The Dyer-Lashof algebra and the hit problems

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Abstract. We use work of Curtis and Wellington on \(\mathcal{A}\)-annihilated classes in \(H_*Q^0\) together with Priddy’s computations on the action of the Dyer-Lashof algebra on \(H_*BO\) to provide new examples of \(\mathcal{A}\)-annihilated classes in \(H_*BO\) as well \(H_*BO(2^r - 1)\). We then consider the Becker-Gottlieb transfer associated to \(O(1)^{\infty} \to O(n)\) and speculate on the possible applications of our computations to obtain new examples of \(\mathcal{A}\)-annihilated classes in \(H_*BO(1)^n\). Our results, all at the prime \(p = 2\), provide new numerical conditions which seem to show up for the first time in this context.

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1. Introduction and statement of results

The hit problem of Peterson is about finding a basis for the vector space \(\mathbb{Z}/2 \otimes \mathcal{A} H^*X\) when \(X = \mathbb{RP}^k\); \(\mathbb{Z}/2\) is an \(\mathcal{A}\)-module concentrated in degree 0 [40, Section 7] (see also [36],[37] for a very recent account on the problem). We work at the prime 2 writing \(\mathbb{Z}/2\) for the field of two elements, \(\mathcal{A}\) and \(\mathcal{R}\) for the mod 2 Steenrod and Dyer-Lashof algebras respectively, \(H^*\) (resp. \(H_*\)) for \(H^*(-; \mathbb{Z}/2)\) (resp. \(H_*(-; \mathbb{Z}/2)\)). We write \(\mathbb{RP}\) for the infinite dimensional real projective space \(\mathbb{RP}^\infty\). For a pointed space \(X\), \(X^{\times k}\) and \(X^{\times k}\) denote the \(k\)-fold Cartesian and smash products of \(X\) with itself, respectively.
An equivalent problem, in a homological setting, is to study the submodule of \(\mathcal{A}\)-annihilated elements of \(H_*X\) where \(x \in H_*X\) is called \(\mathcal{A}\)-annihilated if \(Sq^ix = 0\) for all \(i > 0\). Here, duality of vector spaces over \(\mathbb{Z}/2\), provided by the Universal Coefficient Theorem over \(\mathbb{Z}/2\), allows to consider \(Sq^i : H_*X \to H_*X\) as the dual operation to \(Sq^i : H^*X \to H^{*+i}X\). Our aim in this paper is to study this problem by means of looking at the dual problem for \(H_*BO\) which is the stable case of the symmetric hit problem of Janfada and Wood [16], [17].

Conventions and notations. For most of the paper we work with sequences of nonnegative/positive integers. For this purpose, we fix some notations to be used through the paper. We write \(\phi = (\ )\) for the empty sequence. Given \(I = (i_1, \ldots, i_r)\) and \(J = (j_1, \ldots, j_s)\), we abbreviate \((I, J) = (i_1, \ldots, i_r, j_1, \ldots, j_s)\) allowing \(I\) or \(J\) or both to be the empty sequence. If \(I = (i)\) we write \((i, J)\); we use similar notation if \(J = (j)\). For \(I = (i_1, \ldots, i_r)\) a sequence of nonnegative integers, we define and denote excess of \(I\) by \(ex(I) = i_1 - (i_2 + \cdots + i_r)\). dimension of \(I\) by \(|I| = i_1 + \cdots + i_r\), and length of \(I\) by \(l(I) = r\); if necessary for the empty sequence \(\phi = (\ )\) we use the conventions that \(ex(\phi) = +\infty\), \(|\phi| = l(\phi) = 0\). We shall say \(I\) is admissible if either \(l(I) > 1\) and \(i_j \leq 2l_{j+1}\) for all \(1 \leq j < r - 1\) or \(l(I) = 1\) or \(l(\phi) = 1\). For a nonempty sequence \(I = (i_1, \ldots, i_r)\) we write \(I = I_0\) and \(I_j = (i_{j+1}, \ldots, i_r)\) for \(j < r\), and define \(ex_j(I) = ex(I_j)\); note that if \(I\) is admissible so is \(I_j\) for all \(j < r\). We allow ourselves to use the abbreviation \(ex_j\) instead of \(ex_j(I)\) if there is no confusion about the sequence \(I\).

The following observation is a consequence of definitions and conventions above which we wish to highlight due to its usefulness. We defer its proof to the following sections.

**Lemma 1.1.** (i) \(I = (i_1, \ldots, i_r)\) is admissible if and only if \(ex_j \leq ex_{j+1}\) for \(j = 0, \ldots, r - 1\).
(ii) Suppose \(I = (i_1, \ldots, i_r)\) is an admissible sequence such that all of its entries are odd. Then the sequence \((ex_0, ex_1, \ldots, ex_{r-1})\) is strictly increasing.
(iii) If \(I = (i_1, \ldots, i_r)\) is admissible with \(ex(I) > 0\) then \(i_j > i_{j+1}\) for all \(j = 1, \ldots, r - 1\).

Next, we describe \(H_*(\mathbb{Z} \times BO)\) which is required to state our first result. Write \(t : \mathbb{R}P \to \{1\} \times BO \to \mathbb{Z} \times BO\) for the inclusion and \(a_i \in H_i\mathbb{R}P\) for a generator with \(i > 0\), and let \(e_i = t_*a_i\) for \(i > 0\). There is a certain infinite loop map \(\chi : QS^0 \to \mathbb{Z} \times BO\) (see Section 3 for more details) and certain elements \([1], [-1] \in H_0QS^0\) with \([1]^{-1} = [-1]\) (see comments after Remark 2.1 for the definition of these classes). Let \(e_0 = \chi_*[1]\) and \(e_0^{-1} = \chi_*[-1]\). The space \(\mathbb{Z} \times BO\) is an infinite loop space by Bott periodicity which furnishes \(H_*(\mathbb{Z} \times BO)\) with a ring structure. It is known that \([29]\) \(H_*(\mathbb{Z} \times BO) \cong \mathbb{Z}/2[e_0, e_0^{-1}, e_i : i > 0, \deg e_i = i]\). Sometimes, in order to stress the role of multiplication by some specific element \(\xi\), we choose to write \(*\) for the product operation in homology.
and $\ast \xi$ means multiplying by $\xi$. We also assume that while writing an arbitrary monomial $e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}$ the variables are ordered so that $i_1 < i_2 < \cdots < i_r$. Although, sometimes it makes it easy to allow cases with $i_1 \leq i_2 \leq \cdots \leq i_r$ which will be specified whenever it happens. Our first result reads as follows.

**Theorem 1.2.** (i) Suppose $I = (i_1, \ldots, i_r)$ is a sequence of positive integers satisfying

1. $0 < \exp(I) < 2^\phi(i_r)$,
2. $0 \leq 2i_{j+1} - i_j < 2^\phi(i_{j+1})$

if $r > 1$, and (1) if $r = 1$. Then there exists an $A$-annihilated class, say $\xi_I^{Z \times BO} \in H_1[I](\mathbb{Z} \times BO)$ so that

$$
\xi_I^{Z \times BO} = e_0^k e_{\omega_0} e_{\omega_1}^{2^r} \cdots e_{\omega_{r-1}}^{2^r} + \text{other terms } \neq 0
$$

with

$$
0 < \exp_0 < \exp_1 < \cdots < \exp_{r-1}.
$$

(ii) If $|I| = r$ then the class $\xi_I^{BO} := e_0 e_{\omega_0} e_{\omega_1}^{2^r} \cdots e_{\omega_{r-1}}^{2^r} \xi_I^{Z \times BO}$ pulls back to $H_* BO(2^r - 1)$.

(iii) The classes $\xi_I^{Z \times BO}$ where $I$ ranges over all admissible sequences are linearly independent. A similar statement holds for the classes $\xi_I^{BO}$ in $H_* BO(2^r - 1)$

Here, for $n \in \mathbb{N}$ with binary expansion $n = \sum_{i=0}^{+\infty} n_i 2^i$ with $n_i \in \{0, 1\}$ we define $\phi(n) = \min\{i : n_i = 0\}$.

We prove the above Theorem in Section 3.

**Note 1.3.** An important question is about the existence of the sequences mentioned in Theorem 1.2 of a given dimension and length. The tables provided by [39] give a list of such sequences up to dimension 200 where it seems that in these dimensions, omitting sequences with $\exp(I) = 0$, there is only at most one sequence in each dimension satisfying conditions of Theorem 1.2. However, this does not appear to be true in all dimensions as in a work in progress [11] and with computer assisted computations, we have determined all such sequences, with $\exp(I) > 0$, up to dimension $1.1 \times 10^7$ (the highest dimension which we did manage to get with our laptop running for 30 hours). It is observed that in this range there exist less than $8 \times 10^6$ number of such sequences. As an outcome of our computations, we have obtained sequences of the same dimension, some of the same length and some with different lengths. For instance, the sequences

$$(1091, 547, 287, 159, 95), (1091, 547, 275, 139, 127)$$

are both of dimension 2179 and length 5 and satisfy our conditions. As another example, in a low dimension such as 4353 there exist only three such sequences, namely

$$
(2177, 1089, 545, 287, 255), \\
(2177, 1089, 545, 273, 143, 79, 47), (2177, 1089, 545, 273, 137, 69, 63).
$$

The methodological outcome is that our computations in this paper could produce more precise bounds on the dimension of modules related to the symmetric and nonsymmetric hit problems.
As another application of Priddy’s formula, we obtain yet another family of $\mathcal{A}$-annihilated classes. For a nonempty sequence of nonnegative integers $K = (k_1, \ldots, k_n)$ we write $e^K$ to denote $e_1^{k_1} \cdots e_n^{k_n}$ in $H_*(\mathbb{Z} \times BO)$. We use the convention that $e_i^0 = 1$; for instance if $K = (1, 0, 2)$ then $e^K = e_1 e_2^0 e_3^2 = e_1 e_3^2$.

We define and denote the dimension of $e^K$ by $|e^K| = \sum_{j=1}^{n} j k_j$. We also define length of $e^K$ by $l(e^K) = \sum_{i=1}^{d} k_i$ which corresponds to the length filtration function on $H_*(\mathbb{Z} \times BO)$ (see comments after Remark 2.3 for more discussions). We refer to a positive integer $n$ as a spike if $n = 2^t - 1$ for some $t > 0$. Given a natural number, we may consider its partition into spikes as $n = \sum_{i=1}^{d} (2^{\lambda_i} - 1)$ where $d > 0$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$ where unlike [17, Section 2] we consider only positive $\lambda_i$ in order to avoid trivial cases. Note that any such partition corresponds to a decreasing sequence $(\lambda_1, \ldots, \lambda_d)$; we define $d$ to be length of our partition. We define the spike partition number of $n$, denoted $sp(n)$, to be the number of distinct partitions of $n$ into spikes where two partitions are assumed to be distinct if either they are of different lengths or they have the same length but differ at least in one term. More precisely, define

$$\text{Spike}_d^d(n) = \{ (\lambda_1, \ldots, \lambda_d) \in \mathbb{Z}^+ : \sum_{i=1}^{d} (2^{\lambda_i} - 1) = n \}$$

and note that the symmetric group on $d$ elements $\Sigma_d$ acts on this set. It follows that

$$sp(n) = \sum_{d=1}^{+\infty} |\text{Spike}_d^d(n)/\Sigma_d|.$$ 

We shall use this function in the statement of our next result. We urge the reader to note that our use of spikes in studying the dual hit problem is different from its applications in the hit problem (compare to [17]). For nonempty sequences $K = (k_1, \ldots, k_n)$ and $M = (m_1, \ldots, m_n)$, we shall write $K = 2M$ if $k_j = 2 m_j$ for all $j$. For positive integers $k, m$ if $k$ divides $m$ then we write $k|m$. For $i > 0$, we shall write $\bar{e}_i = e_0^{-i} e_1$ noting that $H_*BO \simeq \mathbb{Z}/2[\bar{e}_i : i > 0]$ (see Section 2.4 for more details). Our next result reads as follows.

**Theorem 1.4.** (i) For $n > 0$ let $\mathcal{K}_n$ be the set of all sequences of nonnegative integers $K = (k_1, \ldots, k_n)$ such that

$$\left( \frac{l(e^K)}{k_1! \cdots k_n!} \right) \equiv 1 \mod 2, \sum_{i=1}^{n} i k_i = n.$$ 

For $t > 0$, there exists a nonzero $\mathcal{A}$-annihilated class $e_t^{\mathbb{Z} \times BO} \in H_{2^t - 1}(\mathbb{Z} \times BO)$ such that

$$e_{t}^{\mathbb{Z} \times BO} = \sum_{K \in \mathcal{K}_{2^t - 1}} e_0^{-l(e^K) - 2} e^K.$$
In particular, the sum involves the terms $e_0^{-(2^i-1)-2}e_1^{2^i-1}$ and $e_0^{-3}e_{2^i-1}$. Consequently, we have a nonzero $A$-annihilated class $\xi_t^{BO} \in H_{2^i-1}BO$ given by

$$\xi_t^{BO} = e_0^2 \xi_1^{Z\times BO}.$$  

In particular, the sum involves terms such as $\bar{e}_1^{2^i-1}$ and $\bar{e}_{2^i-1}$. Moreover, this class is in the image of $H_*BO(2^i-1) \rightarrow H_*BO$ and it does not pull back to $H_*BO(s)$ with $s < 2^i - 1$.

(ii) Suppose $n > 0$ is given. Then, for each partition of $n$ into spikes $n = \sum_{i=1}^d (2^{\lambda_i} - 1)$ with $\lambda_1 \geq \cdots \geq \lambda_d > 0$ there exists a nonzero $A$-annihilated class $\xi_{\lambda_1, \cdots, \lambda_d}^{Z\times BO} = \prod_{i=1}^d \xi_{\lambda_i}^{Z\times BO}$.

In particular, up to multiplication by a power of $e_0$, $\xi_{\lambda_1, \cdots, \lambda_d}^{Z\times BO}$ involves terms of the form $e_1^n$ and $\prod_{i=1}^d e_{2^{\lambda_i}-1}$. Moreover, there are $sp(n)$ number of linearly independent nonzero $A$-annihilated classes, living in $H_n(\mathbb{Z} \times BO)$, that are obtained in this way. Furthermore, by a translation map, we obtain $sp(n)$ number of nonzero $A$-annihilated classes of the form

$$\xi_{\lambda_1, \cdots, \lambda_d}^{BO} = \prod_{i=1}^d \xi_{\lambda_i}^{BO}$$

so that each $\xi_{\lambda_1, \cdots, \lambda_d}^{BO}$ involves terms of the form $e_1^n$ and $\prod_{i=1}^d e_{2^{\lambda_i}-1}$.

The above Theorem allows to identify $A$-annihilated classes in $H_*(\mathbb{Z} \times BO)$, and consequently $H_*BO$, which are not products of classes of the form $e_{2^i-1}$. For instance, consider the class $\xi_4^{Z\times BO} \in H_{12}BO$. We can see that up to a factor of $e_0$ we have

$$\xi_4^{BO} = e_1^{15} + e_1 e_7 + e_3 e_5 + (e_3^2 + e_3 e_5^2) + \text{other terms}$$

where the first four terms are products of classes of the form $e_{2^i-1}$. Now, the nontrivial interesting class in this case is $e_3^2 + e_3 e_5^2$ where $S\xi_2^2(e_3^2) = S\xi_2^2(e_3 e_5^2) = e_3 e_5^2$ so $S\xi_2^2(e_3^2 + e_3 e_5^2) = 0$. One can show that in fact $e_3^2 + e_3 e_5^2$ is $A$-annihilated which is a nontrivial example. In fact, whenever $2^i - 1$ is not prime, we have nontrivial such terms in $\xi_t^{Z\times BO}$.

