRu^* = R \). \( \square \)

The following crucial ingredient is an adaptation of [9, Lemma 5.7.3].

Proposition 3.3. Let \( \text{tr} \) denote the restriction of \( \text{tr}_M \) on \( N' \cap M \). Then, \( N' \cap M \) has a two-sided Pimsner-Popa basis over \( C \) with respect to \( \text{tr} \).

Proof. Let \( \vec{n} = [n_1, n_2, \ldots, n_k] \) denote the dimension vector of \( N' \cap M \) and \( \vec{t} \) denote the trace vector of \( \text{tr} \). Consider the path algebra model \( B_{-1} \subseteq B_0 \subseteq B_1 \) for the inclusion \( C \subseteq N' \cap M \) as recalled in Section 2.2.2. Since \( (C \subseteq N' \cap M) \cong (B_0 \subseteq B_1) \), it is enough to show that \( B_0 \subseteq B_1 \) admits a two-sided basis with respect to the tracial state (on \( B_1 \)) determined by the trace vector \( \vec{t} \). Let
\[ J := \{ (\kappa, \beta) : \kappa, \beta \in \Omega_1 \text{ such that } r(\kappa) = r(\beta) \}. \]
Then, by [9, Proposition 5.4.1(iv)] (or see Section 2.2.2), \( \{ e_{\kappa, \beta} : (\kappa, \beta) \in J \} \)
is a system of matrix units for \( B_1 \). So, by [9, Proposition 5.4.3 (iii)], we easily deduce that
\[ E_{B_0}(e_{\kappa, \beta}(e_{\kappa', \beta'})^*) = \delta_{(\kappa, \beta),(\kappa', \beta')}\vec{t}_r(\kappa) \]
for all \( (\kappa, \beta), (\kappa', \beta') \in J \).

Then, defining
\[ \lambda_{\kappa, \beta} = \frac{1}{\sqrt{\vec{t}_r(\kappa)}} e_{\kappa, \beta} \text{ for } (\kappa, \beta) \in J, \]
we obtain
\[ \sum_{(\kappa', \beta') \in I} E_{B_0}(e_{\kappa, \beta}(\lambda_{\kappa', \beta'})^*) \lambda_{\kappa', \beta'} = e_{\kappa, \beta} \text{ for all } (\kappa, \beta) \in J. \]
In particular, since \( \{ e_{\kappa, \beta} : (\kappa, \beta) \in J \} \) is a system of matrix units for \( B_1 \), we have
\[ \sum_{(\kappa', \beta') \in J} E_{B_0}(x(\lambda_{\kappa', \beta'})^*) \lambda_{\kappa', \beta'} = x \text{ for all } x \in B_1, \]
that is, \( B := \{ \lambda_{\kappa', \beta'} : (\kappa', \beta') \in J \} \) is a left Pimsner-Popa basis for \( (N' \cap M)/\mathbb{C} \). Hence, being a self-adjoint set, \( B \) is in fact a two-sided Pimsner-Popa basis for \( B_1 \) over \( \mathbb{C} \). \( \square \)

Lemma 3.4. \( \mathcal{R} \) has a two-sided basis over \( N \) contained in \( N' \cap M \).
Proposition 3.6. Let $\theta$ be an automorphism of $\mathcal{R}$ such that its restriction to $N$ is an outer automorphism of $N$. Then, $\theta$ is a free automorphism of $\mathcal{R}$.

Proof. Suppose $\theta$ is not a free automorphism of $\mathcal{R}$. Then, by definition, there exists a non-zero $r \in \mathcal{R}$ such that

$$rx = \theta(x)r \quad \text{for all } x \in \mathcal{R}. \quad (3.1)$$

By Lemma 3.4, there exists a basis $\{\lambda_1, \ldots, \lambda_n\}$ for $\mathcal{R}/N$ contained in $N' \cap M$. Since $\sum_{i=1}^k \lambda_i E_N(\lambda_i^* r) = r \neq 0$, we must have $E_N(\lambda_i^* r) \neq 0$ for at least one $\lambda_j$. Thus, multiplying both sides of Equation (3.1) by $\lambda_j^*$ on the left, we obtain

$$\lambda_j^* rx = \lambda_j^* \theta(x)r = \theta(x)\lambda_j^* r \quad \text{for all } x \in N. \quad (3.2)$$

