Flow equivalence of topological Markov shifts and Ruelle algebras

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Abstract. In this paper we study discrete flow equivalence of two-sided topological Markov shifts by using extended Ruelle algebra. We characterize flow equivalence of two-sided topological Markov shifts in terms of conjugacy of certain actions weighted by ceiling functions of two-dimensional torus on the stabilized extended Ruelle algebras for the Markov shifts.

Contents

1. Introduction 1375
2. Preliminaries 1379
3. Bilateral dimension groups 1382
4. Dimension quadruplets and AF-algebras 1386
5. Gauge actions with potentials 1394
6. Flow equivalence 1400
7. Flow equivalence and topological conjugacy 1408
References 1412

1. Introduction

Flow equivalence relation in two-sided topological Markov shifts is one of the most interesting and important equivalence relations in symbolic dynamics as seen in many papers [2], [3], [9], [22], etc. Let \((\tilde{X}_A, \tilde{\sigma}_A)\) be the two-sided topological Markov shift defined by an \(N \times N\) irreducible matrix \(A = [A(i, j)]_{i,j=1}^N\) with entries in \(\{0, 1\}\). The shift space \(\tilde{X}_A\) consists of bi-infinite sequences \((x_n)_{n \in \mathbb{Z}} \in \{1, \ldots, N\}^\mathbb{Z}\) of \(\{1, \ldots, N\}\) such that \(A(x_n, x_{n+1}) = 1\) for all \(n \in \mathbb{Z}\). Take and fix a real number \(\lambda_o\) such as \(0 < \lambda_o < 1\). The space \(\tilde{X}_A\) is a...
compact metric space by the metric defined by for \( x = (x_n)_{n \in \mathbb{Z}}, y = (y_n)_{n \in \mathbb{Z}} \) with \( x \neq y \)

\[
d(x, y) = \begin{cases} 
1 & \text{if } x_0 \neq y_0, \\
(l \alpha)^m & \text{if } m = \max \{ n | x_k = y_k \text{ for all } k \text{ with } |k| < n \}.
\end{cases}
\]

The homeomorphism of the shift transformation \( \sigma_A \) on \( X_A \) is defined by

\[
\sigma_A((x_n)_{n \in \mathbb{Z}}) = (x_{n+1})_{n \in \mathbb{Z}}.
\]

Two topological Markov shifts \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) are said to be flow equivalent if they are realized as cross sections with their first return maps of a common flow space. Parry–Sullivan in [22] proved that \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) are flow equivalent if and only if they are realized as discrete cross sections with their first return maps of a common topological Markov shift. Cuntz–Krieger have first found that there is an interesting relation between flow equivalence of topological Markov shifts and certain purely infinite simple \( C^* \)-algebras called Cuntz–Krieger algebras that they introduced in [7]. For an irreducible matrix \( A \) with entries in \( \{0, 1\} \), let \( \mathcal{O}_A \) be the Cuntz–Krieger algebra and \( D_A \) its canonical maximal abelian \( C^* \)-subalgebra of \( \mathcal{O}_A \). We denote by \( \mathcal{K} \) and \( \mathcal{C} \) the \( C^* \)-algebra of compact operators on the separable infinite dimensional Hilbert space \( l^2(\mathbb{N}) \) and its commutative \( C^* \)-subalgebra of diagonal operators on \( l^2(\mathbb{N}) \), respectively. Cuntz–Krieger proved that for irreducible non-permutation matrices \( A \) and \( B \), if \( (X_A, \sigma_A) \) and \( (X_B, \sigma_B) \) are flow equivalent, then there exists an isomorphism \( \Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K} \) of \( C^* \)-algebras such that \( \Phi(D_A \otimes \mathcal{C}) = D_B \otimes \mathcal{C} \). Its converse implication holds by [21] (for more general matrices a similar assertion is shown in [5]). They also proved in [7] that the extension group \( \text{Ext}(\mathcal{O}_A) \), which is isomorphic to the \( K \)-group \( K_0(\mathcal{O}_A) \) as groups, appears as the Bowen–Franks group \( BF(A) \) defined by Bowen–Franks in [2], that is an invariant of flow equivalence of \( (X_A, \sigma_A) \) ([2]).