Next, we wish to investigate the possible applications of the above computations to study $A$-annihilated classes in $H_*BO(1)^{\otimes k}$. By the Künneth formula over $\mathbb{Z}/2$ we have $H_*X^{\otimes k} \simeq (H_*X)^{\otimes k}$ where the latter denotes the $k$-fold tensor product of $H_*X$ with itself. Recall that the Becker-Gottlieb transfer map associated to the fibre bundle $i : BO(1)^{\otimes n} \rightarrow BO(n)$, induced by the diagonal embedding $O(1)^{\otimes n} \rightarrow O(n)$, is a stable map $t : \Sigma\infty BO(n)_+ \rightarrow \Sigma\infty BO(1)^{\otimes n}$ where $X_+$ is the union of $X$ with a disjoint base-point $[4]$ (see also [30] and [1, Chapter 4]). In particular, it satisfies $H_*\Sigma\infty X_+ \simeq \tilde{H}_*X_+ \simeq H_*X$. The representation $\Sigma_n \rightarrow \text{GL}_n(\mathbb{Z}/2)$, sending each permutation to its corresponding
permutation matrix, allows to define \( \Sigma_n = \sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma) \sigma \in \mathbb{Z}/2[\text{GL}_n(\mathbb{Z}/2)] \) where \( \text{sgn} \) is the sign function. It is known that \( i^* t^* = \Sigma_n \) [25, Lemma 6.2]. In particular, this shows that \( t^* \neq 0 \) and consequently \( t_* \neq 0 \). For the transfer \( t_* \) we have identified \( \ker(t_*) \) and \( \text{coker}(t_*) \) as well as a submodule \( H_* BO(k) \) of \( H_* BO(k) \) on which the restriction of \( t_* \) is a monomorphism (see Proposition 4.2 and Corollary 4.3). Assuming that \( Q^l e_0 \) has some terms which belong to this submodule, we automatically obtain some nonzero \( A \)-annihilated classes in \( H_* \mathbb{R}P^{\times n} \). Let us write \( a_1 \otimes \cdots \otimes a_k \) for the elements of \( H_* X^{\times k} \) which we denote by \( a_i^{\otimes k} \) if \( i_1 = \cdots = i_k = i \). Using this notation, we have the following.

**Theorem 1.5.** (i) Suppose \( I = (i_1, \ldots, i_r) \) is a sequence of positive integers satisfying

\[
(1) \ 0 < \text{ex}(I) < 2\#(I), \ (2) \ 0 \leq 2i_{j+1} - i_j < 2\#(I)
\]

if \( r > 1 \), and (1) if \( r = 1 \). If the pullback of \( \xi_i^{BO} \) to \( H_* BO(2^r - 1) \) projects nontrivially into \( H_* BO(2^r - 1) = \text{coker}(t_*) \) then there exists a nonzero \( A \)-annihilated class \( \xi_I \in H_{|I|} \mathbb{R}P^{\times (2^r - 1)} \) so that for any permutation \( \tau \in \Sigma_{2^r - 1} \) we have \( \tau \xi_I = \xi_I \).

(ii) Suppose \( k = \sum_{j=1}^r (2^{i_j} - 1) \) so that for each \( j \) there exists a sequence \( I^j \) of positive integers with \( l(I^j) = r_j \) and \( \text{ex}(I^j) > 0 \) satisfying the conditions in part (i). Then, for \( \sigma \in \Sigma_t \) there exists a nonzero \( A \)-annihilated class \( \xi_\sigma \in H_d \mathbb{R}P^{\times k} \) with \( d = \sum |I^j| \) which is invariant under the action of \( \Sigma_{2^{i_1} - 1} \times \cdots \times \Sigma_{2^{i_r} - 1} \). Moreover, depending on the partition chosen for \( k \), subject to the existence of sequences \( I^j \), this would lead to at least \( 1! \) distinct \( A \)-annihilated classes in \( H_* \mathbb{R}P^{\times k} \).

The \( A \)-annihilated classes that we have obtained in the above theorems seem to provide new families of \( A \)-annihilated classes in the homology of the relevant spaces of which we don’t know of any published account. Note that earlier versions of this paper included a claim that the classes \( \xi_I \) are linearly independent when dimension and length of the sequences is fixed, which we believe to be true. However, the provided proof was based on the wrong assumption that the transfer \( t_* : H_* BO(n) \to H_* BO(1)^{\times n} \) is a monomorphism. Besides this, the numerical assumptions of Theorem 1.2 allowed us to do machine based computations [11] which suggest that sequences \( I \) leading to these classes are often very rare; so far we have found 4 sequences living in the same dimension with our search going up to dimension \( 1.1 \times 10^7 \). Hence, computationally this claim seems rather trivial and we hope to come back to it in another work.

**Previous similar works.** Pengelley and Williams have considered using the infinite loop space structure on \( BO \) as well as \( BU \) to study the problem [26] (see also [28] as well as [27]), however, it seems that they have not used Wellington’s computations [39], [38] in their work, neither have they considered work of Priddy [29]. So, our methodology and the results we have obtained, in terms of the numerical conditions, is different from theirs. Singer [33] considers an extension of the Steenrod algebra, denoted by \( \mathcal{H} \), together
with its action on $H_\ast BO(k)$; the algebra $\mathcal{H}$ is a bigraded algebra with generators $Sq^i$ for $i > -1$, subject to the Adem relations formally the same as $A$, with $Sq^0$ treated in the same level as $Sq^i$ (not necessarily as the identity operator). The bigrading comes from degree and length so that $Sq^i$ lives in bidegree $(i, 1)$ of $\mathcal{H}$. According to Singer (see comments after [33, Theorem 1.4]) there is an action of $\mathcal{H}$ on $\bigoplus H_\ast BO(s)$ which in terms of grading shifts is given by

$$Sq^i : H_n BO(s) \to H_{2n+i-s} BO(s+i).$$

Clearly, this action, if likely to give an action of $\mathcal{H}$ on $H_\ast BO$ or $H_\ast (\mathbb{Z} \times BO)$, will increase the homological degree if $i < s + n$. More precisely, the homological dimension of $Sq^i(x)$ is always greater than $n$ for any $x \in H_n BO(s)$ if $i < s + n$. However, in our setting given by the left-action of $A^{op}$ or the right-action of $A$ on $H_\ast BO(s)$ which is expressed in terms of the $Sq^i$ operations, for $x \in H_n BO(s)$ the class $Sq^i x$ is of dimension $n - i$ and for the action to be non-trivial we always need $i < n$ which is definitely included in the cases $i < n + s$. Therefore, we argue that in Singer’s work [33], the action of $\mathcal{H}$ most of the time increases the homological degree whereas in our case we have a decrease in the homological degree. Another related work is by Ault and Singer [3]. From our point of view, their approach is mainly algebraic where the authors consider the collection $\{H_\ast \mathbb{R} P_{xk}\}_{k \geq 0}$ as well as $\{H_\ast \mathbb{R} P_{xk}^\ast\}_{k \geq 0}$ as a graded algebra with the multiplicative structure coming from natural pairing $\mathbb{R} P_{xk} \times \mathbb{R} P_{xl}^\ast \to \mathbb{R} P_{x(l+k)}$. They then consider the actions of $Sq^2$ operations between various gradings of this graded algebra and prove a kind of freeness result for it, generalising a result of Anick. The geometry hidden in this work comes from the James splitting of $\Omega \Sigma X$ which is stably weak homotopy equivalent to $\bigvee_{k=1}^{\infty} X^k$. Hence, this work is different from ours in the sense that in our approach we mainly consider the homology that is arising from geometry of the spaces. We note that there also exists extensive literature on the hit problem for the Dickson algebra such as [10],[14],[13] where the objects under study are rings such as $(H_\ast \mathbb{R} P_{xk})^{Gl_\ast (\mathbb{Z}/2)} \cong D^k_\ast$ and the interest is in their $A$-module structure, hence relating Dickson algebras $D_k$ to the dual of the Dyer-Lashof algebra by work of Madsen [21, Corollary 3.3]. The difference of such works with ours is that we work prior to the quotient maps under the $Gl_\ast (\mathbb{Z}/2)$-action and we work in the dual setting and our starting point is the action of the Dyer-Lashof algebra rather than the algebra itself. Finally, we have to recall the very much related work of Repka and Selick [32] which is in the same line as our work trying to identify $A$-annihilated subalgebras of $H_\ast \mathbb{R} P_{xk}$; our results differ from theirs in numerical conditions as well as the methods we have employed to prove our result.

**Final comments.** We note that a general approach to the hit problems, other than providing a $\mathbb{Z}/2$-basis for the vector spaces $Q(k) : = \mathbb{Z}/2 \otimes A H^{\ast} BO(k)$ and $\mathbb{Z}/2 \otimes A H^\ast BO(k)$, is to find upper/lower bounds on the dimension of these spaces. Most notably, we have conjectures of Peterson and Kameko on bounds.
on \( \dim_{\mathbb{Z}/2} Q(k) \) [41] (see [35, Theorem 1.2] for a negative answer to Kameko’s conjecture for \( k > 3 \)). It is possible that our work allows one to improve the known lower bounds. However, in this work, we have not studied this side of the problem.

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2. Preliminaries

2.1. The action of the Steenrod algebra. For an arbitrary space \( X \), the Steenrod square \( Sq^i : H^nX \to H^{n+i}X \) is a linear map of \( \mathbb{Z}/2 \)-vector spaces. Since we work over the field \( \mathbb{Z}/2 \), then by duality we have \( Sq^i_* : H_nX \to H_{n-i}X \). The left action of the Steenrod algebra \( \mathcal{A} \) on \( H_*X \), through the operations \( Sq^i_* \), furnishes the homology \( H_*X \) with a right \( \mathcal{A} \)-action, or equivalently a left action of the opposite algebra \( \mathcal{A}^{op} \) on \( H_*X \). We abuse the language to say that \( \mathcal{A} \) acts on \( H_*X \) referring to the action given by the \( Sq^i_* \) operations. Below, we list some of the properties of these operations, or the formulae that these operations satisfy, which will be used throughout the paper implicitly or explicitly.

Unstable \( \mathcal{A}^{op} \)-modules. The instability condition for the action of \( \mathcal{A} \) on \( H_*X \), \( Sq^i_\xi = 0 \) for \( i > \dim \xi \), in the homological setting reads as \( Sq^i_* x = 0 \) if \( 2t > n \) with \( x \in H_nX \). This can be simply verified using Kronecker pairing with \( \langle x, Sq^i_\xi \rangle = \langle Sq^i_* x, \xi \rangle \) (see also [39]). This could be used to define unstable \( \mathcal{A}^{op} \)-modules in purely algebraic terms where an \( \mathcal{A}^{op} \)-module \( M \) is unstable if \( Sq^i_* x = 0 \) for any \( x \in M \) with \( 2t > \dim x \). In particular, \( H_*X \) is an unstable \( \mathcal{A}^{op} \)-module.

Cartan formulae. For space \( X \) and \( Y \), the external Cartan formula is give by

\[
Sq^i_* (x \otimes y) = \sum_{t_1 + t_2 = t} (Sq^{t_1}_* x) \otimes (Sq^{t_2}_* y).
\]  

(1)

If we are equipped with a pairing \( m : X \times Y \to Z \) then the naturality of the Steenrod operations together with the above formula implies that

\[
Sq^i_* (xy) = \sum_{t_1 + t_2 = t} (Sq^{t_1}_* x)(Sq^{t_2}_* y)
\]

where \( xy := m_(x \otimes y) \). In particular, for \( X = Y = Z \) being an \( H \)-space with \( m \) the multiplication of the \( H \)-space, the above formula proves to be useful (see also [39, Chapter 5]). We may refer to this latter relation as the internal Cartan...
2.2. The Dyer-Lashof algebra. We give a brief account on the Kudo-Araki operations over $\mathbb{Z}/2$ and the Dyer-Lashof algebra generated by these operations. The Kudo-Araki operations, first noted by Kudo and Araki [19] (see also [20]), are additive operations which act on $\mathbb{Z}/2$-homology of iterated loop spaces amongst which we are interested in infinite loop spaces such as $\mathbb{Z} \times BO$, $BO$, and $QS^0 = \text{colim} \Omega^1 S^1$ [23]; the infinite loop space structure on $\mathbb{Z} \times BO$ as well as $BO$ arise from the Whitney sum of vector bundles. These operations fit into an algebra over $\mathbb{Z}/2$, known as the Dyer-Lashof algebra denoted by $\mathcal{R}$; $\mathcal{R}$ can be constructed from the associative algebra generated by the operations $Q^i$ (see [39] for a detailed description and construction of this algebra, see also [9, Section 3.2]). The action of the Kudo-Araki operations on homology of an infinite loop space $X$ induces an action of $\mathcal{R}$ on $H_* X$ which turns $H_* X$ into a left $\mathcal{R}$-module. This action has many useful properties of which we list some below (see [6, Part I, Theorem 1.1] for a full list of these properties).

Suppose $X$ is an infinite loop space. A Kudo-Araki operation $Q^n : H_* X \to H_{*+n} X$ is an additive homomorphism such that

(i) $Q^n$ is natural with maps of infinite loop spaces;

(ii) if $n < \dim x$ then $Q^n x = 0$;

(iii) if $n = \dim x$ then $Q^n x = x^2$ where the squaring is with respect to the Pontrjagin product on $H_* X$ induced by the addition of loops.

(iv) For $x, y \in H_* X$, $Q^n(xy)$ is computed by the Cartan formula given by

$$Q^n(xy) = \sum_{i+j=n} (Q^i x)(Q^j y).$$

We note that there are various other properties which we do not use in this paper and refer the reader to [6] for more details.

Remark 2.1. (i) For $x_1, \ldots, x_r \in H_* X$ the Cartan formula implies that

$$Q^n(x_1 \cdots x_r) = \sum_{i_1 + \cdots + i_r = n} (Q^{i_1} x_1) \cdots (Q^{i_r} x_r).$$

(ii) As we work over $\mathbb{Z}/2$, for an arbitrary class $\xi \in H_* X$ we have

$$Q^{2n+1} \xi^2 = 0, \quad Q^{2n} \xi^2 = (Q^n \xi)^2.$$ 

(iii) By the above properties if $I$ is a sequence so that $\text{ex}(I) < \dim x$ then $Q^I x = 0$ in $H_* X$.

Note that $\pi_i QS^0 \cong \pi_i^+ \mathbb{Z}$ and in particular $\pi_0 QS^0 \cong \mathbb{Z}$. Write $(QS^0)_n$ for the path component of $QS^0$ corresponding to $n \in \pi_0 QS^0$. If $n : S^0 \to QS^0$ corresponds to $n \in \pi_0 QS^0 \cong \mathbb{Z}$, as $QS^0$ is an infinite loop space, then we may extend $n$ to an infinite loop map $n : QS^0 \to QS^0$ (in fact a homotopy equivalence whose inverse is $-n$) so that if $\alpha \in (QS^0)_m$ then $\alpha \ast n \in (QS^0)_{m+n}$. Here, in this subsection, by abuse of notation $\ast$ denotes the loop sum in $QS^0$ and the Pontrjagin product induced by it in $H_* QS^0$. We write for the translation
map * [n] : \(H_\ast QS^0 \to H_\ast QS^0\) induced by \(n : QS^0 \to QS^0\) which when restricted to the component \((QS^0)_m\) looks like \(* [n] : H_\ast (QS^0)_m \to H_\ast (QS^0)_{m+n}\). Let’s note that we may consider a generator \([n] \in H_0((QS^0)_n; \mathbb{Z}/2) \cong \mathbb{Z}/2\) and think of \(* [n]\) as multiplication by \([n]\). The classes \([n]\) (or the homomorphisms \(* [n]\)) have the property that \([n] * [m] = [n + m]\) in \(H_\ast QS^0\) with \([0]\) playing the role of neutral element [21]. According to Dyer and Lashof [8, Section 5, Corollary 2] (see also [34, Page 33]) \(H_\ast QS^0\) is a polynomial algebra (under the Pontrjagin product coming from the loop sum) generated by elements of the form \(Q^1[1] \in H_1(QS^0)\) where \(I\) is any nonempty admissible sequence and \([1], [-1] \in H_0 QS^0\) subject to \([1]^{-1} = [-1]\) (see also [6, Part I] for more details).

In this paper, for the purpose of applications in \(H_\ast (\mathbb{Z} \times BO)\), we are interested in the submodule of \(A\)-annihilated classes in \(H_\ast QS^0\) whose complete description is unknown. But, there are some sufficient conditions that allow one to identify some of these classes. The following is due to Curtis [7, Lemma 6.2, Theorem 6.3] (see also Wellington [38, Theorem 5.6] as well as [39]).

**Theorem 2.2.** Define \(\phi : \mathbb{N} \to \mathbb{N} \cup \{0\}\) by \(\phi(n) = \min \{i : n_i = 0\}\) for \(n = \sum_{i=0}^{\infty} n_i 2^i\) with \(n_i \in \{0, 1\}\). For a generator \(Q^I[1]\) of \(H_\ast QS^0\), suppose \(I = (i_1, \ldots, i_s)\) with \(s > 1\) is a sequence so that \(ex(I) < 2^{\phi(I)}\) and \(0 \leq 2i_{j+1} - i_j < 2^{\phi(I)}\) for \(1 \leq j \leq s - 1\). Then \(Q^I[1]\) is \(A\)-annihilated. If \(I = (i)\) with \(i < 2^{\phi(I)}\), i.e. \(i = 2^s - 1\) for some \(t > 0\), then \(Q^I[1]\) is \(A\)-annihilated. Here, \(ex(Q^I x) = i_1 - (i_2 + \cdots + i_s)\).

The above theorem can be generalised to describe \(A\)-annihilated monomials \(Q^I x\) in homology of \(QX\). We refer the reader to [42, Theorem 2] for more details (see also [43]).