Then, taking conditional expectation $E_N$ on both sides of Equation (3.2), we get

$$E_N(\lambda_j^* r)x = \theta(x)E_N(\lambda_j^* r) \quad \text{for all } x \in N.$$

This shows that $\theta|_N$ is not free. But a free automorphism of a factor is outer ([10], [9, §A.4]). Hence, we have a contradiction as $\theta|_N$ is given to be outer. \hfill \Box

Lemma 3.5. Let $\theta$ be an automorphism of $\mathcal{R}$ such that its restriction to $N$ is an outer automorphism of $N$. Then, $\theta$ is a free automorphism of $\mathcal{R}$.

Proposition 3.6. Let $G$ denote the generalized Weyl group of $N \subseteq M$. Then, any set of coset representatives $\{u_g : g = [u_g] \in G\}$ of $G$ in $N_M(N)$ forms a two-sided orthonormal system for $M/\mathcal{R}$.

Proof. Let $w \in N_M(N)$. We first assert that

$$E_{\mathcal{R}}(w) = 0 \quad \text{if and only if } w \in N_M(N) \setminus U(N)U(N' \cap M).$$

Necessity is obvious. Conversely, suppose $w \notin U(N)U(N' \cap M)$. Note that, by Lemma 3.2, $wxw^* \in \mathcal{R}$ for all $x \in \mathcal{R}$. So, $\beta : \mathcal{R} \to \mathcal{R}$ defined by $\beta(x) = wxw^*$ is an automorphism of $\mathcal{R}$, which restricts to an outer automorphism on $N$ (since $w \notin U(N)U(N' \cap M)$). Thus, by Proposition 3.5, $\beta$ is a free automorphism of $\mathcal{R}$. Then, applying $E_{\mathcal{R}}$ on both sides of the equation $wx = \beta(x)w$, we obtain $E_{\mathcal{R}}(w)x = \beta(x)E_{\mathcal{R}}(w)$ for all $x \in \mathcal{R}$. Since $\beta$ is free, we must have $E_{\mathcal{R}}(w) = 0$. This proves the assertion.

Now, fix a set of coset representatives $\{u_g : g = [u_g] \in G\}$ of $G$ in $N_M(N)$. Then, by above assertion, we have

$$E_{\mathcal{R}}(u_g u_h^*) = 0 = E_{\mathcal{R}}(u_g^* u_h) \quad \text{if and only if } g \neq h. \quad (3.3)$$
Hence, \( \{ u_g : g \in G \} \) forms a two-sided orthonormal system for \( M \) over \( \mathcal{R} \).

**Proposition 3.7.** Let \( \mathcal{P} := \mathcal{N}_M(N)^{\prime\prime} \) and \( \{ u_g : g \in G \} \) be an orthonormal system for \( M/\mathcal{R} \) as in Proposition 3.6. If \( p \) denotes the support of \( \{ u_g : g \in G \} \), then \( p = e\mathcal{P} \).

In particular, if \( N \subset M \) is regular, then \( \{ u_g : g \in G \} \) forms a two-sided orthonormal basis for \( M \) over \( \mathcal{R} \).

**Proof.** We have \( p = \sum_g u_g e_{\mathcal{R}^\prime} u_g^* \in \langle \mathcal{M}, e_{\mathcal{R}} \rangle \in B(L^2(\mathcal{M})) \) (see Definition 2.4). We first assert that \( p|_{L^2(\mathcal{P})} = id \).