There is another kind of construction of \( C^* \)-algebras from two-sided topological Markov shifts by using groupoids and regarding the Markov shifts as Smale spaces ([1], [30], [33], etc.). The construction was initiated by D. Ruelle [30], [31] and I. Putnam [23], [24]. I. Putnam in [23] constructed several kinds of groupoids from each Smale space. Each of the groupoids yields a \( C^* \)-algebra. In this paper, we focus on asymptotic groupoids \( G_A^a \) among several groupoids studied in [12], [23], [24], [25], etc. and their semi-direct products defined below. The asymptotic étale groupoid \( G_A^a \) for \( (X_A, \sigma_A) \) is defined by

\[
G_A^a := \{(x, y) \in X_A \times X_A | \lim_{n \to -\infty} d(\sigma_A^n(x), \sigma_A^n(y)) = \lim_{n \to -\infty} d(\sigma_A^n(x), \sigma_A^n(y)) = 0 \}
\]

with natural groupoid operations and topology (see [23]). It has been shown in [24] that the groupoid \( G_A^a \) is amenable and its \( C^* \)-algebra \( C^*(G_A^a) \) is stably isomorphic to the tensor product \( \mathcal{F}_{A'} \otimes \mathcal{F}_A \) of the canonical AF-subalgebras \( \mathcal{F}_{A'} \) and \( \mathcal{F}_A \) inside the Cuntz–Krieger algebras \( \mathcal{O}_{A'} \) and \( \mathcal{O}_A \), respectively. The
The semi-direct product $G_A^t \rtimes \mathbb{Z}$ is defined by
\[ G_A^t \rtimes \mathbb{Z} := \{(x, k - l, y) \in \tilde{X}_A \times \mathbb{Z} \times \tilde{X}_A \mid (\sigma_A^k(x), \sigma_A^l(x)) \in G_A^t \} \]
with natural groupoid operations and topology (see [23]). It is étale and amenable. The groupoid $C^*$-algebra $C^*(G_A^t \rtimes \mathbb{Z})$ is called the Ruelle algebra for the Markov shift $(\tilde{X}_A, \tilde{\sigma}_A)$ and written $\mathcal{R}_A$. Since the unit space $(G_A^t \rtimes \mathbb{Z})^0$ is \{(x, 0, x) \in G_A^t \rtimes \mathbb{Z} \mid x \in \tilde{X}_A\} that is identified with the shift space $\tilde{X}_A$, the algebra $\mathcal{R}_A$ has the commutative $C^*$-algebra $C(\tilde{X}_A)$ of continuous functions on $\tilde{X}_A$ as a maximal commutative $C^*$-subalgebra. It is the crossed product $C^*(G_A^t \rtimes \mathbb{Z})$ induced by the automorphism of the shift $\tilde{\sigma}_A$, and hence has the dual action written $\rho^t_A$, $t \in \mathbb{T}$. See [26] for the construction of $C^*$-algebras from groupoids.

Following [23], let us consider the groupoids $G_A^s$ and $G_A^u$ defined by stable equivalence relation and unstable equivalence relation on $(\tilde{X}_A, \tilde{\sigma}_A)$, respectively, which are defined by
\[ G_A^s := \{(x, y) \in \tilde{X}_A \times \tilde{X}_A \mid \lim_{n \to \infty} d(\tilde{\sigma}_A^n(x), \tilde{\sigma}_A^n(y)) = 0\}, \]
\[ G_A^u := \{(x, y) \in \tilde{X}_A \times \tilde{X}_A \mid \lim_{n \to -\infty} d(\tilde{\sigma}_A^n(x), \tilde{\sigma}_A^n(y)) = 0\}. \]