### 2.3. \(H_\ast (\mathbb{Z} \times BO)\)

We follow [29] to complete our description of \(H_\ast (\mathbb{Z} \times BO)\). Write \(\iota : \mathbb{R}P^\infty \to [1] \times BO \to \mathbb{Z} \times BO\) for the inclusion and \(a_i \in H_i \mathbb{R}P^\infty\) for a generator with \(i > 0\), and let \(e_i = \iota_* a_i\) for \(i > 0\). For \(S^0 = \{0, 1\}\), let \(\chi : S^0 \to \mathbb{Z} \times BO\) send 0 into \(\{0\} \times BO\) and 1 into \(\{1\} \times BO\). By the infinite loop space structure of \(\mathbb{Z} \times BO\), provided by Bott periodicity, there exists an infinite loop map \(\bar{\chi} : QS^0 \to \mathbb{Z} \times BO\) [23]. Setting \(e_0 = \bar{\chi}_* [1]\), \(\bar{\chi}_* [-1] = e_0^{-1}\), we have an isomorphism of algebras

\[
H_\ast (\mathbb{Z} \times BO) \cong \mathbb{Z}/2[e_0, e_0^{-1}, e_1 : \text{deg } e_1 = 1].
\]

The product in this case is coming from the loop structure induced by the Whitney sum. The elements \(e_0^n, n \in \mathbb{Z}\), provide translation maps between different path components given by \(* e_0^n : H_\ast ([m] \times BO) \to H_\ast ([m + n] \times BO)\) which is simply multiplying by \(e_0^n\) whose inverse is \(* e_0^{-n}\); this is analogous to the role of \([n]\) in \(H_\ast QS^0\). We may also write \(e_0^n *\) for \(* e_0^n\) when it is more convenient with * referring to the product on \(H_\ast (\mathbb{Z} \times BO)\) under the additive structure.
2.4. \( H_* BO \) inside \( H_* (\mathbb{Z} \times BO) \). The aim of this section is to clear up some subtlety which arises from geometric considerations. The subtlety of presenting \( H_* BO \) as a subalgebra of \( H_* (\mathbb{Z} \times BO) \) is that the space \( \mathbb{Z} \times BO \) is an \( E_\infty \) ring space [24] with two different products coming from the Whitney sum of virtual vector bundles (the additive structure) and tensor product of virtual vector bundles (the multiplicative structure); the delooping of \( \mathbb{Z} \times BO \) provided by the Bott periodicity corresponds to the additive structure. At the prime \( p = 2 \) these provide two different loop structures on \( \mathbb{Z} \times BO \), consequently two different ring structures on \( H_* (\mathbb{Z} \times BO) \) [22]. Now, the base point component of \( \mathbb{Z} \times BO \) when equipped with the additive structure is \( \{0\} \times BO \) (denoted by \( BO_0 \) in the literature) on which the product comes from Whitney sum of virtual vector bundles of dimension 0, whereas the basepoint component of \( \mathbb{Z} \times BO \) under the multiplicative structure is \( \{1\} \times BO \) (denoted by \( BO_\otimes \) in the literature) on which the product is the tensor product of virtual vector bundles of dimension 1. Moreover, the translation map \( * e_0 : H_* BO_0 \to H_* BO_\otimes \) is not a ring map and \( H_* (\{0\} \times BO) \) and \( H_* (\{1\} \times BO) \) are not isomorphic as rings (although they are additively isomorphic under \( * e_0 \)). Hence, since we equip \( H_* (\mathbb{Z} \times BO) = \mathbb{Z} / 2[ e_0, e_0^{-1}, e_i : i > 0 ] \) with the product induced by the Whitney sum then \( H_* (\{1\} \times BO) \) is not an honest subalgebra of \( H_* (\mathbb{Z} \times BO) \) under the additive structure. So, from now on we fix the additive structure on \( \mathbb{Z} \times BO \).

Our favourite copy of \( BO \) in \( \mathbb{Z} \times BO \) is \( \{0\} \times BO \). Following [29, Section 2], for \( i > 0 \), we set \( \overline{e_i} = e_0^{-1} e_i \) which allows us to identify

\[ H_* BO \cong H_* (\{0\} \times BO) \cong \mathbb{Z} / 2[ \overline{e_i} : i > 0 ] \]

as a subalgebra of \( H_* (\mathbb{Z} \times BO) \).

**Remark 2.3.** For the sake of presentation, it could be more natural to take \( H_* (\{0\} \times BO) \cong \mathbb{Z} / 2[ \overline{e_i} : i > 0 ] \) with \( \text{deg} \overline{e_i} = i \) and then write \( H_* (\mathbb{Z} \times BO) \cong H_* (\{0\} \times BO)[ e_0, e_0^{-1} ] \) as a Laurent polynomial algebra which is similar to [34, Page 33]. However, we have followed the tradition in our presentation.

Note that additively \( H_* (\mathbb{Z} \times BO) \cong \bigoplus_{n \in \mathbb{Z}} H_* ([n] \times BO) \). Since we work with the additive infinite loop structure on \( \mathbb{Z} \times BO \), the ring structure provides various compatible pairings

\[ H_* ([n] \times BO) \otimes H_* ([m] \times BO) \to H_* ([n + m] \times BO). \]

In particular, for \( e_{i_1}^{k_1} \cdots e_{i_r}^{k_r} \in H_* (\mathbb{Z} \times BO) \) with \( 0 \leq i_1 < \cdots < i_r \) we have \( e_{i_1}^{k_1} \cdots e_{i_r}^{k_r} \in H_* ([k] \times BO) \) where \( k = \sum k_j \). Note that here only \( k_1 \) is allowed to be negative if \( i_1 = 0 \) and \( k_j > 0 \) for all \( j > 1 \). This induces a filtration on \( H_* (\mathbb{Z} \times BO) \), say the length filtration, which assigns length \( k = \sum k_j \) to a monomial \( e_{i_1}^{k_1} \cdots e_{i_r}^{k_r} \) which we denote by \( l(e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}) = k \). We may also introduce a reduced length filtration on \( H_* (\mathbb{Z} \times BO) \) which is defined as follows

\[
\tilde{l}(e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}) = \begin{cases} k & \text{if } i_1 > 0, \\ \sum_{j > 1} k_j & \text{if } i_1 = 0. \end{cases}
\]
Let $\xi = \sum_{\text{finite}} \xi_k$ be given where $\xi_k \in H_*(\mathbb{Z} \times BO)$ is nonzero for all $k$. We write $l(\xi) = l$ if and only if $l(\xi_k) = l$ for all $k$. We use a similar convention for the reduced length filtration. The following is about the relation between the action of $\mathcal{A}^{op}$ on $H_*(\mathbb{Z} \times BO)$ and these filtrations.

**Lemma 2.4.** (i) The action of $\mathcal{A}^{op}$ on $H_*(\mathbb{Z} \times BO)$ respects the length filtration in the sense that if $\text{Sq}^t_1 \xi \neq 0$ and $l(\xi) = k$ then $l(\text{Sq}^t_1 \xi) = k$ where $\xi$ is any monomial in $H_*(\mathbb{Z} \times BO)$.

(ii) The action of $\mathcal{A}^{op}$ on $H_*(\mathbb{Z} \times BO)$ preserves the reduced length filtration in the sense that if $\text{Sq}^t_1 \xi \neq 0$ and $\bar{l}(\xi) = r$ then $\bar{l}(\text{Sq}^t_1 \xi) = r$ where $\xi$ is any monomial in $H_*(\mathbb{Z} \times BO)$.

**Proof.** (i) This is evident from the Cartan formula.

(ii) By definition, for $\xi = e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}$ we have

$$\bar{l}(\xi) = \begin{cases} l(\xi) & \text{if } i_1 > 0, \\ l(e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}) & \text{if } i_1 = 0. \end{cases}$$

So we only have to verify the claim in the case $i_1 = 0$. Note that $\text{Sq}^t_1 e_{i_1}^{k_1} = 0$ for all $t > 0$. Hence, by the Cartan formula, $\text{Sq}^t_1 \xi = e_{i_1}^{k_1} \text{Sq}^t_1 (e_{i_2}^{k_2} \cdots e_{i_r}^{k_r})$. Our claim, now follows from part (i) noting that by instability conditions it is impossible to have $e_{i_j}^{k_j}$ with $i_j > 0$ such that $\text{Sq}^t_1 e_{i_j}^{k_j} = e_{i_j}^{\alpha_j}$ for some nonnegative integer $\alpha_j$.

**Remark 2.5.** Since $H_*(\mathbb{Z} \times BO)$ is a polynomial algebra then any $\xi \in H_*(\mathbb{Z} \times BO)$ can be written as a sum of finite number of monomials of the form $e_I^K := e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}$ with $r > 0$ where $I = (i_1, \ldots, i_r)$ and $K = (k_1, \ldots, k_r)$. So, for any $\xi \in H_*(\mathbb{Z} \times BO)$ we may write

$$\xi = \sum e_I^K(\xi) e_I^K$$

where the sum runs over all sequences $I = (i_1, \ldots, i_r)$ with $0 \leq i_1 < \cdots < i_r$ as well as all sequences of integers $K = (k_1, \ldots, k_r)$ so that $k_1$ is allowed to be negative if $i_1 = 0$ and $k_j \geq 0$ for all $j > 1$. Here, $e_I^K = e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}$ and $e_I^K(\xi) \in \mathbb{Z}/2$ so that except for finitely many $I$ and $K$ we have $e_I^K(\xi) = 0$. We may rearrange terms and use the length filtration to write

$$\xi = \sum_{k \in \mathbb{Z}} \sum_{k_1 + \cdots + k_r = k} e_I^K(\xi) e_I^K.$$

We write $\xi(k) = \sum_{k_1 + \cdots + k_r = k} e_I^K(\xi) e_I^K$ to be the sum of all terms of $\xi$ of length $k$ so $\xi = \sum \xi(k)$. Moreover, among terms of the same length $k$, we may impose the reduced length filtration, and set

$$\xi(k, l) = \sum_{k_1 + \cdots + k_r = k} e_I^K(\xi) e_I^K$$

where the sum runs over all terms of $\xi$ with length equal to $k$ and reduced length equal to $l$. In these terms, we have $\xi(k) = \sum_{l \in \mathbb{Z}} \xi(k, l)$. 

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The splitting of terms in the above remark has an immediate consequence which we record as follows.

**Lemma 2.6.** (i) Suppose \( \xi = \sum \xi(k) \) as in Remark 2.5. Then \( \xi \) is \( \mathcal{A} \)-annihilated if and only if \( \xi(k) \) is for all \( k \).
(ii) Suppose \( \xi(k) = \sum_{l \in \mathbb{Z}} \xi(k, l) \) as in Remark 2.5. Then, \( \xi(k) \) is \( \mathcal{A} \)-annihilated if and only if \( \xi(k, l) \) is for all \( l \).

**Proof.** (i) By the Cartan formula, the action of \( \text{Sq}^i \) operations respects length. Therefore if \( \xi \) is \( \mathcal{A} \)-annihilated so is \( \xi_r \). The converse follows from the additivity of \( \text{Sq}^i \) operations.
(ii) The action of the Steenrod algebra on \( H^*X \) respects reduced length. This implies our claim in one direction now follows. The converse follows from the additivity of \( \text{Sq}^i \) operations. \( \square \)

**Remark 2.7.** Note that \( H_*(\mathbb{Z} \times BO) \) is equipped with three filtrations provided by dimension, length and reduced length. This is compatible with the filtration of \([33]\) and the new ingredient here is the reduced length filtration. The reduced length of a monomial \( e_{i_1}^K \cdots e_{i_r}^K \) tells us its “length” when translated back into \( H_*BO_{\mathbb{Q}} \) and written in terms of \( \overline{e}_i \) classes. In fact, if \( l(e_{i_1}^K) = m \) then \( \overline{e}_{i_1}^K = e_0^{-m}e_{i_1}^K \in H_*BO_{\mathbb{Q}} \). Moreover, if \( \overline{l}(e_{i_1}^K) = n \) then \( \overline{e}_{i_1}^K \) is in the image of \( H_*BO(n) \to H_*BO \) (see Section 2.5 below for more discussion on this).

Finally, we wish to record the following logarithmic property of length and reduced length functions.

**Lemma 2.8.** Given two monomials \( \xi, \eta \in H_*(\mathbb{Z} \times BO) \) we have

\[
l(\xi \eta) = l(\xi) + l(\eta), \quad \overline{l}(\xi \eta) = \overline{l}(\xi) + \overline{l}(\eta).
\]

The proof is immediate from definitions.

**2.5. Presenting \( H_*BO(k) \) inside \( H_*BO \).** In order to avoid any confusion with the existing literature, we need to fix our notations and choose our presentations of homologies very carefully. As mentioned at the introduction, while writing monomials \( e_{l_1}^{i_1} \cdots e_{l_r}^{i_r} \) in \( H_*(\mathbb{Z} \times BO) \), we fix the order \( 0 \leq i_1 < \cdots < i_r \) on variables where the role of \( e_0 \) is to change the path component. Since we have chosen \( \{0\} \times BO \) as our favourite copy of \( BO \) in \( \mathbb{Z} \times BO \), we need to describe \( H_*BO(k) \) in accordance with this choice.

Recall that \( \overline{e}_i := e_0^{-1}e_i \) for \( i > 0 \). We write \( e_0 \in H_0BO(k) \simeq \mathbb{Z}/2 \) for a generator. Note that we may define \( \overline{e}_0 = e_0^{-1}e_0 = 1 \) where 1 is the unit element of the ring \( H_*(\mathbb{Z} \times BO) \), and by abuse of notation write \( \overline{e}_0 \in H_0BO(k) \) for a generator which maps to \( 1 \in H_0(\mathbb{Z} \times BO) \) under the inclusion \( BO(k) \to \{0\} \times BO \to \mathbb{Z} \times BO \). This latter would be equivalent to defining \( \overline{e}_0 = \chi \cdot [0] \) which would imply \( Q^i\overline{e}_0 = 0 \) for all \( i > 0 \). Any of these definitions for \( \overline{e}_0 \) would be fine.
For any $k > 0$ the inclusion $i_k : BO(k) \to BO$ induces a monomorphism of $\mathcal{A}^{op}$-modules in homology. In this case, $H_*BO(k)$ is identified with the $\mathcal{A}^{op}$-submodule of $H_*BO$ generated by all monomials $e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}$ with $0 \leq i_1 < \cdots < i_r$ and length $\sum_{i=1}^r k_i \leq k$ (compare to [2, Page 154] noting that their $e_i$ coincides with our $\overline{e_i}$).

The point of choosing $\overline{e_0}$ is that since it is the unit element in $H_*(Z \times BO)$ then $\overline{e_0} = e_0$. Hence, given an element $\overline{e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}}$ which we know falls into the image of $(t_n)_* : H_*(BO(n)) \to H_*(BO)$ for any $n$ with $k := \sum k_j \leq n$, we may regard this monomial as of full length, i.e. of length $n$, just by adding the required number of $e_0$ in front of it as

$$\overline{e_0^{n-k}e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}} = \overline{e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}}$$

in $H_*(Z \times BO)$ as well as $H_*BO$. We use this convention which allows to think of elements of $H_*BO(n)$ to always have length $n$.

3. The stable symmetric hit problem

3.1. Preparatory observations. We prove Theorem 1.2 in a few steps. We begin by proving Lemma 1.1.

**Proof of Lemma 1.1.** (i) By definition of $ex_j$ it follows that $ex_j - ex_{j-1} = 2i_{j+1} - i_j$. The immediately implies the $I$ is admissible if and only if $ex_j \leq ex_{j+1}$ for all $j = 0, ..., r - 1$.

(ii) Since all entries of $I$ are odd then $i_j = 2i_{j+1}$ cannot happen and the admissibility condition $i_j < 2i_{j+1}$ implies that $2i_{j+1} - i_j \geq 1$. Noting that $ex_j = ex_{j-1} + (2i_{j+1} - i_j)$ we deduce that $ex_j > ex_{j-1}$. This completes the proof.

(iii) By part (i) $ex_{j-1} \geq ex(I) > 0$ for all $j = 1, ..., r - 1$. It follows that $i_j > (i_{j+1} + \cdots + i_r) > i_{j+1}$ as we work with sequence only having positive entries.

Next, we record some formulae on the action of Kudo-Araki operations. The following is a corollary of Theorem 2.2 and the above description of $H_*(Z \times BO)$.

**Lemma 3.1.** Suppose $I$ is a sequence satisfying the conditions of Theorem 2.2. Then, $Q^I e_0, Q^I e_0^{-1}$ are $\mathcal{A}$-annihilated classes in $H_*(Z \times BO)$.