Let \( A = \text{span}(\mathcal{N}_M(N)) \). Then, \( \mathcal{P} = A^{\prime\prime} \) and since \( A \) is a unital \(*\)-subalgebra of \( \mathcal{P} \), by Double Commutant Theorem, we have \( A^{\prime\prime} = A^{\text{SOT}} \). Let \( x \in \mathcal{P} \). Then, there exists a net \( \langle x_i \rangle \subset A \) such that \( x_i \) converges to \( x \) in SOT. Thus, \( x_i \Omega \) converges to \( x \Omega \) in \( L^2(\mathcal{M}) \). So, it suffices to show that \( p(u\Omega) = u\Omega \) for every \( u \in \mathcal{N}_M(N) \) for then we will have

\[
p(x\Omega) = \lim_i p(x_i\Omega) = \lim_i x_i\Omega = x\Omega.
\]

Let \( u \in \mathcal{N}_M(N) \). Then, \( [u] = [u_g] \) for a unique \( g \in G \). So, \( u = u_g v \) for some \( v \in \mathcal{U}(\mathcal{N}) \mathcal{U}(\mathcal{N}^\prime \cap \mathcal{M}) \). Thus,

\[
p(u\Omega) = \sum_{t \in G} u_t e_{\mathcal{R}^\prime} u_t^* u\Omega = \sum_{t \in G} u_t E_{\mathcal{R}^\prime}(u_t^* u)\Omega = \sum_{t \in G} u_t E_{\mathcal{R}^\prime}(u_t^* u_g)v\Omega = u_g v\Omega = u\Omega,
\]

where the second last equality holds because of Equation 3.3.

Now, it just remains to show that

\[
p|_{(L^2(\mathcal{P}))^\perp} = 0.
\]

For this, it suffices to show that, for all \( y \in M \) satisfying \( \text{tr}_M(x^* y) = 0 \) for all \( x \in \mathcal{P} \), we must have \( p(y\Omega) = 0 \), that is, we just need to show that \( \sum_{g \in G} u_g E_{\mathcal{R}}(u_g^* y)\Omega = 0 \) for any such \( y \). In fact, we assert that \( E_{\mathcal{R}}(u_g^* y) = 0 \) for all \( g \in G \).

For \( z \in \mathcal{U}(\mathcal{N}) \mathcal{U}(\mathcal{N}^\prime \cap \mathcal{M}) \), \( u_g z^* \in \mathcal{P} \) so that \( \text{tr}_M(z u_g^* y) = 0 \) for all \( g \in G \). Further, since

\[
\mathcal{R} = \overline{\text{span}\{ \mathcal{U}(\mathcal{N}) \mathcal{U}(\mathcal{N}^\prime \cap \mathcal{M}) \}}^{\text{SOT}}
\]

and \( \text{tr}_M \) is SOT-continuous on bounded sets, it follows that \( \text{tr}_M(r u_g^* y) = 0 \) for all \( r \in \mathcal{R} \) and \( g \in G \). Hence, by the trace preserving property of the conditional expectation, we deduce that \( E_{\mathcal{R}}(u_g^* y) = 0 \) for all \( g \in G \). This completes the proof.

The following two elementary observations turn out to be catalytic in proving the existence of two-sided basis for an arbitrary regular subfactor of type \( II_1 \) with finite index.
**Lemma 3.8.** Let $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$ be an inclusion of finite von Neumann algebras with a faithful tracial state $\text{tr}$ on $\mathcal{M}$ and $\{\lambda_i : 1 \leq i \leq m\}$ be a basis for $\mathcal{P}/\mathcal{N}$. Then, for any $u \in \mathcal{N}_M(\mathcal{P}) \cap \mathcal{N}_M(\mathcal{N})$, $\{u\lambda_i u^* : 1 \leq i \leq m\}$ is also a basis for $\mathcal{P}/\mathcal{N}$.