In [19] and [20], the author introduced the groupoid $G_A^{s,u} \rtimes \mathbb{Z}^2$ defined by
\[ G_A^{s,u} \rtimes \mathbb{Z}^2 := \{(x, p, q, y) \in \tilde{X}_A \times \mathbb{Z} \times \mathbb{Z} \times \tilde{X}_A \mid (\tilde{\sigma}_A^p(x), y) \in G_A^s, (\tilde{\sigma}_A^q(x), y) \in G_A^u\} \]
which has a natural groupoid operations and topology making it étale and amenable. The groupoid $C^*$-algebra $C^*(G_A^{s,u} \rtimes \mathbb{Z}^2)$ is called the extended Ruelle algebra written $\widehat{\mathcal{R}}_A$. Since the unit space $(G_A^{s,u} \rtimes \mathbb{Z}^2)^0$ is \{(x, 0, 0, x) \in G_A^{s,u} \rtimes \mathbb{Z}^2 \mid x \in \tilde{X}_A\} that is identified with the shift space $\tilde{X}_A$, the algebra $\widehat{\mathcal{R}}_A$ has $C(\tilde{X}_A)$ as a maximal abelian $C^*$-subalgebra. As in [19] and [20], there exists a projection $E_A$ in the tensor product $\mathcal{O}_A$ that is naturally isomorphic to the algebra $\widehat{\mathcal{R}}_A$, so that the $C^*$-algebra $\widehat{\mathcal{R}}_A$ is regarded as a version of the bilateral Cuntz–Krieger algebra. Let $\alpha_A$ denote the gauge action on the Cuntz–Krieger algebra $\mathcal{O}_A$. Under the identification between $E_A(\mathcal{O}_A \otimes \mathcal{O}_A)E_A$ and $\widehat{\mathcal{R}}_A$, the tensor product $\alpha_A^{(1)} \otimes \alpha_A^{(1)}$ for $(r, s) \in \mathbb{T}^2$ yields an action of $\mathbb{T}^2$ written $\gamma_A((r, s), (r, s)) \in \mathbb{T}^2$. In [20, Theorem 1.1], it was shown that the triplet $\widehat{(\mathcal{R}_A, C(\tilde{X}_A), \rho^t_A)}$ is a complete invariant for the topological conjugacy class of $(\tilde{X}_A, \tilde{\sigma}_A)$. For a continuous function $f : \tilde{X}_A \to \mathbb{N}$, we may define an action $\gamma_A^f$ weighted by $f$ on $\mathcal{R}_A$. In this paper, we will characterize the flow equivalence class of $(\tilde{X}_A, \tilde{\sigma}_A)$ in terms of the stabilized version of $\widehat{\mathcal{R}}_A$ with the weighted action $\gamma_A^{f,t}$. The continuous function $f : \tilde{X}_A \to \mathbb{N}$ exactly corresponds to a ceiling function of a discrete suspension. The main result of this paper is the following theorem.

**Theorem 1.1** (Theorem 6.8). Let $A, B$ be irreducible, non-permutation matrices with entries in $\{0, 1\}$. The two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and
\((X_B, \sigma_B)\) are flow equivalent if and only if there is an irreducible non-permutation matrix \(C\) with entries in \([0, 1]\) and continuous functions \(f_A, f_B : X_C \to \mathbb{N}\) with values in the positive integers such that \(\mathcal{R}_A \otimes \mathcal{K}\) and \(\mathcal{R}_B \otimes \mathcal{K}\) are isomorphic to \(\mathcal{R}_C \otimes \mathcal{K}\) via isomorphisms \(\Phi_A\) and \(\Phi_B\) satisfying

\[
\Phi_A \circ (y^{A, f}_A \otimes \text{id}) = (y^{C, f_A}_C \otimes \text{id}) \circ \Phi_A,
\]

\[
\Phi_B \circ (y^{B, f}_B \otimes \text{id}) = (y^{C, f_B}_C \otimes \text{id}) \circ \Phi_B,
\]

for \((r, s) \in \mathbb{T}^2\).

The above statement exactly corresponds to the situation that the topological Markov shift \((X_A, \sigma_A)\) is realized as a discrete suspension of \((X_C, \sigma_C)\) by ceiling function \(f_A\), and \((X_B, \sigma_B)\) is realized as a discrete suspension of \((X_C, \sigma_C)\) by ceiling function \(f_B\). As a corollary we have the following.