**Proof.** Since $\overline{\lambda}$ is an infinite loop map, then $\overline{\lambda} Q^I[1] = Q^I e_0$ and $\overline{\lambda} Q^I[-1] = Q^I e_0^{-1}$. The mapping $\overline{\lambda}$ is an $\mathcal{A}^{op}$-module homomorphism by naturality of the Steenrod operations. The classes $Q^I[1]$ and $Q^I[-1]$ satisfying the conditions of Theorem 2.2 are $\mathcal{A}$-annihilated in $H_*(QS^0)$. Therefore, $Q^I e_0 = \overline{\lambda} Q^I[1]$ and $Q^I e_0^{-1} = \overline{\lambda} Q^I[-1]$ are $\mathcal{A}$-annihilated in $H_*(Z \times BO)$. □

The proof of Theorem 1.2(i) follows from evaluating $Q^I e_0$ in $H_*(Z \times BO)$ for $I$ admissible for which we appeal to Priddy’s computations (see Theorem 3.2).
together with an application of Lemma 2.6. The proof of Theorem 1.2(ii) is immediate upon applying the length filtration mentioned in Note 2.5 and having part (i). The proof of Theorem 1.2(iii) regarding the linear independence needs a bit of work which will follow later on.

The evaluation of the $Q^i$ operations on the generators of $H_*(\mathbb{Z} \times BO)$ has been carried out by Priddy [29, Theorem 1.1 and Corollary 2.3] (see also work of Kochman [18] and compare to [34, Chapter 1, Proposition 5.11]).

**Theorem 3.2.** (i) For $n > k > 0$ we have

$$Q^n e_k = \sum_{u=0}^{k} \binom{n-k+u-1}{u} e_{n-u}e_k^{u}.$$

(ii) For $n > 0$ we have

$$Q^n e_0^{-1} = \sum \binom{k}{k_1, \ldots, k_n} e_{k_1}^{i_1} \cdots e_{k_n}^{i_n} e_0^{k-2}$$

where the sum is over all sequences $k_1, \ldots, k_n$ of nonnegative integers with $\sum i_k = n$, $k = \sum k_i$ and

$$\binom{k}{k_1, \ldots, k_n} = \frac{k!}{k_1! \cdots k_n!}.$$

By properties recalled in Section 2.2 it follows immediately that

$$Q^k e_k = e_k^2, \quad Q^0 e_0^{-1} = e_0^{-2}$$

which are the obvious cases that are not considered by part (i) and part (ii) of the above theorem, respectively.

**Remark 3.3.** By Theorem 3.2(i), by choosing the term with $u = 0$ in Priddy’s formula, for any $n > k > 0$

$$Q^n e_k = e_k e_n$$

modulo terms $e_k e_{n+u}$, hence modulo terms of the form $e_k e_j$ with $j > n$ and $i < k$ corresponding to the cases with $u > 0$. Consequently, none of the other terms cancels out with $e_k e_n$ showing that $Q^n e_k \neq 0$.

Next, we show that for $I$ admissible $Q^I e_0 \neq 0$. We freely use the observation and notations of Lemma 1.1 throughout the statement of the lemma and its proof.

**Lemma 3.4.** Suppose $I = (i_1, \ldots, i_r)$ is an admissible sequence with $\text{ex}(I) > 0$. Then, in $H_*(\mathbb{Z} \times BO)$ we have

$$Q^I e_0 = e_0 e_{e_0 e_{e_1}^2} \cdots e_{e_{e_{r-1}^{r-1}}}^{e_{e_{r-1}^{r-1}}} + O_I \neq 0.$$

For $r = 1$, $O_I = 0$. For $r > 1$,

$$O_I = \sum \epsilon_l e_L$$

where $\epsilon_l \in \mathbb{Z}/2$ and the sum runs over all nondecreasing sequences of nonnegative integers $L = (l_1, \ldots, l_{2r})$ so that $l_{2r} > \text{ex}_{r-1} = i_r$ and $e_L = e_{l_1} \cdots e_{l_{2r}}$. 
**Proof.** For $r = 1$, $I = (i_r)$ and by Theorem 3.2(i) we have $Q^i_0 e_0 = e_0 e_i_r \neq 0$. In particular, there are no other terms as $u \in \{0, k\}$ with $k = 0$ in Priddy’s formula. So, the statement holds for $r = 1$.

We now prove the case for $r > 1$. We proceed by induction, and in order to make an illustration, we do the cases $r = 2, 3$. For $r = 2$ with $I = (i_{r-1}, i_r)$, using the Cartan formula and the computation for the case $r = 1$, we compute that

$$Q^{i_r} - 1 Q^{i_r} e_0 = Q^{i_r - 1} (e_0 e_{i_r}) = \sum_{\beta_{r-1}=0}^{i_r} (Q^{i_{r-1}} - \beta_{r-1} e_0) (Q^{\beta_{r-1}} e_{i_r}).$$

We compute each term $(Q^{i_{r-1}} - \beta_{r-1} e_0)(Q^{\beta_{r-1}} e_{i_r})$ of the above sum as follows. By Priddy’s formula, Theorem 3.2(i), $Q^{i_{r-1}} - \beta_{r-1} e_0 = e_0 e_{i_{r-1} - \beta_{r-1}}$. For the term $Q^{\beta_{r-1}} e_{i_r}$, we proceed as follows. By Lemma 1.1(iii), as $I$ is admissible of positive excess, $i_{r-1} \geq i_r$. In the above sum, for $\beta_{r-1}$ running from 0 to $i_{r-1}$, if $\beta_{r-1} \leq i_r$ then $Q^{\beta_{r-1}} e_{i_r} = 0$ by basic properties of Kudo-Araki operation recalled in Section 2.2. Hence, in order to get nontrivial terms in the above sum, we have to restrict to $\beta_{r-1} \geq i_r$. Again by the properties recalled in Section 2.2, for $\beta_{r-1} = i_r$ we have $Q^{i_{r-1}} e_{i_r} = e_{i_r}^2$. Consequently, after separating the term with $\beta_{r-1} = \text{ex}_{r-1} = i_r$, hence $i_{r-1} - \beta_{r-1} = i_{r-1} - i_r = \text{ex}_{r-2}$, we obtain

$$Q^{i_r} - 1 Q^{i_r} e_0 = e_0 e_{\text{ex}_{r-2}} e_{i_r} + \sum_{\beta_{r-1}=i_{r-1}+1}^{i_r} e_0 e_{i_{r-1} - \beta_{r-1}} (Q^{\beta_{r-1}} e_{i_r})$$

where by Remark 3.3 we see that $Q^{\beta_{r-1}} e_{i_r} = e_{i_r} e_{\beta_{r-1}}$ modulo terms $e_{i_{r-1}} e_{i_r}$ with $l_r > \beta_{r-1}$ and $l_r - 1 < i_r$. Note that by our choice of $\beta_{r-1}$ these inequalities combine and yield

$$l_r > \beta_{r-1} \geq i_r + 1 > i_r.$$

Our claim now follows.

For $r = 3$ with $I = (i_{r-2}, i_{r-1}, i_r)$, using our computations for the case of $r = 2$, we have

$$Q^{i_{r-2}} Q^{i_{r-1}} - 1 Q^{i_r} e_0 = Q^{i_{r-2}} (e_0 e_{\text{ex}_1(I)} e_{2}^{\text{ex}_2(I)} + \sum e_I e_L) = Q^{i_{r-2}} (e_0 e_{\text{ex}_1(I)} e_{2}^{\text{ex}_2(I)}) + Q^{i_{r-2}} (\sum e_I e_L)$$

where $L = (l_1, l_2, l_3, l_4), e_L \in \mathbb{Z}/2$, and $l_4 > i_r = \text{ex}_2(I)$. Applying the Cartan formula (see Remark 2.1), we see that the first term of (3) can be written as

$$Q^{i_{r-2}} (e_0 e_{\text{ex}_1(I)} e_{2}^{\text{ex}_2(I)}) = \sum_{j_1+j_2+j_3=\text{ex}_2(I)} (Q^{j_1} e_0)(Q^{j_2} e_{\text{ex}_1(I)})(Q^{j_3} e_{\text{ex}_2(I)}).$$

First we extract the leading term of our expression. Choose

$$j_3 = \dim(e_{\text{ex}_2(I)}) = 2 \text{ex}_2(I) = 2 l_r,$$
$$j_2 = \dim(e_{\text{ex}_1(I)}) = i_{r-1} - i_r.$$

Note that by our conventions, for $I = (i_{r-2}, i_{r-1}, i_r)$ we have $I_1 = (i_{r-1}, i_r)$ and $I_2 = (i_r)$ which yield $\text{ex}_1(I) = \text{ex}(I_1) = i_{r-1} - i_r$ and $\text{ex}_2(I) = i_r$. Moreover, by
the hypothesis and Lemma 1.1(i) we have $0 < \text{ex}(I) \leq \text{ex}_1(I) \leq \text{ex}_2(I)$. These choices together with $\sum j_k = i_{r-2}$ give $j_1 = i_{r-2} - (i_{r-1} + i_r) = \text{ex}_{r-2}(I)$. As recalled in Remark 2.1 we have $Q^{i+1}i^2 = 0$ and $Q^2i^2 = (Q^i\xi)^2$. This implies that for the above choices of $j_1, j_2, j_3$ we have

$$(Q^{j_1}e_0)(Q^{j_2}e_{\text{ex}_2(I)}(Q^{j_3}e_{\text{ex}_3(I)}) = e_0 e_{\text{ex}_d(I)} e^2_{\text{ex}_2(I)} e_{\text{ex}_3(I)}$$

which is our leading term. For the other terms of (4), note that in order to have a nontrivial term we need $j_3 > \dim e^2_{\text{ex}_d(I)} = 2i_r$. By the Cartan formulae, we see that

$$Q^{j_3}(e^2_{\text{ex}_d(I)}) = \begin{cases} (Q^{j_3}e_{\text{ex}(I)})^2 & \text{if } j_3 = 2j_3', \\ 0 & \text{otherwise} \end{cases}$$

where $j_3 > 2i_r$ implies that $j_3' > i_r$. Hence, all of the terms in the other terms of (4) are of the claimed forms. Note that by Priddy’s formula, applying one single operation $Q^n$ doubles the length, that is $Q^n e_k$ is a sum of terms of length 2. Hence, taking care of the powers of variables, all terms in the equation (4) are of length $(2^2)^2$. In order to complete our verification, consider the second term of (3). By the Cartan formula, for any single $L$, we have

$$Q^{l-2}e_L = \sum_{\sum j_k = l-2} (Q^{j_1}e_{i_1}) \cdots (Q^{j_k}e_{i_k}).$$

It is immediate that this leads to an expression with terms of length $(2^2)^2$. Moreover, if a term in the above expression is nontrivial then $j_4 > l_4 > i_4$. Consequently, all terms in this expression have the claimed form.

For the general case, suppose $I' = (i', i_1, \ldots, i_r)$ is an admissible sequence with $\text{ex}(I') > 0$. This means $I = (i_1, \ldots, i_r)$ is also admissible with $\text{ex}(I) > 0$ and by the induction hypothesis we have

$$Q^i e_0 = e_0 e_{\text{ex}(I)} e^2_{\text{ex}(I)} \cdots e^{2^{r-1}}_{\text{ex}(I)} + \sum e_L e_L$$

with $l(L) = 2^r$ and $l_{2^r} > i_r$ which allows us to write

$$Q^i e_0 = Q^i Q^i e_0 = Q^i (e_0 e_{\text{ex}(I)} e^2_{\text{ex}(I)} \cdots e^{2^{r-1}}_{\text{ex}(I)} + \sum e_L e_L) = Q^i (e_0 e_{\text{ex}(I)} e^2_{\text{ex}(I)} \cdots e^{2^{r-1}}_{\text{ex}(I)} + \sum e_L e_{L'}).$$

By the Cartan formula for the first term we have

$$Q^i (e_0 e_{\text{ex}(I)} e^2_{\text{ex}(I)} \cdots e^{2^{r-1}}_{\text{ex}(I)}) = \sum_{i_0 + \cdots + i_{r-1} = i'} (Q^{i_0} e_0) \cdots (Q^{i_{r-1}} e^{2^{r-1}}_{\text{ex}(I)}).$$

We may write $i'$ as

$$2^{r-1}i_r + 2^{r-2}(i_{r-1} - i_r) + \cdots + (i_1 - i_2 - \cdots - i_r) + (i' - i_1 - i_2 - \cdots - i_r) = 2^{r-1} \text{ex}_{r-1}(I) + \cdots + \text{ex}_0(I) + \text{ex}(I').$$
which allows us to separate one term of the sum and the other terms and write
\[ Q^l(e_0e_{\text{ex}(I)}^2e_{\text{ex}_1(I)}^2 \cdots e_{\text{ex}_{r-1}(I)}^{2r-1}) = e_0e_{\text{ex}(I')}e_{\text{ex}_1(I)}^2 \cdots e_{\text{ex}_{r-1}(I)}^{2r-1} + \sum(Q^l e_0) \cdots (Q^{l'-1} e_{\text{ex}_{r-1}(I)}^{2r-1}) \]  
where the sum is running over all sequences \((t_0, \ldots, t_{r-1})\) of nonnegative integers with \(\sum t_j = l'\) so that
\[(t_0, t_1, \ldots, t_{r-1}) \neq (\text{ex}(I'), \text{ex}_0(I), 2\text{ex}_1(I), \ldots, 2^{r-1}\text{ex}_{r-1}(I)).\]
The term \(e_0 e_{\text{ex}(I')} e_{\text{ex}_0}^2 \cdots e_{\text{ex}_{r-1}}^{2r-1}\) is the ‘leading term’ as claimed. For the second sum, for any single term to be nontrivial we need \(Q^{l'-1} e_{\text{ex}_{r-1}(I)}^{2r-1} \neq 0\) which having excluded the cases with \(t_{r-1} = 2^{r-1}\text{ex}_{r-1}(I) = 2^{r-1}i_r\), leaves us with the cases \(t_{r-1} > 2^{r-1}i_r\). An iterated application of the Cartan formula \(Q^{2l+1} = 0\) shows that if there exists \(j \in \{0, \ldots, r-1\}\) such that \(2^j\) does not divide \(t_j\) then \(Q^{l/2} e_{\text{ex}_j(I)}^{2^j} = 0\) and consequently
\[(Q^{l/2} e_0) \cdots (Q^{l'-1} e_{\text{ex}_{r-1}(I)}^{2r-1}) = 0.\]
Therefore, the only nontrivial terms in the above sum correspond to \(r\)-tuples \((i_0, \ldots, i_{r-1})\) such that \(2^j\) divides \(t_j\) for all \(0 \leq j \leq r-1\). Moreover, the Cartan formula \(Q^{2l+1} = (Q^l)^2\) shows that if \(t_j = 2^j p_j\) for some positive integer \(p_j\) then \(Q^{l/2} e_{\text{ex}_j(I)}^{2^j} = (Q^l e_{\text{ex}_j(I)})^{2^j}\). Notice that \(t_{r-1} > 2^{r-1}i_r\) which implies that \(p_{r-1} > i_r\). Hence, the only possible nontrivial terms in the above sum must include a product factor as
\[ Q^{l/2} e_{i_r} = e_{i_r} e_{p_{r-1}} \text{ modulo terms } e_{a} e_{b} \text{ with } b > p_{r-1} > i_r. \]
Consequently, any term in \(Q^l e_L\) would include a factor \(e_{i_j}\) with \(l_j > i_r\). Hence, so far, \(Q^{l/2} e_0\) satisfies the claimed expression. Next, we deal with the terms in \(\sum e_L Q^l e_L\) where by the inductive assumption there exists \(j\) so that \(l_j > i_r\); for each \(L\) we denote such a choice of \(j\) by \(j_0\). By the Cartan formula, for each term \(Q^{l/2} e_L\) we have
\[ Q^{l/2} e_L = \sum_{t_1 + \cdots + t_{l'} = l'} (Q^{l/2} e_{i_1}) \cdots (Q^{l/2} e_{i_{l'}}). \]
If a term in the above sum is to be nontrivial we need \(t_{j_0} > l_{j_0}\). As above, the factor \(Q^{l/2} e_{i_{j_0}}\) gives the desired factors \(e_{a} e_{b}\) with \(b > i_r\). The last claim that \(l_{2r} > \text{ex}_{r-1}(I) = i_r\) can be shown by induction on \(r\) and a similar analysis as we did above which we leave to the reader. This completes the proof.

**Note 3.5.** By iterated application of Cartan formula together with Theorem 3.2(i), also recorded in Lemma 3.4, all terms of \(Q^l e_0\) are of the same length filtration, namely filtration \(2^l\). But, different terms may have different reduced length filtration. For example consider \(I = (5, 3)\). By iterated application of Priddy’s formula, we compute that
\[ Q^5 Q^3 e_0 = e_0 e_2 e_5^2 + e_0 e_4 e_2 e_4 + e_0 e_4 e_2 e_5 + e_0 e_4 e_2 e_3 + e_0 e_4 e_3 e_5. \]
The last term has reduced length equal to 2 whereas all of other terms are of reduced length 3.