**Proof.** Note that the map $\theta : \mathcal{P} \to \mathcal{P}$ given by $\theta(x) = uxx^*$ is a $\text{tr}_\mathcal{M}$ (and hence $\text{tr}_\mathcal{P}$) preserving automorphism of $\mathcal{P}$ that keeps $\mathcal{N}$ invariant. Then, a routine verification shows that $\{u\lambda_i u^* : 1 \leq i \leq m\}$ is also a basis for $\mathcal{P}/\mathcal{N}$, which we leave to the reader. □

**Proposition 3.9.** Let $\mathcal{N} \subset \mathcal{P} \subset \mathcal{M}$ be as in Lemma 3.8. Suppose $\mathcal{P}/\mathcal{N}$ has a two-sided basis $\{\lambda_i : 1 \leq i \leq m\}$ and $\mathcal{M}/\mathcal{P}$ has a two-sided basis $\{\mu_j : 1 \leq j \leq n\}$ contained in $N_M(\mathcal{P}) \cap N_M(\mathcal{N})$. Then, $\{\mu_j \lambda_i : 1 \leq i \leq m, 1 \leq j \leq n\}$ is a two-sided basis for $\mathcal{M}/\mathcal{N}$.

**Proof.** Let $\lambda'_{i,j} := \mu_j \lambda_i \mu_j^*$, $1 \leq i \leq m, 1 \leq j \leq n$. Then, by Lemma 3.8, $\{\lambda'_{i,j} : 1 \leq i \leq m\}$ is a basis for $\mathcal{P}/\mathcal{N}$ for each $j$. Similarly, $\{(\lambda'_{i,j})^* : 1 \leq i \leq m\}$ is also a basis for $\mathcal{P}/\mathcal{N}$. Since $\{\lambda_i\}$ is a basis for $\mathcal{P}/\mathcal{N}$, we have $\sum \lambda_i e_1 \lambda_i^* = e_\mathcal{P}$ (see Section 2.2.1). So, by Lemma 2.6, we obtain $\sum_{i,j} \mu_j \lambda_i e_1 \lambda_i^* \mu_j^* = \sum_j \mu_j e_\mathcal{P} \mu_j^* = 1$. Therefore, by Lemma 2.6 again, $\{\mu_j \lambda_i\}$ is a basis for $\mathcal{M}/\mathcal{N}$. On the other hand, we have

$$\sum_{i,j} \lambda_i^* \mu_j^* e_1 \mu_j \lambda_i = \sum_{i,j} \mu_j^*(\lambda_i')^* e_1 \lambda_i' \mu_j = \sum_j \mu_j^* e_\mathcal{P} \mu_j = 1,$$

where the second last equality holds because $\{\lambda_{i,j} : 1 \leq i \leq m\}$ is a basis for $\mathcal{P}/\mathcal{N}$ and the last equality follows because $\{\mu_j^* : 1 \leq j \leq n\}$ is a basis for $\mathcal{M}/\mathcal{P}$. Thus, we conclude that $\{(\mu_j \lambda_i)^*\}$ is also a basis for $\mathcal{M}/\mathcal{N}$. This completes the proof. □

We are now all set to deduce the main theorem of this article.

**Theorem 3.10.** Let $N \subset M$ be a regular subfactor of type $II_1$ with finite index. Then, $M$ admits a two-sided Pimsner-Popa basis over $N$.

**Proof.** We observed in Lemma 3.4 that $\mathcal{R} := N \vee (N' \cap M)$ admits a two-sided basis, say, $\{\lambda_i\}$, over $N$. Further, we readily deduce, from Proposition 3.7, that $M$ also admits a two-sided basis, say, $\{\mu_j\}$, over $\mathcal{R}$, which is contained in $N_M(N)$. By Lemma 3.2, we know that $N_M(N) \subseteq N_M(\mathcal{R})$. Hence, by Proposition 3.9, we conclude that $\{\mu_j \lambda_i\}$ is a two-sided Pimsner-Popa basis for $\mathcal{M}$ over $N$. □

In view of Proposition 3.1, we obtain the following:

**Corollary 3.11.** Every regular subfactor of type $II_1$ with finite index is extremal.

It is well known to the experts that every regular subfactor has integer index; for instance, there is a mention of this fact in [5, Page 150] (without a proof). As final application of some of the results proved above, we deduce
this fact along with a precise expression for the index of such a subfactor.
We will use Watatani’s notion of index of a conditional expectation to do so.