**Corollary 1.2** (Corollary 6.9). Let \(A, B\) be irreducible, non-permutation matrices with entries in \([0, 1]\). Two-sided topological Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are flow equivalent if and only if there exist continuous functions \(f_A : X_A \to \mathbb{N}\) and \(f_B : X_B \to \mathbb{N}\) with values in the positive integers, and an isomorphism \(\Phi : \mathcal{R}_A \otimes \mathcal{K} \to \mathcal{R}_B \otimes \mathcal{K}\) of \(C^*\)-algebras such that

\[
\Phi \circ (y^{A, f_A}_A \otimes \text{id}) = (y^{B, f_B}_B \otimes \text{id}) \circ \Phi, \quad (r, s) \in \mathbb{T}^2.
\]

The organization of the paper is the following.

In Section 2, we will briefly recall basic notation and terminology on groupoid \(C^*\)-algebras, Cuntz-Krieger algebras, Ruelle algebras and flow equivalence of topological Markov shifts.

In Section 3, a bilateral version of the Krieger’s dimension group for topological Markov shifts will be studied and called the dimension quadruplet that will be shown to be invariant for shift equivalence of the underlying matrices.

In Section 4, the dimension quadruplet is described by the \(K\)-group of the AF-algebra \(C^*(G^a_A)\) of the groupoid \(G^a_A\). As a result, a sufficient condition under which the two-sided Markov shifts \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are flow equivalent is given in terms of the stabilized action \(y^A \otimes \text{id}\) of \(\mathbb{T}^2\) on \(\mathcal{R}_A \otimes \mathcal{K}\) (Proposition 4.6).

In Section 5, the action \(y^{A, f}\) with potential function \(f\) on the algebra \(\mathcal{R}_A\) is introduced.

In Section 6, we characterize the flow equivalence of two-sided topological Markov shifts in terms of the actions with potential functions of two-dimensional torus on the extended Ruelle algebras \(\mathcal{R}_A\).

In Section 7, we reformulate Theorem 1.1 and Corollary 1.2 by describing their statements including not only flow equivalence but also topological conjugacy of two-sided topological Markov shifts (Theorem 7.2 and Theorem 7.3).

Throughout the paper, we denote by \(\mathbb{Z}_+\) and \(\mathbb{N}\) the set of nonnegative integers and the set of positive integers, respectively.
2. Preliminaries

In this section, we briefly recall basic notation and terminology on the C*-algebras of étale groupoids, Cuntz-Krieger algebras, Ruelle algebras and flow equivalence of topological Markov shifts. In what follows, a square matrix $A = [A(i, j)]_{i,j=1}^N$ is assumed to be an $N \times N$ irreducible, non-permutation matrix with entries in $\{0, 1\}$.

2.1. C*-algebras of étale groupoids. Let us construct C*-algebras from étale groupoids. The general theory of the construction of groupoid C*-algebras was initiated and studied by Renault [26] (see also [27], [28]). The construction will be used in the following sections. Let $G$ be an étale groupoid with its unit space $G^\circ$ and range map, source map $r, s : G \to G^\circ$ and $C_c(G)$ denote the *-algebra of continuous functions on $G$ with compact support having its product and *-involution defined by

$$(f * g)(\gamma) = \sum_{\eta; r(\eta) = s(\gamma)} f(\eta)g(\eta^{-1}\gamma), \quad f^*(\gamma) = \overline{f(\gamma^{-1})}$$

for $f, g \in C_c(G)$, $\gamma \in G$. We denote by $C_0(G^\circ)$ the commutative C*-algebra of continuous functions on $G^\circ$ vanishing at infinity. The algebra $C_c(G)$ has a structure of right $C_0(G^\circ)$-module with $C_0(G^\circ)$-valued right inner product given by

$$(\xi g)(\gamma) = \xi(\gamma)g(s(\gamma)), \quad <\xi, \zeta>(\gamma) = \sum_{\eta; r(\eta) = s(\gamma)} \overline{\xi(\eta)}\zeta(\eta)$$