The implicit property which seems to be useful in some of our computations is that the reduced length of each term in $Q^2Q^3e_0$ is bounded below. We wish to have this property in full generality. We don’t have a proof in the general case, however, and we provide a proof when $l(I)$ is small.

Lemma 3.6. Suppose $I = (i_1, \ldots, i_r)$ is an admissible sequence of positive dimension, having only odd entries, $ex(I) > 0$ and $r \in \{1, 2\}$. If $Q^i e_0 = \sum \epsilon_K e_K$ where $K$ runs over all increasing sequences with $l(K) = 2^r$ and $\epsilon_K \in \mathbb{Z}/2$ then $l(\epsilon_K) \geq 2^{r-1}$ whenever $\epsilon_K = 1$.

We remark that a partial proof we had for $l(I) = 3$ depends on too much detailed analysis, hence does not seem too hopeful to be generalised in an inductive manner.

Proof. By Lemma 3.4 we have $Q^i e_0 \neq 0$. Hence, there exists a $K$ such that $\epsilon_K = 1$.

Case of $l(I) = 1$. For any $I = (i)$ with $i > 0$, we have $Q^i e_0 = e_0 e_i$ which satisfies $l(e_0 e_i) = 1 = 2^{\delta(I)}$. Hence, the above inequality holds in this case.

Case of $l(I) = 2$. Suppose $I = (i_1, i_2)$ is an admissible as required by the lemma. Then, by Priddy’s formula and the Cartan formula we have

$$Q^i Q^i e_0 = Q^i(Q^i - 1) e_0 e_i = \sum_{j_1 = j_2}^{i_1 i_2} (Q^i - 1) e_0 e_{i_1} e_{i_2}.$$ 

First, suppose $j_1 < i_1$. In this case $i_1 - j_1 > 0$. Moreover, for any $u$ we have $j_1 + u \geq i_2 > 0$. Hence, any term with $j_1 \neq i_1$ in the above sum has at least two $e_a$’s with $a > 0$. Hence, $l(e_0 e_{i_1 - j_1} e_{i_2 - u} e_{j_1 + u}) \geq 2^{l(I)}$. Next, suppose $j_1 = i_1$ where the corresponding terms are $(i_1 - j_1 + u - 1)^{2^r} e_{i_1} e_{j_1 - u} e_{j_1 + u}$ with $0 \leq u \leq i_2$. Obviously, for $u < i_2$ any term, if with nonzero coefficient, has reduced length equal to 2. For $u = i_2$ the term is $(i_1 - j_1 + u - 1)^2 e_{i_1} e_{j_1 - u}$ and the binomial coefficient is trivial since $i_1$ and $i_2$ are both odd by hypothesis. Hence, any term in the expression for $Q^i Q^i e_0$ is either of reduced length 3 or 2. In particular, this proves our claim on the lower bound for the reduced length. \qed

3.2. Proof of Theorem 1.2. This section is devoted to the proof of Theorem 1.2.

Proof of Theorem 1.2. (i) Let $\xi^Z \times \text{BO} \mapsto Q^i e_0 \in H_{|I|} Q(Z \times \text{BO})$ where $I$ is chosen as in Theorem 1.2. By Lemma 3.1 this class is $\mathcal{A}$-annihilated. By Lemma 3.4

$$\xi^Z \times \text{BO} = e_0 e_{e_0^2} e_{e_1^2} \cdots e_{e_{r-1}^2} + O_I \neq 0$$
where \( O_I = 0 \) if \( r = 1 \) and for \( r > 1 \),
\[
O_I = \sum \epsilon_I e_L
\]
where \( \epsilon_I \in \mathbb{Z}/2 \) and the sum runs over all sequences of nonnegative integers \( L = (l_1, \ldots, l_r) \) satisfying conditions of Lemma 3.4. Since \( I \) satisfies conditions of Theorem 1.2 then by Lemma A.2 all entries of \( I \) are odd. It follows from Lemma 1.1(ii) that \( \text{ex}_{j-1} < \text{ex}_j \) for \( 0 < j < r \).

(ii) We note that \( \text{ex}_{j} = \text{ex}_{j-1} + 1 \) for \( 0 < j < r \). Note that by hypothesis \( \text{ex}_0 = \text{ex}(I) > 0 \). Hence,
\[
\xi^B_{\bar{I}} = e_0^e e_0^2 \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}} e_0^{e_{\text{ex}_1}} \cdots e_0^{e_{\text{ex}_{r-1}}} + \text{O}_I.
\]
Let \( e_{\text{ex}_0} = e_0^{-2} \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}} \) which is given by
\[
\xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}} = e_0^{-2} \xi_{\bar{I}} \quad \text{such that} \quad e_{\text{ex}_0} = e_0.
\]
The class \( e_0 \) is also \( A \)-annihilated, so is \( e_0^{-2} \xi_{\bar{I}} \). Hence, by the Cartan formula \( \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}} \) is also \( A \)-annihilated, which obviously lives in \( H_+(\mathbb{Z} \times \text{BO}) \). Now, it is obvious that being a sum of monomials of length \( 2r - 1 \) the class \( \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}} \) is in the image of \( H_{+} \text{BO}(2r - 1) \to H_{+} \text{BO} \), induced by the inclusion \( \text{BO}(2r - 1) \to \text{BO} \), which is a monomorphism of \( A \)-modules, i.e. \( \xi_{\bar{I}}^{\mathbb{Z} \times \text{BO}} \) pulls back to an \( A \)-annihilated class in \( H_+ \text{BO}(2r - 1) \).

(iii) We wish to show that the set \( \{ \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}} : \bar{I} \text{ admissible} \} \) is a linearly independent set. By filtration considerations as well as dimensional reasons, it is enough to show that
\[
\mathcal{B}_{r,d} = \{ \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}} : I \text{ admissible where } I(I) = r, |I| = d \}
\]
is a linearly independent set where \( r \) and \( d \) are arbitrary positive integers. This is equivalent to showing that every finite subset of \( \mathcal{B}_{r,d} \) is a linearly independent set. As we work over \( \mathbb{Z}/2 \), it is enough to show that given a finite set \( \{ \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}_{k}} \}_{k=1}^{n} \subseteq \mathcal{B}_{r,d} \) no nontrivial linear combination of its elements adds up to 0. More precisely, given a finite set \( \{ \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}_{k}} \}_{k=1}^{n} \subseteq \mathcal{B}_{r,d} \) we have to show if \( \{ c_k : c_k \in \mathbb{Z}/2 \}_{k=1}^{n} \) is given so that at least one \( c_k \) is nonzero then
\[
\sum_{k=1}^{n} c_k \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}_{k}} \neq 0.
\]
Obviously, we can drop the terms with \( c_k = 0 \). Since we choose \( c_k \in \mathbb{Z}/2 \) then this boils down to showing that for every finite subset \( \{ \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}_{k}} \}_{k=1}^{n} \subseteq \mathcal{B}_{r,d} \) we have
\[
\sum_{k=1}^{n} \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}_{k}} \neq 0.
\]
By definition of the classes \( \xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}} \), as well as by Lemma 3.4 we have
\[
\xi^{\mathbb{Z} \times \text{BO}}_{\bar{I}} = O^I e_0 = e_0 \bar{e}_{\text{ex},0(I)} e_0^e \bar{e}_{\text{ex},1(I)} \cdots e_0^{e_{\text{ex}_{r-1}}(I)} + \sum \epsilon_I e_L.
\]
so that \( b_r > \text{ex}_{r-1}(I_k) \). Here, \( l(I) = r \) and \(|I| = d \) is fixed. We have two cases.

**Case 1.** Suppose sequences \( I_1, \ldots, I_n \) are given with \( l(I_k) = r \) for all \( 1 \leq k \leq n \) so that there is a unique sequence say \( I_{k_m} \) among them with \( \text{ex}_{r-1}(I_{k_m}) < \text{ex}_{r-1}(I_k) \) for all \( k \neq k_m \). Then the leading term of \( Q^{k_m}e_0 \), namely \( e_0 e_{\text{ex}_0(I_{k_m})} e_2^{\text{ex}_2(I_{k_m})} \cdots e_{2^{r-1}}^{\text{ex}_{2^{r-1}}(I_{k_m})} \), is not cancelled out by the leading term of \( Q^{l_k}e_0 \), namely \( e_0 e_{\text{ex}_0(I_k)} e_2^{\text{ex}_2(I_k)} \cdots e_{2^{r-1}}^{\text{ex}_{2^{r-1}}(I_k)} \), as \( \text{ex}_{r-1}(I_{k_m}) < \text{ex}_{r-1}(I_k) \). Moreover, other terms in \( Q^{l_k}e_0 \) are of the form \( e_k \) with \( l_r > \text{ex}_{r-1}(I_k) > \text{ex}_{r-1}(I_{k_m}) \). Hence, none of these terms also cancels out with the leading term of \( Q^{l_k}e_0 \). This completes the proof in this case.

**Case 2.** Suppose sequences \( I_1, \ldots, I_n \) are given so that sequences \( I_{k_{m_1}}, \ldots, I_{k_{m_s}} \) exist with \( s > 1 \), \( \text{ex}_{r-1}(I_{k_{m_1}}) = \cdots = \text{ex}_{r-1}(I_{k_{m_s}}) \) and \( \text{ex}_{r-1}(I_{k_{m_1}}) < \text{ex}_{r-1}(I_k) \) for all \( k \not\in \{k_{m_1}, \ldots, k_{m_s}\} \). It is enough to show that the sum of leading terms of \( Q^{l_{k_{m_1}}}e_0 \) is nonzero. However, this is obvious since if \( I \neq J \) and \( l(I) = l(J) = r \) then \( (\text{ex}_0(I), \ldots, \text{ex}_{r-1}(I)) \neq (\text{ex}_0(J), \ldots, \text{ex}_{r-1}(J)) \). This implies that in the leading terms there will always be different indices, hence the sum of them is nonzero. This completes the proof in this case.

Finally, to see that the classes \( \xi_I^{BO} \) are linearly independent, notice that \( * e_0^{-2r} \) is an isomorphism of \( \mathbb{Z}/2 \)-modules whose inverse is \( * e_0^{2r} \). Consequently, it sends linearly independent sets to linearly independent sets, and in particular sends \( \{\xi_I^{Z\times BO}\} \) to \( \{\xi_I^{BO}\} \), and the linear independence of the former implies the linear independence of the latter. Finally, note that the inclusion \( \iota_k : BO(k) \to BO \) induces a monomorphism of \( \mathcal{A}_{op} \)-modules. Hence, the pull back of the classes \( \xi_I^{BO} \) to \( H_rBO(2^r - 1) \) are linearly independent. This completes the proof of part (iii). □

The proof has an interesting side result which we wish to record. A careful analysis of the above proof provides the following.

**Corollary 3.7.** For any \( I \) chosen as in Theorem 1.2 with \( l(I) = r \), the class \( \xi_I^{BO} \) is a sum of monomials \( \overline{e_K} \) each of which is not divisible by \( e_0 \) neither by \( \overline{e_0} \). More precisely

\[
Q^{l}e_0 = \sum \varepsilon_K \overline{e_K}
\]

where \( \varepsilon_K \in \mathbb{Z}/2, \overline{e_K} = \overline{e_{k_1}} \cdots \overline{e_{k_r}}, \) where \( K \) runs over all admissible nondecreasing sequences of length \( 2^r \) with \( k_1 > 0 \).

The proof is easy and we leave it to the reader. After Lemma 3.4 one may ask about the cases when \( Q^{l}e_0 = 0 \) in \( H_r(\mathbb{Z} \times BO) \) or in general \( Q^{l}e_k = 0 \). By Lemma 3.4 if \( Q^{l}e_0 = 0 \) then \( I \) is not admissible. It seems to the author that we can answer this question in the same line as [31, Lemma 3.5]. These computations require more topological tools to be recalled or introduced. So, we leave further investigation on this to a future work.

### 3.3. Proof of Theorem 1.4.

The proof we present is an application of Theorem 3.2(ii).
Proof of Theorem 1.4. (i) An application of Lemma 3.1 together with Theorem A.1(i) shows that the class $Q^te_0^{-1}$ is $\mathcal{A}$-annihilated if and only if $n = 2^t - 1$ for some $t > 0$. Let $\xi_{Z \times BO} = Q^{2^t-1}e_0^{-1}$. The equation

$$\xi_{Z \times BO} = \sum_{K \in \mathcal{K}_{2^t-1}} e_0^{-l(e^K) - 2} e^K$$

is merely a restatement of Theorem 3.2(ii). For $i \mid 2^t - 1$ let $K$ be a sequence of length $l(e^K) = 2^t - 1$ so that $k_j = 0$ for all $j \neq i$ and $k_i$ be defined by $2^t - 1 = ik_i$. Obviously the coefficient $c_{k_1, \ldots, k_n}^{(i)} = \delta_{k_i}$. The corresponding term would be $e_0^{-k_i-2} e_i$. In particular, our sum would involve the terms $e_0^{-(2^t-1)2} e_1^{2^t-1}$ and $e_0^{-3} e_i^{2^t-1}$. Hence, $\xi_{Z \times BO}$ involves some nontrivial terms. Also, note that $\xi_{Z \times BO}$ is a sum of monomials $e^K$ when $K$ varies in $\mathcal{K}_{2^t-1}$, hence a sum of different monomials. Therefore, $\xi_{Z \times BO} = 0$. Since $e_i = e_0 e_i$ we see that

$$\xi_{Z \times BO} = \sum_{K \in \mathcal{K}_{2^t-1}} e_0^{-l(e^K) - 2} e_i^{l(e^K)} e^K = \sum_{K \in \mathcal{K}_{2^t-1}} e_0^{-2} e^K$$

where $e^K = e_1^{k_1} \cdots e_{2^{t-1}}^{k_{2^{t-1}}}$. Consequently, $e_0^2$ being an isomorphism of $\mathbb{Z}/2$-vector spaces,

$$\xi_{Z \times BO} = e_0^2 \xi_{Z \times BO}$$

is a nontrivial $\mathcal{A}$-annihilated class. The terms $e_0^{-(2^t-1)2} e_1^{2^t-1}$ and $e_0^{-3} e_i^{2^t-1}$ of $\xi_{Z \times BO}$ translate to $e_1^{2^t-1}$ and $e_{2^{t-1}}$ in $H_\ast BO$ as claimed. By filtration considerations, this class is in the image of $H_\ast BO(2^t - 1) \to H_\ast BO$. Moreover, the presence of $e_1^{2^t-1}$ does not allow this class to pull back to $H_\ast BO(s)$ for $s < 2^t - 1$. (ii) It immediately follows from the Cartan formula that the given classes $\xi_{Z \times BO} \in H_{2^{t-1}}(\mathbb{Z} \times BO)$ which are $\mathcal{A}$-annihilated then the class

$$\xi_{Z \times BO} = \prod_{i=1}^d \xi_{A_i \cdots A_d}$$

is $\mathcal{A}$-annihilated. We only note that for two different partitions of $n$ into spikes such as

$$(\lambda_1, \ldots, \lambda_d), (\mu_1, \ldots, \mu_s)$$

then the fact that the classes $\xi_{A_1 \cdots A_d}$ and $\xi_{Z \times BO}$ involve the leading terms $\prod_{i=1}^d e_2^{n_i}$ and $\prod_{j=1}^s e_2^{n_j-1}$, respectively, shows that $\xi_{A_1 \cdots A_d} \neq \xi_{Z \times BO}$. We therefore obtain $s p(n)$ different $\mathcal{A}$-annihilated classes in $H_\ast BO$. The case of $H_\ast BO$ is similar and we leave it to the reader.

Remark 3.8. (i) The class $\xi := e_0^2 Q^{2^t-1}e_0^{-1}$ lives in $H_\ast BO$ and being expressed in terms of monomials $e^K$ with $l(K) = 2^t - 1$ pulls back to $H_\ast BO(2^t - 1)$. However, by Proposition 4.2, all of its terms, apart from $e_{2^{t-1}}$, maps trivially under
the transfer \( t_* : H_*BO(2^i - 1) \to H_*\mathbb{R}^D(2^i - 1) \). This implies that

\[
t_* \xi_i = \sum_{a \in \Sigma_{2^i - 1}} \tau(a_{2^i - 1} \otimes 1^{2^i - 2}).
\]

(ii) In theory, we may apply Theorem 1.4 to produce more \( A \)-annihilated classes in \( H_*BO \). Suppose \( d \) is given and assume \( d = \Sigma_{j=1}^n (2^j - 1) \). Then, we may look for sequences \( K^j \) with \( |e^{K^j}| = 2^j - 1 \). Then the product of these elements is nontrivial. Moreover, a partition of \( d \) into spikes, i.e. writing \( d \) as \( d = \Sigma_{j=1}^n (2^j - 1) \), is not unique and we may look for various ways that can give us new \( A \)-annihilated classes. This seems to be related to the \( \mu \) function of Wood [41] but we have not exploited the idea in this paper.