Recall, from [20], that, given an inclusion $B \subset A$ of unital $C^*$-algebras, a conditional expectation $E : A \to B$ is said to have finite index if there exists a right Pimsner-Popa basis $\{\lambda_i : 1 \leq i \leq n\}$ for $A$ over $B$ via $E$ and the Watatani index of $E$ is defined as

$$\text{Ind}(E) = \sum_{i=1}^{n} \lambda_i^* \lambda_i,$$

which is independent of the basis $\{\lambda_i\}$ and is an element of $\mathbb{Z}(A)$.

**Theorem 3.12.** Every regular subfactor $N \subset M$ of type $II_1$ with finite Jones index has integer valued index and the index is given by

$$[M : N] = |G| \dim \mathbb{C}(N' \cap M),$$

where $G$ denotes the generalized Weyl group of the inclusion $N \subset M$.

**Proof.** Consider the inclusion $C \subseteq N' \cap M$. Let $\Lambda$ denote its inclusion matrix. Let $\{\lambda_i\} \subset N' \cap M$ be a two-sided basis for $N' \cap M$ over $C$ with respect to $\text{tr}$ as in Proposition 3.3. We observed in Lemma 3.4 that $\{\lambda_i\}$ is a two-sided basis for $R := N \vee (N' \cap M)$ as well over $N$ with respect to $E_{N|R}$.

Further, from Proposition 3.7, $M$ admits a two-sided basis consisting of unitaries, say, $\{\mu_j : 1 \leq j \leq |G|\}$, over $R$, which is contained in $N_M(N)$. As seen in Theorem 3.10, $\{\lambda_i \mu_j\}$ is also a basis for $M$ over $N$, and we obtain

$$[M : N] = \sum_{i,j} \lambda_i^* \mu_j^* \mu_j \lambda_i = |G| \sum_i \lambda_i^* \lambda_i = |G| \sum_i \lambda_i \lambda_i^* = |G| \text{Ind}(\text{tr}),$$

where the second last equality holds because $\{\lambda_i\}$ is a two-sided basis for $\text{tr}$. In particular, $\text{Ind}(\text{tr})$ is scalar-valued. So, if $\Lambda$ denotes the matrix of the inclusion $C \subseteq N' \cap M$ and $\bar{s} = (s_1, \ldots, s_k)$ denotes the trace vector of $\text{tr}$, then by [20, Corollary 2.4.3], there exists a $\beta > 0$ such that $\bar{s} \Lambda \Lambda^t = \beta \bar{s}$ and $\text{Ind}(\text{tr}) = \beta$.

Now, if $[n_1, \ldots, n_k]$ is the dimension vector of $N' \cap M$, then by Watatani’s convention, we have $\Lambda = [n_1, \ldots, n_k]^t$. Since $\sum_{i=1}^{k} s_i n_i = 1$, we obtain

$$\bar{s} \Lambda \Lambda^t = \left( \left( \sum_{i=1}^{k} s_i n_i \right) n_1, \left( \sum_{i=1}^{k} s_i n_i \right) n_2, \ldots, \left( \sum_{i=1}^{k} s_i n_i \right) n_k \right) = (n_1, n_2, \ldots, n_k),$$

which yields $\beta = \frac{n_i}{s_i}$ for all $1 \leq i \leq k$. Thus, if $p_i$ denotes a minimal projection in the $i$-th summand of $N' \cap M$ and $\bar{p}_i$ denotes the $i$-th minimal
central projection, then $\text{tr}(p_i) = s_i = n_i/\beta$ for all $1 \leq i \leq k$; so, $\text{tr}(\tilde{p}_i) = n_i^2/\beta = s_in_i$ for all $1 \leq i \leq k$. This gives

$$1 = \text{tr}(1) = \sum_{i=1}^{k} \text{tr}(\tilde{p}_i) = \sum_{i=1}^{k} n_i^2/\beta;$$

so that $\beta = \sum_{i=1}^{k} n_i^2 = \dim C(N' \cap M)$. Hence,

$$[M : N] = |G| \dim C(N' \cap M).$$

This completes the proof. \hfill \Box

References


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