for $\xi, \zeta \in C_c(G)$, $g \in C_c(G^\circ)$, $\gamma \in G$, $t \in G^\circ$. The completion of $C_c(G)$ by the norm defined by the above inner product is denoted by $\ell^2(G)$, which is a Hilbert C*-right module over $C_0(G^\circ)$. The algebra $C_c(G)$ is represented on $\ell^2(G)$ as bounded adjointable $C_0(G^\circ)$-right module maps by $\pi(f)\xi = f * \xi$ for $f \in C_c(G), \xi \in \ell^2(G)$. The closure of $\pi(C_c(G))$ by the operator norm on $\ell^2(G)$ is denoted by $C^*_r(G)$ and called the (reduced) groupoid C*-algebra for the étale groupoid $G$. The completion of $C_c(G)$ by the universal C*-norm is called the (full) groupoid C*-algebra for $G$. Now we treat the three kinds of groupoids $G_A^a$, $G_A^{a} \rtimes \mathbb{Z}, G_A^{a} \rtimes \mathbb{Z}^2$. They are all étale and amenable, so that the two groupoid C*-algebras $C^*_r(G)$ and $C^*(G)$ are canonically isomorphic for such groupoids. We do not distinguish them, and write them as $C^*(G)$ for $G = G_A^a, G_A^{a} \rtimes \mathbb{Z}, G_A^{a} \rtimes \mathbb{Z}^2$.

2.2. Cuntz-Krieger algebras, Ruelle algebras and extended Ruelle algebras. The Cuntz-Krieger algebra $O_A$ introduced by Cuntz-Krieger [7] is a universal unique C*-algebra generated by partial isometries $S_1, ..., S_N$ subject to the relations:

$$\sum_{j=1}^N S_j S_j^* = 1, \quad S_i^* S_i = \sum_{j=1}^N A(i, j) S_j S_j^*, \quad i = 1, ..., N. \quad (2.1)$$
By the universality for the relations (2.1) of operators, the correspondence \( S_i \mapsto \exp(2\pi\sqrt{-1}t)S_i, \ i = 1, \ldots, N \) for each \( t \in \mathbb{R}/\mathbb{Z} = \mathbb{T} \) yields an automorphism written \( \alpha^A_t \) on the C*-algebra \( O_A \). The automorphisms \( \alpha^A_t, t \in \mathbb{T} \) define an action of \( \mathbb{T} \) on \( O_A \) called the gauge action. It is well-known that the fixed point algebra \( (O_A)^{\alpha^A} \) of \( O_A \) under the gauge action is an AF-algebra written \( F_A \). Let us denote by \( B_m(\tilde{X}_A) \) the set of admissible words in \( \tilde{X}_A \) of length \( m \) and by \( B_*(\tilde{X}_A) \) the set of all admissible words of \( \tilde{X}_A \). For \( \mu = (\mu_1, \ldots, \mu_m) \in B_m(\tilde{X}_A) \), we write \( S_\mu = S_{\mu_1} \cdots S_{\mu_m} \). We denote by \( D_A \) the C*-subalgebra of \( F_A \) generated by projections \( S_\mu S_\mu^* \), \( \mu \in B_*(\tilde{X}_A) \).

As in [7] and [6] (cf. [18], [29]), the extended Ruelle algebra \( O_A \rtimes \alpha^A \mathbb{T} \) is stably isomorphic to the AF-algebra \( F_A \). Hence the dual action \( \Delta^A \) on \( O_A \rtimes \alpha^A \mathbb{T} \) induces an automorphism on \( K_0(F_A) \), that is written \( \delta_A \). The triplet \( (K_0(F_A), K_0^+(F_A), \delta_A) \) appears as the (future) dimension triplet written \( (\Delta_A, \Delta_A^+, \delta_A) \) for \( A \) defined by W. Krieger [16]. For the transposed matrix \( A^t \) of \( A \), we similarly consider the Cuntz–Krieger algebra \( O_{A^t} \) and its AF-subalgebra \( F_{A^t} \). Let us denote by \( T_1, \ldots, T_N \) the generating partial isometries of \( O_{A^t} \) which satisfy the relations:

\[
\sum_{i=1}^N T_i^* T_i = 1, \quad T_j^* T_j = \sum_{i=1}^N A(i, j)T_i^* T_i, \quad j = 1, \ldots, N.
\]