4. The (nonsymmetric) hit problem

This section demonstrates how one can use the action of the Dyer-Lashof algebra to the study of the hit problem as promised at the beginning. The proof of Theorem 1.5 is a standard argument based on transfer and the idea is to use the transfer map \( BO(k)_+ \to \mathbb{R}P^k \) and its effect in reduced homology. The classes that we wish to obtain are the images of classes \( \xi I \) where we simply define \( \xi I = t_* \xi BO \). By stability of the \( Sq^I \) operations \( \xi I \) is \( A \)-annihilated. It remains to show that these classes are nontrivial.

4.1. The transfer \( BO(k)_+ \to BO(1)^{\times k}_+ \). Consider the fibre bundle \( i : BO(1)^{\times k} \to BO(k) \) and the Becker-Gottlieb transfer associated to it which is a stable map \( t : \Sigma BO(k)_+ \to \Sigma BO(1)^{\times k}_+ \) [4] (see also [30] and [1, Chapter 4]). Here, for a space \( X \), \( X_+ \) denotes \( X \) with an added disjoint base point which satisfies \( H_\ast \Sigma^\infty X_+ \cong \tilde{H}_\ast X_+ \cong H_\ast X \). We begin with the following which has to be well known. For instance, it immediately follows from [5, Theorem 3.11].

Lemma 4.1. Suppose \( 0 \leq i_1 < \cdots < i_r \). The map \( t_* : H_*BO(k) \to H_*BO(1)^{\times k} \cong (H_*BO(1))^{\otimes k} \) induced by the transfer satisfies

\[
t_* (e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}) = \sum_{a \in \Sigma_k} \tau(a_{i_1}^{\otimes k_1} \cdots \otimes a_{i_r}^{\otimes k_r})
\]

where \( k_1 + \cdots + k_n = k \) and \( \Sigma_k \) is the permutation group on \( k \) elements which acts by permutation of the factors.

This allows the following description of \( \text{ker} t_* \), as well as an implicit description of \( \text{coker} t_* \).

Proposition 4.2. (i) Consider a monomial \( e_{i_1}^{k_1} \cdots e_{i_r}^{k_r} \in H_*BO(k) \) where \( k = \Sigma_{j=1}^r k_j \) and \( k_j > 0 \) for all \( j \). Suppose \( k_j > 1 \) for some \( j \). Then for the transfer \( t_* : H_*BO(k) \to H_*BO(1)^{\times k} \) we have \( t_* (e_{i_1}^{k_1} \cdots e_{i_r}^{k_r}) = 0 \).

(ii) Let \( k > 0 \). Let \( H_*BO \) be the submodule of \( H_*BO \) generated by all monomials \( e_I = e_{i_1} \cdots e_{i_r} \) with \( I \) being strictly increasing and \( l(I) = k \); in the previous notation this corresponds to the cases with \( k_j = 1 \) for all \( j \) and \( r = k \). Let
$H^*_BO(k) = H_BO(k) \cap H^*_BO$. Then the restriction of $t_* : H^*_BO(k) \to H_*\mathbb{R}P^{\times k}$ is a monomorphism.

**Proof.** (i) Suppose $\tau(a_{i_1}^{k_1} \otimes \cdots \otimes a_{i_r}^{k_r})$ is a term of the above expression for $t_*(\overline{e}_{i_1}^{k_1} \cdot \overline{e}_{i_r}^{k_r})$ where $\tau \in \Sigma_k$. For $j_0$ with $k_{j_0} > 1$ and let $\sigma \in \Sigma_{k_{j_0}} \cdots \Sigma_{k_r}$ be a permutation which on the blocks coming from $\Sigma_{k_j}$ with $j \neq j_0$ acts as identity and on the $j_0$-th block a nonidentity permutation; in matrix representation it is a block diagonal matrix $\sigma = \text{diag}(1_{k_1 \times k_1}, \cdots, \sigma_{k_{j_0}} \cdots, 1_{k_r \times k_r})$ where $\sigma_{k_{j_0}} \in \Sigma_{k_{j_0}}$ is a nonidentity element. It is evident that

$$\tau(a_{i_1}^{k_1} \otimes \cdots \otimes a_{i_r}^{k_r}) = \tau \sigma(a_{i_1}^{k_1} \otimes \cdots \otimes a_{i_r}^{k_r}).$$

Hence, the terms corresponding to $\tau$ and $\tau \sigma$ cancel out in the expression for $t_*(\overline{e}_{i_1}^{k_1} \cdot \overline{e}_{i_r}^{k_r})$. Moreover, we can do this for any $\tau \in \Sigma_k$. These together with the fact that the above expression has even number of terms, proves our claim.

(ii) This is immediate from the formula. Since $S$ is strictly increasing, then for any $\sigma, \tau \in \Sigma_k$ we have $\tau(a_i \otimes \cdots \otimes a_i) = \sigma(a_i \otimes \cdots \otimes a_i)$. In particular, $\tau(a_i \otimes \cdots \otimes a_i) = a_i \otimes \cdots \otimes a_i$ for any $\tau \in \Sigma_k$. This together with the above formula completes the proof. 

**Corollary 4.3.** Consider the transfer homomorphism $t_* : H_*BO(k) \to H_*\mathbb{R}P^{\times k}$. Then $\ker t_*$ is precisely the submodule of $H_*BO(k)$ that is generated by all monomials $\overline{e}_{i_1}^{k_1} \cdots e_{i_r}^{k_r} \in H_*BO(k)$ where $k = \sum_{j=1}^{r} k_j$ and $k_j > 0$ for all $j$ such that $k_j > 1$ for some $j$. Let $H^*_BO(k) = \text{coker}(t_*)$ which is defined by the following short exact sequence

$$0 \to \ker t_* \to H_*BO(k) \xrightarrow{p} H^*_BO(k) \to 0.$$

Then $t_*$ extends to a nontrivial monomorphism $\overline{t}_* : H^*_BO(k) \to H_*\mathbb{R}P^{\times k}$.

**Proof.** This immediately follows from Proposition 4.2. 

It would be ideal to determine all admissible sequences $I = (i_1, \ldots, i_r)$ satisfying conditions of Theorem 1.5 so that $\xi^BO_I$ satisfies $p(\xi^BO_I) \neq 0$ in $H^*_BO(k)$, hence also mapping nontrivially under $t_* : H_*BO(k) \to H_*\mathbb{R}P^{\times k}$. Notice that for $l(I) = r$ the class $\xi^BO_I = e_0^{-2r} Q^l e_0$ is a sum of (formal) monomials in $\overline{e}_i$'s which has a square-free term if and only if $Q^l e_0$ has a square-free term up a multiplication by $e_0^{-2r}$. Therefore, it is enough to have some criteria for $Q^l e_0$ having a square-free term, up to multiplication by $e_0^{-2r}$. Any class $Q^l e_0$ can be expressed as a sum of classes $e_K$ with $K$ a non-decreasing sequence such that $l(K) = 2^r$ if $l(I) = r$. The map $* e_0^{-2r}$ then rewrites these classes in $H_*([-1] \times BO)$ as a sum of monomials in $\overline{e}_i$ variables. It is easy to show that the length and reduced length are related by $l(e_0^{-2r} e_K) = l(e_K)$. Note that $* e_0^{-2r}$ is a ring map. The Cartan formulae $Q^{2n+1} \xi^2 = 0$ and $Q^{2n} \xi^2 = (Q^n \xi)^2$ imply that if the part of $K$ consisting of nonzero entries is not strictly increasing, i.e. $e_K$ has a square, then depending
on the parity of $n$, $Q^n e_K$ is either 0 or has a square in it. This reduces our search for square-free terms to the following cases.

**Lemma 4.4.** Suppose $0 < i_1 < \cdots < i_k$ and $n > i_1 + \cdots + i_k$. If $Q^n(e_{i_1} \cdots e_{i_k})$ has a square-free term then $n > (i_1 + \cdots + i_k) + k - 1$.

**Proof.** A proof by contradiction argument, applying the Cartan formula 2.1(i), proves the Lemma. We leave the details to the reader. □

4.2. Proof of Theorem 1.5. For any $\xi = e_{i_1} \cdots e_{i_k}$ with $0 < i_1 < \cdots < i_k$ if $Q^n \xi$ has a square-free term then

$$n \geq \dim \xi + \text{number of variables in } \xi = \dim \xi + l(\xi).$$

In fact, what $k$ counts is the number of variables $e_i$ with $i > 0$. This immediately proves the following.

**Corollary 4.5.** Suppose $I$ is an increasing sequence of nonnegative integers. If $Q^n(e_i)$ has a square-free term then $n > \dim(e_i) + \bar{l}(e_i) - 1$. In particular, for $I = (i_1, \ldots, i_s)$ if $s \geq 2$ is the largest integer such that $i_s = 0$ then $(i_{s+1}, \ldots, i_r)$ is strictly increasing.

We wish to use this result to decide about the square-free terms of $Q^l e_0$, albeit up to a power of $e_0$. We note that the case of $l(I) = 1$ is a trivial case as $Q^l e_0 = e_0 e_i$ which after translation by $e_0$ pulls back to $e_i$ in $H_\ast BO(1)$. It is known that the transfer in the case of $n = 1$ is the same as the identity. Hence, we focus on the cases with $l(I) \geq 2$. We have the following.

**Lemma 4.6.** Suppose $I = (i_1, \ldots, i_r)$ is an admissible sequence with $r \geq 2$. For $I_1 = (i_2, \ldots, i_r)$, suppose $Q^l e_0 = \sum e_K e_K$ where $e_K \in \Z / 2$ and $K$ runs over all increasing sequences of non-negative integers of length $2^r - 1$. If $Q^l e_0$ has a square-free term, up to a multiplication by a power of $e_0$, then there exists a $K$ with $e_K = 1$ and

$$\text{ex}(I) > \bar{l}(e_K) - 1.$$  

**Proof.** For any $e_K$ in the expression for $Q^l e_0$ we have $\dim(e_K) = \dim(I_1) = i_2 + \cdots + i_r$. Noting that $\text{ex}(I) = i_1 - (i_2 + \cdots + i_r)$ the claimed inequality follows the inequality of Corollary 4.5. This completes the proof. □

4.2. Proof of Theorem 1.5. The above observations on the transfer map $t_\ast : H_\ast BO(k) \to H_\ast \R P^{\infty}$ allow us to prove Theorem 1.5(i). We have the following.

**Lemma 4.7.** Suppose $I$ is a nonempty sequence, with $l(I) > 1$, as in Theorem 1.5(i). Then $\xi_I \neq 0$. Moreover, the class $\xi_I$ is invariant under the action of $\Sigma_{2^r - 1}$.

**Proof.** The point that $\xi_I \neq 0$ is obvious from the definitions. The class $\xi_I^{BO} = e_0^{2^r \xi_I^{BO}} e_0^{2^r \xi_I^{BO}}$ pulls back to $H_{|I|} BO(2^r - 1)$ and by definition $\xi_I = t_\ast \xi_I^{BO}$. Obviously, $p(e_0^{2^r \xi_I^{BO}}) = 0$ is the same as $\xi_I \neq 0$. We proceed to show that invariance
under the action of $\Sigma_{2^r-1}$. We may always force the expression of Lemma 3.4 to split and write

$$\xi I^{z \times BO} = Q^i e_0 = e_0 e_{ex_0}^2 e_{ex_1}^2 \cdots e_{ex_{r-1}}^{2^r-1} + \sum_S e_S e_S' + O$$

where $e_S \in \mathbb{Z}/2$ and $S = (s_1, \ldots, s_{2^r})$ is running over all strictly increasing sequences with $l(S) = 2^r$ such that $s_1 = 0$, and $O$ consists of the remaining terms. Note that for $r = 1$ the middle sum and $O$ are empty. For $r > 1$, the assumption that $p(\xi) \neq 0$ in $H^*_S BO(2^r - 1)$ guarantees $\xi$ has some terms which belong to $H^*_S BO(2^r - 1)$ so that the middle sum is not empty, i.e. if $r > 1$ then at least for one $S$ we have $e_S = 1$. Moreover, the sum over $S$ runs over all increasing (but not strictly) sequences of nonnegative integers $L = (l_1, \ldots, l_{2^r})$ so that $l_{2^r} > ex_{r-1}(l) = i_r$ and $e_L = e_{l_1} \cdots e_{l_{2^r}}$. Using $\bar{e}_i = e_0^{-1} e_i$, this implies that

$$\xi I^{z BO} = \frac{e_{ex_0}^2 e_{ex_1}^2 \cdots e_{ex_{r-1}}^{2^r-1}}{e_{ex_0} e_{ex_1} \cdots e_{ex_{r-1}}} + \sum_S e_S e_S' + e_0^{-2^r} O$$

where $S_1 = (s_2, \ldots, s_{2^r})$ by the notation introduced at the beginning and $\bar{e}_{S_1} = \frac{e_{ex_1} e_{ex_2} \cdots e_{ex_{r}}}{e_{ex_1} e_{ex_2} \cdots e_{ex_{r}}}$. It is now easy to see that

$$\xi I^{z BO} = \sum_S e_S e_S' + R$$

where $R$ is a sum of terms of the form $\bar{e}_L$ with $L$ increasing but not strictly. By Proposition 4.2 all terms that have an index with at least two repeated entries map trivially under $t_*$, i.e. $t_* R = 0$, and we have

$$\xi I = t_* \sum_S e_S e_S'.$$

Moreover, the sum over $S$'s includes at least one nontrivial term. With all sequences being strictly increasing, it is evident that if $S, S'$ are two distinct strictly increasing sequences then $S \neq \tau S'$ for all $\tau \in \Sigma_{2^r-1}$ where the permutation group acts by permuting the factors. Moreover, $S \neq \tau S$ for all nonidentity permutations $\tau \in \Sigma_{2^r-1}$. This then proves that $\xi I \neq 0$. Finally, note that by the formula for $t_*$ as in Lemma 4.1 the element $\xi I = t_* \xi I^{z BO}$ with $\xi I^{z BO}$ being a sum of monomials satisfies the relation $\tau \xi I = \xi I$, so $\xi I$ is invariant under the action of $\Sigma_{2^r-1}$.

Next, we turn to using these classes to construct more $A$-annihilated classes. Let’s note that there exist obvious pairings $\mathbb{R} P^{\infty} \times \mathbb{R} P^{\infty} \to \mathbb{R} P^{\infty} \times (n+m)$ which induce $H_* \mathbb{R} P^{\infty} \otimes H_* \mathbb{R} P^{\infty} \to H_* \mathbb{R} P^{\infty} \times (n+m)$. These also induce pairings $\mathbb{R} P^{\infty} \wedge \mathbb{R} P^{\infty} \to \mathbb{R} P^{\infty} \times (n+m)$ inducing $H_* \mathbb{R} P^{\infty} \otimes H_* \mathbb{R} P^{\infty} \to H_* \mathbb{R} P^{\infty} \times (n+m)$. Both pairings on the level of homology are given by juxtaposition of elements. The geometry behind this is provided by James’ splitting that for a path connected space $X$ we have $\Sigma \Omega \Sigma X$ is weak homotopy equivalent to $\bigwedge_{r=1}^{\infty} \Sigma X^{\wedge r}$ [15]. Moreover, we have $H_* \Omega \Sigma X \simeq T(\tilde{H}_* X)$, the tensor algebra generated by $\tilde{H}_* X$, where the latter
isomorphism holds in $\Omega$-homology. Consequently, working with such pairings is quite natural and corresponds to working in the homology ring of $\Omega\Sigma X$. We have the following.

**Theorem 4.8.** Suppose $k = \sum_{j=1}^{l}(Z^{j} - 1)$ so that for each $j$ there exists a sequence $I^{j}$ of positive integers with $l(I^{j}) = r_{j}$ and $\text{ex}(I^{j}) > 0$ satisfying the conditions of Theorem 1.5. Then, for any $\sigma \in \Sigma_{l}$ there exists an $A$-annihilated class $\xi_{\sigma} \in H_{d}R P^{l_{k}}$ with $d = \sum |I^{j}|$ which is invariant under the action of $\Sigma'_{\sigma(1)} \times \cdots \times \Sigma'_{\sigma(l)}$. Moreover, depending on the partition chosen for $k$, subject to the existence of sequences $I^{j}$, this would lead to at least $\prod$ distinct $A$-annihilated classes in $H_{d}R P^{l_{k}}$.

**Proof.** Since $I^{j}$ satisfies the conditions of Theorem 1.5, and consequently satisfies the conclusions of Lemma 4.7, then we have nontrivial $A$-annihilated classes, namely $\xi_{\sigma} = H_{d}R P^{l_{k}}$ with $d = \sum |I^{j}|$ which is invariant under the action of $\Sigma'_{\sigma(1)} \times \cdots \times \Sigma'_{\sigma(l)}$. Moreover, depending on the partition chosen for $k$, subject to the existence of sequences $I^{j}$, this would lead to at least $\prod$ distinct $A$-annihilated classes in $H_{d}R P^{l_{k}}$.

As the pairing is induced by a mapping of spaces, hence by naturality of the Steenrod operations the class $\xi_{\sigma}$ is an $A$-annihilated class. The invariance under the action of $\Sigma'_{\sigma(1)} \times \cdots \times \Sigma'_{\sigma(l)}$ is clear since each factor in $\xi_{\sigma}$ is invariant under the action of the relevant permutation group. Since $H_{d}R P^{l_{k}}$ is a tensor algebra, it is not commutative. So, while taking the product of various elements the order matters. Because of the invariance under $\Sigma'_{\sigma(1)} \times \cdots \times \Sigma'_{\sigma(l)}$ the only order changing permutations can arise from $\Sigma_{l}$. By suitable choices for the sequences $I^{j}$, say choosing them to be from different lengths and dimensions, and having no entries in common, one can see that the number $\prod$ distinct classes can be attained. We leave the details to the reader.  

Finally, we turn to some computations in low lengths. The results of the appendix (see Theorem A.1) determine all sequences $I$ of positive excess with $l(I) \leq 3$ with $Q^{l}e_{0}$ being $A$-annihilated, hence all nontrivial classes $\xi_{I}^{BO}$ classes with $l(I) \leq 4$. We may ask if in these cases $\xi_{I} = t_{e}\xi_{I}^{BO} \neq 0$? We provide an answer to this when $l(I) \leq 3$. First, note that it is a general fact that the transfer associated to the identity is the identity. Hence, in the case of $n = 1$ we have the $A$-annihilated classes $\tilde{\alpha}_{2l-1} \in H_{2l-1}BO$ which pull back to $a_{2l-1} \in H_{2l-1}BO(1)$ with $t_{e}a_{2l-1} = a_{2l-1}$. According to Theorem A.1 there is no sequence with $l(I) = 2$ so that $Q^{l}e_{0}$ is $A$-annihilated. Hence, we do not expect any $\xi_{I}$ class with $l(I) = 2$. The following lemma eliminates existence of $\xi_{I}$ classes with $l(I) = 3$.

**Lemma 4.9.** Suppose $I$ is an admissible sequence of positive integers with $l(I) = 3$ such that $Q^{l}e_{0}$ is $A$-annihilated, that is $I$ satisfies conditions of Theorem 1.2 which are the same as conditions of Theorem 2.2. Then the pull back of $e_{0}^{-2}Q^{l}e_{0}$
to $H_*BO(2^{l(I)} - 1)$ maps trivially under the transfer $t_* : H_*BO(2^{l(I)} - 1) \to H_*\mathbb{R}P^{2^{l(I)}-1}$.

**Proof.** The proof is by contradiction. Suppose there exists an admissible sequence $I = (i_1, i_2, i_3)$ so that the pullback of $e_0^{2^{l(I)}} Q^l e_0$ maps nontrivially under the transfer $t_*$. By Proposition 4.2 and Corollary 4.3 and the explanations afterwards any elements non-square-free monomial in $H_*BO(k)$ maps trivially under the transfer $t_* : BO(k) \to H_*\mathbb{R}P^{2^{l(I)}k}$. Hence, $\xi_I \neq 0$ implies that $Q^l e_0$ must have at least one square-free term. We show that the existence of any square-free term would contradict Lemma 4.6. By Lemma 4.6 if $Q^l e_0$ is a sum of monomials $e_K$ then there exists $K$ with $l(K) = 2^2$ such that $\tilde{l}(e_K) - 1 < \text{ex}(I)$, i.e. $\tilde{l}(e_K) \leq \text{ex}(I)$. On the other hand, by Theorem A.1, there exist positive integers $m$ and $n$ with $m < n - 1$ so that

$$I = (2^{n+1} + 2^m - 1, 2^n + 2^m - 1, 2^n - 1).$$

Consequently $\text{ex}(I) = 1$. However, since $I$ satisfies conditions of Theorem 1.2 then by Lemma A.2 all of its entries are odd allowing us to use Lemma 3.6 to $I_1 = (i_2, i_3)$ with $l(I_1) = 2$ which shows that $\tilde{l}(e_K) \geq 2^{l(I_1)-1} = 2$. This contradicts the inequality $\tilde{l}(e_K) \leq \text{ex}(I)$ and completes the proof. \qed

**Appendix A. Closed forms for low lengths**

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The contents of this appendix are identical to some parts of [12] which we have put online only to share and publicise our result; in particular [12] is not intended to be published. We refer the interested reader to [12] for more details as well as tables which computes sequences satisfying conditions of Theorem 2.2 up to dimension $2^{17}$.

It is possible to use the conditions of Theorem 2.2 to put more restrictions on the sequences $I$. From computational point of view putting more restrictions on the set of all such sequences is the same as making the space in which we have to search for such sequences smaller. Our main result in this appendix determines all sequences of small length.

**Theorem A.1.** Suppose $I$ is an admissible sequence of positive excess satisfying conditions of Theorem 2.2.

(i) If $l(I) = 1$ then $I = (2^t - 1)$ for some $t > 0$. Moreover, any $I = (2^t - 1)$ satisfies
conditions of Theorem 2.2.
(iii) If \( l(I) = 3 \) then there exist positive integers \( m \) and \( n \) with \( m \leq n - 1 \) so that
\[
I = (2^{n+1} + 2^m - 1, 2^n + 2^m - 1, 2^n - 1).
\]
Moreover, any sequence \( I \) of the above form satisfies conditions of Theorem 2.2.
(iv) If \( l(I) = 4 \) if \( I \) satisfies conditions of Theorem 2.2 then \( I \) has one of the following forms
\[
I = (2^{n+3} + 2^{n-1} - 1, 2^{n+2} + 2^{n-1} - 1, 2^{n+1} + 2^{n-1} - 1, 2^n + 2^{n-1} - 1) \\
\text{with } n > 2,
\]
\[
I = (2^{n+3} + 2^{m+1} + 2^{m'} - 1, 2^{n+1} + 2^n + 2^{m'} - 1, 2^n + 2^m - 1, 2^n - 1) \\
\text{with } n > m > m' > 1.
\]
Moreover, any sequence of the above forms satisfies conditions of Theorem 2.2.

Let’s note that part (i) and (ii) of the above Theorem are almost trivial and were known to Wellington. We provide a proof for the case of \( l(I) = 3 \) as an illustration. We leave the case of \( l(I) = 4 \) to the reader as it is lengthy and tedious. We only prove the assertions in the direction that takes us from knowing that \( I \) satisfies conditions of Theorem 2.2 with \( l(I) = r \) to getting closed form for \( I \). The proof in the other direction is immediate as from given closed forms one can readily verify that the satisfy conditions of Theorem 2.2; we leave the details to the reader. Next, we give a list of elementary properties that these sequence enjoy and will be use in the coming proofs.

**Lemma A.2.** (i) For \( r > 1 \), if \( I = (i_1, \ldots, i_r) \) is an admissible sequence with \( \text{ex}(I) > 0 \) then \( I \) is strictly decreasing, i.e. \( i_j > i_{j+1} \) for all \( 1 \leq j < r \).

(ii) Suppose \( I = (i_1, \ldots, i_r) \) is a sequence with \( \text{ex}(I) > 0 \) which satisfies conditions of Theorem 2.2. Then, all entries of \( I \) are odd.

**Proof.** (i) This easily follows by induction from the admissibility and positivity of excess.

(ii) It is straightforward to see that if \( s = 1 \) then \( I = (2^t - 1) \) for some \( t > 0 \).
Suppose \( s > 0 \) and \( I = (i_1, \ldots, i_r) \) is an admissible sequence with \( \text{ex}(I) > 0 \) which satisfies condition of Theorem 2.2. First, note that if \( i_1 \) is even then \( \phi(i_1) = 0 \) which together with condition \( 0 < \text{ex}(I) < 2^{\phi(i_1)} \) shows that \( 0 < \text{ex}(I) < 1 \) which is a contradiction. Hence, \( i_1 \) must be odd. For \( j > 1 \) suppose \( i_j \) is even, hence \( \phi(i_j) = 0 \). The condition \( 0 \leq 2i_j - i_{j-1} < 2^{\phi(i_j)} = 1 \) implies that \( i_{j-1} = 2i_j \).
By iterating this process we see that \( i_1 \) must be even which is a contradiction. Hence, \( I \) consists of only odd entries. \( \square \)

The following is now evident.

**Corollary A.3.** If \( I \) satisfies conditions of Theorem 2.2 then \( l(I) \) and \( |I| \) have the same parity.

Next, note that for a sequence \( I = (i_1, \ldots, i_r) \) we have its dimension \( |I| = i_1 + \cdots + i_r \). We have the following.
Lemma A.4. Suppose \( I = (i_1, \ldots, i_r) \) is an admissible sequence of positive excess. Then \( I \) satisfies conditions of Theorem 2.2 if and only if for \( (i_0, I) := (i_0, i_1, \ldots, i_r) \) we have \( 0 < 2l_{j+1} - i_j < 2^{\phi(l_{j+1})} \) for all \( j \in \{0, \ldots, r - 1\} \) where \( i_0 = |I| \).

Proof. The inequality \( 0 < \text{ex}(I) < 2^{\phi(l_i)} \) is the same as \( 0 < 2l_{i} - i_0 < 2^{\phi(l_i)} \) if we replace \( \text{ex}(I) \) by \( 2l_{i} - i_0 \). The result now follows.

The above observations are important as they replace two conditions of Theorem 2.2 only with one condition, so that search for sequences satisfying condition of Lemma A.4 would also provide us with all sequences which satisfy conditions of Theorem 2.2 and the refinement would be \( i_0 = |I| = i_1 + \cdots + i_r \).

The main ingredient of Theorem 2.2 is the function \( \phi \) which allows us to express the result in a combinatorial manner. We are therefore interested in studying properties of this function. To begin with, let’s recall the following which is implicit in Curtis’s work and follows by looking at the binary expansion of numbers.

Lemma A.5. Suppose \( I = (i_1, \ldots, i_r) \) is an admissible sequence with \( \text{ex}(I) > 0 \) satisfying conditions of Theorem 2.2. Then,

\[
\phi(i_1) \leq \cdots \leq \phi(i_r).
\]

We also define \( \psi : \mathbb{N} \to \mathbb{N} \) by

\[
\psi(n) = \max\{i : n_i = 1\} + 1 = \min\{i : \forall j \geq i, n_j = 0\}.
\]

In fact, writing \( n \in \mathbb{N} \) in binary expansion \( \cdots n_1 n_0 \), that is \( n = \sum_{i=0}^{\infty} n_i 2^i \) with \( n_i \in \{0, 1\} \), the function \( \psi \) assigns to \( n \) the ‘length’ of its binary expansion.

We call \( n \) a spike if \( n = 2^t - 1 \) for some \( t > 0 \). The following is immediate.

Lemma A.6. (i) For \( n \in \mathbb{N} \), \( \phi(n) \leq \psi(n) \). Moreover, \( \phi(n) = \psi(n) \) if and only if \( n = 2^t - 1 \) with \( t = \psi(n) \).

(ii) If \( n \) is a non-spike then \( \phi(n) < \psi(n) - 1 \).

Proof. Both parts immediately follow from definitions. We only note that, for (ii), by definition \( \psi(n) - 1 \) is the last place in the binary expansion of \( n \) which a 1 appears whereas \( \phi(n) \) is the first place in the binary expansion a 0 shows up, hence \( \phi(n) \neq \psi(n) - 1 \).

We also have the following characterisation of non-spike integers.

Lemma A.7. (i) Let \( n \) be a non-spike positive integer, that is \( n \neq 2^t - 1 \) for all \( t \). Then, there exists a natural number \( N_n > 1 \) so that \( n = 2^{\phi(n)} N_n + 2^{\phi(n)} - 1 \). In particular, \( N_n \) is an even number.

(ii) If \( n \) is a non-spike then there exists a nonnegative integer \( B(n) \) with either \( B(n) = 0 \) or \( B(n) > 2^{\phi(n)} \) such that

\[
n = 2^{\phi(n) - 1} + B(n) + 2^{\phi(n)} - 1.
\]
Proof. (i) Note that for any positive integer \( k \) we have \( 2^k - 1 = \sum_{j=0}^{k-1} 2^j \). This together with definition of \( \phi(n) \) shows that for \( n = \sum_{i=0}^{+\infty} n_i 2^i \) we have

\[
N_n = (n - 2^{\phi(n)} - 1)/2^{\phi(n)} = \sum_{i=\phi(n)}^{+\infty} n_i 2^{i-\phi(n)}.
\]

If \( N_n \) is an odd number then \( n_{\phi(n)} = 1 \) which contradicts definition of \( \phi(n) \). In particular, \( N_n \) cannot be 1.

(ii) by definition of \( \psi \) we have \( n_{\psi(n)-1} = 1 \) and \( n_i = 0 \) for all \( i \geq \psi(n) \). For \( n = \sum_{i=0}^{+\infty} n_i 2^i \) written in binary form, \( B(n) = \sum_{i=\phi(n)+1}^{\psi(n)-2} n_i 2^i \) is the required value. \( \square \)

Sometimes we refer to \( B(n) \) as the (undetermined-) block of \( n \).

The functions \( \phi \) and \( \psi \) provide some useful upper and lower bounds for non-spike numbers. Before proceeding further, let’s recall that for all \( i > 0 \) we have

\[
2^i - 1 = \sum_{j=0}^{i-1} 2^j.
\]

Most of what we say below are consequences of this equality. For instance, we have the following.

Lemma A.8. (i) For \( n \in \mathbb{N} \), \( n_{\psi(n)} = 0 \) and \( n_{\psi(n)-1} = 1 \).

(ii) For any non-spike positive integer \( n \in \mathbb{N} \) we have

\[
2^{\psi(n)-1} + 2^{\phi(n)} - 1 \leq n < 2^{\psi(n)} - 1 - 2^{\phi(n)} < 2^{\psi(n)} - 1.
\]

(iii) For any non-spike positive integer \( n \) we have \( \phi(n) < \psi(n) - 1 \).

Proof. (i) This follows from the definition of \( \psi \).

(ii) Using the equality \( 2^i - 1 = \sum_{j=0}^{i-1} 2^j \) the inequality is immediate.

(iii) This is also immediate from the definitions. \( \square \)

The following provides an application of the inequalities of Lemma A.8.

Lemma A.9. Suppose \( I = (i_1, \ldots , i_r) \) is an admissible sequence with \( \text{ex}(I) > 0 \) which satisfies conditions of Theorem 2.2. If \( i_j = 2^n - 1 \) for some \( n > 0 \) then \( j = r \).

Proof. By Lemma A.2(i) the sequence \( I \) is strictly decreasing. Suppose \( j < r \) and \( i_j = 2^n - 1 \) for some \( n > 0 \). Then \( i_{j+1} < 2^n - 1 \). We compare \( \psi(i_{j+1}) \) to \( n \) and proceed to show that any possible choice for \( \psi(i_{j+1}) \) leads to a contradiction. We consider the following cases.

Case of \( \psi(i_{j+1}) \geq n + 1 \). In this case, \( \psi(i_{j+1}) - 1 \geq n \). If \( i_{j+1} \) is non-spike then by Lemma A.8(ii) we have

\[
i_{j+1} = 2^{\phi(i_{j+1})-1} + 2^{\psi(i_{j+1})} - 1 > 2^{\phi(i_{j+1})-1} - 1 \geq 2^n - 1 = i_j.
\]
which is a contradiction. If \( i_{j+1} \) is a spike then by Lemma A.6 \( \phi(i_{j+1}) = \psi(i_{j+1}) \) and
\[
i_{j+1} = 2^{\psi(i_{j+1})} - 1 > 2^n - 1 = i_j
\]
which is a contradiction.

**Case of** \( \psi(i_{j+1}) \leq n - 1 \). Using the inequalities of Lemma A.8(ii) we see that
\[
i_{j+1} \leq 2^{\psi(i_{j+1})} - 1 \leq 2^{n-1} - 1
\]
which together with \( i_j = 2^n - 1 \) and the admissibility condition implies that
\[
2i_{j+1} \leq 2^n - 2 = i_j - 1 < i_j \leq 2i_{j+1}
\]
which is an obvious contradiction.

**Case of** \( \psi(i_{j+1}) = n \). First, suppose \( \phi(i_{j+1}) < \psi(i_{j+1}) \), i.e. \( i_{j+1} \) is not a spike. By Lemma A.8(ii) we have
\[
i_{j+1} \geq 2^{\phi(i_{j+1})} - 1 + 2^{\phi(i_{j+1})} - 1 = 2^{n-1} + 2^{\phi(i_{j+1})} - 1.
\]
Noting that \( i_j = 2^n - 1 \), and multiplying both sides of this inequality by 2, we have
\[
2i_{j+1} \geq 2^n + 2^{\phi(i_{j+1})+1} - 2
\]
which implies that \( 2i_{j+1} - i_j \geq 2^{\phi(i_{j+1})+1} - 1 \). For \( I \) satisfying conditions of Theorem 2.2 we have \( 2i_{j+1} - i_j \leq 2^{\phi(i_{j+1})} \). Moreover, by Lemma A.2(ii) all entries of \( I \) are odd and this latter inequality reads as \( 2i_{j+1} - i_j < 2^{\phi(i_{j+1})} \) which is the same as \( 2i_{j+1} - i_j \leq 2^{\phi(i_{j+1})} - 1 \). It is impossible to have both inequalities
\[
2i_{j+1} - i_j \geq 2^{\phi(i_{j+1})+1} - 1, \quad 2i_{j+1} - i_j < 2^{\phi(i_{j+1})} - 1
\]
holding together. Hence, we have a contradiction.

The only possible remaining case is the case of \( \phi(i_{j+1}) = \psi(i_{j+1}) = n \). In this case, by Lemma A.6, \( i_{j+1} = 2^n - 1 = i_j \). But this contradicts Lemma 1.1(iii), unless \( j = r \). This completes the proof. \( \square \)

A very immediate corollary of the proof is the following.

**Corollary A.10.** If \( I = (i_1, \ldots, i_r) \) is an admissible sequence of positive excess, satisfying conditions of Theorem 2.2, with \( i_r = 2^{\phi(i_r)} - 1 \) then \( \phi(i_{r-1}) < \phi(i_r) \).

**Proof.** By Lemma A.5 \( \phi(i_{r-1}) \leq \phi(i_r) \). We have to eliminate the possibility of \( \phi(i_{r-1}) = \phi(i_r) \). By Lemma A.8(ii), \( i_{r-1} \geq 2^{\psi(i_{r-1})} - 1 + 2^{\phi(i_{r-1})} - 1 \). Also, using the upper bound of Lemma A.8(ii) for \( i_r \), noting that by Lemma A.6(i) \( \phi(i_r) = \psi(i_r) \), we see that
\[
2i_r - i_{r-1} \leq 2^{\phi(i_r)} - 1 - 2 - (2^{\phi(i_{r-1})} - 1 + 2^{\phi(i_{r-1})} - 1) = 2^{\phi(i_r)} - 2^{\phi(i_{r-1})} - 1.
\]
By Lemma A.9 \( i_{r-1} \) is not a spike. Hence, by Lemma A.6(ii), \( \phi(i_{r-1}) < \psi(i_{r-1}) - 1 \). If \( \phi(i_r) = \phi(i_{r-1}) \) then the above inequality reads as
\[
2i_r - i_{r-1} \leq 2^{\phi(i_r)} - 2^{\psi(i_{r-1})} - 1 = 2^{\phi(i_{r-1})} - 2^{\phi(i_{r-1})} - 1 < -1
\]
which contradicts the admissibility of \( I \). Hence, it is not possible to have \( \phi(i_r) = \phi(i_{r-1}) \). This completes the proof. \( \square \)
We can prove a more practical criterion of the sequence of $\psi$’s. We have the following.

**Theorem A.11.** Suppose $I = (i_1, ..., i_r)$ is an admissible sequence of positive excess, satisfying conditions of Lemma 2.2. Then for all $j \in \{2, ..., r\}$ we have $\psi(i_j) = \psi(i_{j-1}) - 1$.

**Proof.** We eliminate the cases with $\psi(i_j) \leq \psi(i_{j-1}) - 2$ and $\psi(i_j) \geq \psi(i_{j-1})$ as follows.

**Case of $\psi(i_j) \leq \psi(i_{j-1}) - 2$.** We have

$$i_j \leq 2^{\psi(i_j)} - 1 \leq 2^{\psi(i_{j-1})} - 2 < 2^{\psi(i_{j-1})} - 1$$

which implies that $2i_j \leq 2^{\psi(i_{j-1})} - 2$. Now, for any $j \in \{2, ..., r\}$ by Lemma A.9 $i_{j-1}$ is not a spike. Hence, $i_{j-1} \geq 2^{\psi(i_{j-1})} + 2^{\psi(i_{j-1})} - 1$. These together imply that

$$2i_j - i_{j-1} < -2^{\psi(i_{j-1})} < 0$$

which is a contradiction.

**Case of $\psi(i_j) = \psi(i_{j-1})$.** First, suppose $i_j$ is not a spike. Then,

$$i_j \geq 2^{\psi(i_j)} - 1 = 2^{\psi(i_{j-1})} - 1$$

Moreover, since by Lemma A.9 $i_{j-1}$ is not a spike, then $i_{j-1} < 2^{\psi(i_{j-1})} - 1$. These together imply that $2i_j - i_{j-1} > 2^{\psi(i_j)}$ which contradicts the hypothesis of $I$ satisfying conditions of Theorem 2.2. Hence, $i_j$ has to be a spike and $j = r$ by Lemma A.9, that is $i_r = 2^{\psi(i_r)} - 1$. In this case, as $i_{j-1}$ is not a spike, we have

$$i_{j-1} \leq 2^{\psi(i_{j-1})} - 1 - 2^{\psi(i_{j-1})}.$$ These together imply that

$$2i_r - i_{r-1} \geq 2^{\psi(i_r)} + 2^{\psi(i_{r-1})} - 1 > 2^{\psi(i_r)}$$

which contradicts conditions of Theorem 2.2. Note that the possibility of $2^{\psi(i_{r-1})} - 1 = 0$, hence $\phi(i_{r-1}) = 0$ or equivalently $i_{r-1}$ being even, is eliminated by Lemma A.2(ii) as $I$ satisfying conditions of Theorem 2.2 consists only of odd entries.

Finally, note that it is impossible to have $\psi(i_j) > \psi(i_{j-1})$ as this would imply that $i_j > i_{j-1}$ which contradicts Lemma A.2(i). Consequently, we are left with $\psi(i_j) = \psi(i_{j-1}) - 1$ as the only choice. This completes the proof. \(\square\)

**A.1. Case of $l(I) = 1, 2$.**

**Lemma A.12.** Suppose $I = (i_1, ..., i_s)$ is an admissible sequence.

(i) For $l(I) = 1$, $I$ satisfies conditions of Theorem 2.2 if and only if $I = (2^t - 1)$ for some $t > 0$.

(ii) If $l(I) = 2$ then $I$ does not satisfy conditions of Theorem 2.2.
Proof. (i) It is clear from binary expansion.
(ii) By Lemma A.7, for any non-spike positive integer $i$, we have $i = 2^{\phi(i)} - 1 + N_i 2^{\phi(i)}$ for some positive integer $N_i \neq 1$ (if $i$ is a spike then the expression still is valid with $N_i = 0$). If $I = (i_1, i_2)$ is a sequence satisfying conditions of Theorem 2.2 then by Lemma A.5 $\phi(i_1) \leq \phi(i_2)$. The conditions

$$0 \leq 2i_2 - i_1 < 2^{\phi(i_2)}, \quad i_1 - i_2 < 2^{\phi(i_1)}$$

imply that

$$i_2 = 2^{\phi(i_2)} - 1 + N_{i_2} 2^{\phi(i_2)} < 2^{\phi(i_2) + 1} \Rightarrow N_{i_2} \leq 1.$$ 

If $i_2$ is not a spike then this is a contradiction by Lemma A.7. The only remaining case is that $i_2$ is a spike which corresponds to the case $N_{i_2} = 0$ and $i_2 = 2^{\phi(i_2)} - 1$. Note that $i_1 = 2^{\phi(i_1)} - 1 + N_{i_1} 2^{\phi(i_1)}$. Using the conditions of the theorem simultaneously, yields

$$i_1 - i_2 < 2^{\phi(i_2)} \Rightarrow N_{i_1} < 2^{\phi(i_2)} - \phi(i_1),$$

$$2i_2 - i_1 < 2^{\phi(i_2)} \Rightarrow N_{i_1} > 2^{\phi(i_2)} - \phi(i_1) - 1.$$ 

This gives the desired contradiction. \hfill \Box

A.2. Case of $I(I) = 3$.

Lemma A.13. If $I = (i_1, i_2, i_3)$ is an admissible sequence which satisfies conditions of Theorem 2.2 then $i_3 = 2^n - 1$ for some positive integer $n$.

Proof. Suppose $I = (i_1, i_2, i_3)$ is a sequence which satisfies conditions of Theorem 2.2. Note that by Lemma A.2(ii) all entries of $I$ are odd, consequently $\phi_j > 0$ for all $j$. Since all entries of $I$ are odd, we may sharpen the inequalities of Theorem 2.2 and write

\begin{align*}
(1) \quad i_1 - (i_2 + i_3) &< 2^{\phi_1}, \\
(2) \quad 2i_2 - i_1 &< 2^{\phi_2}, \\
(3) \quad 2i_3 - i_2 &< 2^{\phi_3}
\end{align*}

as the left side of either inequality is meant to be odd. Here, we write $\phi_j = \phi(i_j)$ for brevity. By adding the three inequalities above, we have

$$i_3 < 2^{\phi_1} + 2^{\phi_2} + 2^{\phi_3}.$$ 

By Lemma A.8(ii) we have

$$2^{\psi_3 - 1} + 2^{\phi_3} - 1 \leq i_3.$$ 

These together imply that

$$2^{\psi_3 - 1} + 2^{\phi_3} - 1 \leq i_3 < 2^{\phi_1} + 2^{\phi_2} + 2^{\phi_3} \Rightarrow 2^{\psi_3 - 1} - 1 < 2^{\phi_1} + 2^{\phi_2} \Rightarrow 2^{\phi_3 + 1} \leq 2^{\phi_1} + 2^{\phi_2}.$$ 

Suppose $i_3$ is not a spike. By Lemma A.6(ii) $\phi_3 < \psi_3 - 1$ or equivalently $\phi_3 + 1 \leq \psi_3 - 1$. Consequently,

$$2^{\phi_3 + 1} \leq 2^{\phi_1} + 2^{\phi_2}.$$ 

Note that by Lemma A.5 $\phi_1 \leq \phi_2 \leq \phi_3$ which implies that $2^{\phi_3 + 1} \geq 2^{\phi_1} + 2^{\phi_2}$. Consequently,

$$2^{\phi_3 + 1} = 2^{\phi_1} + 2^{\phi_2}.$$
which is possible if and only if $\phi_1 = \phi_2 = \phi_3$. Since $i_1, i_2, i_3$ are not spike then applying Lemma A.7 we have

$$\text{ex}(I) = i_1 - (i_2 + i_3) = (N_1 - N_2 - N_3)2^{\phi_1} - 2^{\phi_1} + 1.$$  

Since $N_1, N_2, N_3$ are all even then either $N_1 - N_2 - N_3 \leq 0$ or $N_1 - N_2 - N_3 \geq 2$. If $N_1 - N_2 - N_3 \geq 2$ then

$$\text{ex}(I) \geq 2^{\phi_1 + 1} - 2^{\phi_1} - 1 = 2^{\phi_1} + 1 > 2^{\phi_1}$$

which contradicts (1), hence contradicting $I$ satisfying conditions of Theorem 2.2. If $N_1 - N_2 - N_3 \leq 0$ then bearing in mind that $\phi_1 > 0$ we see that $\text{ex}(I) < 0$ which contradicts the hypothesis. We then conclude that $i_3$ is a spike. $\square$

Next, we compute $i_1$ and $i_2$.

**Lemma A.14.** Suppose $I = (i_1, i_2, i_3)$ is an admissible sequence satisfying conditions of Theorem 2.2. Then, for some $m \leq n - 1$

$$I = (2^{n+1} + 2^m - 1, 2^n + 2^m - 1, 2^n - 1).$$

**Proof.** We shall write $\psi_j = \psi(i_j)$ and $\phi_j = \phi(i_j)$. By Lemma A.13 $i_3 = 2^n - 1$ for some $n > 0$. It follows that $\phi_3 = \psi_3 = n$. By Lemma A.11 it follows that $\phi_2 = n + 1$ and $\psi_1 = n + 2$. Moreover, since $i_3$ is a spike then by Lemmata A.10 and A.5 we have $\phi_1 \leq \phi_2 < \phi_3 = n$. By Lemma A.9 both $i_1$ and $i_2$ are non-spike. Applying Lemma A.6(ii) we see that

$$i_1 = 2^{n+1} + B_1 + 2^{\phi_1} - 1, \quad i_2 = 2^n + B_2 + 2^{\phi_2} - 1$$

where $B_1 = \sum_{k=1}^{n} \alpha_{k1}2^{k1}$ and $B_2 = \sum_{k=\phi_1}^{n-1} \alpha_{k2}2^{k2}$. We claim that $B_1 = B_2 = 0$. We proceed as follows.

First, suppose $B_2 > 0$. The conditions $0 < 2i_2 - i_1 < 2^{\phi_2}$ and $0 < \text{ex}(I) < 2^{\phi_1}$ imply that

$$2^{n+1} + 2B_2 + 2^{\phi_2} - 1 - (2^{n+1} + B_1 + 2^{\phi_1} - 1) < 2^{\phi_2}$$

$$2B_2 - B_1 + 2^{\phi_2} - 1 < 2^{\phi_1},$$

$$0 < 2^{n+1} + B_1 + 2^{\phi_1} - 1 - (2^n + B_2 + 2^{\phi_2} - 1 + 2^n - 1) < 2^{\phi_1}.$$ 

By adding the resulting inequalities we have $B_2 - 2 < 2^{\phi_1} < 2^{\phi_2}$, so from $B_2$ being even we deduce that $B_2 \leq 2^{\phi_2}$. But, this is a contradiction as if $B_2 > 0$ then from its expression $B_2 > 2^{\phi_2}$. Hence, $B_2 = 0$ and consequently $i_2 = 2^n + 2^{\phi_2} - 1$. Therefore, for some $m \leq n - 1$ we have $\phi_2 = m$ and

$$I = (2^{n+1} + B_1 + 2^{\phi_1} - 1, 2^n + 2^m - 1, 2^n - 1)$$

where $\phi_1 \leq m$. Next, we turn to $B_1$.

First, suppose $B_1 = 0$. In this case if $\phi_1 < m$ then $\text{ex}(I) < 0$ which is a contradiction. Hence, in the case of $B_1 = 0$ we have $\phi_1 = m$ and

$$I = (2^{n+1} + 2^m - 1, 2^n + 2^m - 1, 2^n - 1).$$
We show that the assumption that $B_1 > 0$ leads to a contradiction. The condition $0 < 2i_2 - i_1 < 2^{\phi_1} = 2^m$ implies that

$$2^m \leq B_1 + 2^{\phi_1} < 2^{m+1}.$$ Consider the binary expansion of $B_1$ as $B_1 = \sum_{k_1=\phi_1+1}^{\psi_1-2} \alpha_{k_1} 2^{k_1}$ bearing in mind that $\psi_1 - 2 = n$. The above inequalities show that

$$\alpha_{k_1} = 0 \text{ for all } k_1 \geq m + 1.$$ Hence, $B_1 = \sum_{k_1=\phi_1+1}^{m} \alpha_{k_1} 2^{k_1}$. If $\phi_1 < m$ then the condition $B_1 + 2^{\phi_1} \geq 2^m$ reads as $\sum_{k_1=\phi_1+1}^{m} \alpha_{k_1} 2^{k_1} + 2^{\phi_1} \geq 2^m$ which is impossible if $\alpha_m = 0$. Hence, $\alpha_m = 1$. Therefore,

$$i_1 = 2^n + 2^m + \sum_{k_1=\phi_1+1}^{m-1} \alpha_{k_1} 2^{k_1} + 2^{\phi_1} - 1.$$ The condition $\text{ex}(I) < 2^{\phi_1}$ reads as

$$\sum_{k_1=\phi_1+1}^{m-1} \alpha_{k_1} 2^{k_1} + 2^{\phi_1} + 1 < 2^{\phi_1}$$ which is an obvious false inequality. Hence, $B_1 = 0$. This completes the proof. \(\square\)

As an example where the boundary value $m = n - 1$ could be attained, consider $I = (19, 11, 7)$ where $11 = 2^3 + 2^{3-1} - 1$. The process of eliminating $B_1$ in the above proof is more of an intuitive nature and would be immediate if one has the experience and passion for working with binary expansions.

References


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