For \( \xi = (\xi_1, \ldots, \xi_k) \in B_k(\tilde{X}_{A^t}) \), we denote by \( \tilde{T} \) the transposed word \((\xi_k, \ldots, \xi_1)\) which belongs to \( B_k(\tilde{X}_A) \), and write \( T_{\tilde{T}} = T_{\tilde{T}_k} \cdots T_{\tilde{T}_1} \).

Define the projection \( E_A \in F_{A^t} \otimes F_A \) by setting

\[
E_A = \sum_{j=1}^N T_j^* T_j \otimes S_j S_j^*
\]

which coincides with \( \sum_{i=1}^N T_i^* T_i \otimes S_i S_i^* \) because of the equalities (2.1) and (2.2).

Let \( G_A^0, G_A^1 \rtimes \mathbb{Z} \) denote the étale amenable groupoids stated in Section 1. For reference, we state [20, Proposition 2.1] as

**Lemma 2.1.**

(i) The groupoid C*-algebra \( C^*(G_A^0) \) is canonically isomorphic to the C*-subalgebra of \( F_{A^t} \otimes F_A \) generated by elements \( \sum_{j=1}^N T_j^* T_j \otimes S_j S_j^* \) where \( \mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_*(\tilde{X}_A), \xi = (\xi_k, \ldots, \xi_1), \eta = (\eta_1, \ldots, \eta_l) \in B_*(\tilde{X}_A) \) satisfying \( A(\xi_k, \mu_1) = A(\eta_1, \nu_1) = 1 \) and \( k = l, m = n \). Hence \( C^*(G_A^0) \) is canonically isomorphic to the C*-algebra \( E_A(F_{A^t} \otimes F_A)E_A \).

(ii) The Ruelle algebra \( R_A = C^*(G_A^1 \rtimes \mathbb{Z}) \) is canonically isomorphic to the C*-subalgebra of \( O_{A^t} \otimes O_A \) generated by elements \( \sum_{j=1}^N T_j^* T_j \otimes S_j S_j^* \) where \( \mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_*(\tilde{X}_A), \xi = (\xi_k, \ldots, \xi_1), \eta = (\eta_1, \ldots, \eta_l) \in B_*(\tilde{X}_A) \) satisfying \( A(\xi_k, \mu_1) = A(\eta_1, \nu_1) = 1 \) and \( m + k = n + l \).

(iii) The extended Ruelle algebra \( \tilde{R}_A = C^*(G_A^1 \rtimes \mathbb{Z}^2) \) is canonically isomorphic to the C*-subalgebra of \( O_{A^t} \otimes O_A \) generated by elements \( \sum_{j=1}^N T_j^* T_j \otimes S_j S_j^* \) where \( \mu = (\mu_1, \ldots, \mu_m), \nu = (\nu_1, \ldots, \nu_n) \in B_*(\tilde{X}_A), \xi = (\xi_k, \ldots, \xi_1), \eta = (\eta_1, \ldots, \eta_l) \in B_*(\tilde{X}_A) \) satisfying \( A(\xi_k, \mu_1) = A(\eta_1, \nu_1) = 1 \) and \( m + k = n + l \).
Lemma 2.2.

(i) The restriction of the action $\rho^A_t, t \in \mathbb{T}$ to the subalgebra $\mathcal{R}_A$ is regarded as the dual action on $\mathcal{R}_A$ under a natural identification between $\mathcal{R}_A$ and the crossed product $C^*(G_A^0) \rtimes \mathbb{Z}$. Hence the fixed point algebra $(\mathcal{R}_A)^{\mathcal{R}_A}$ is isomorphic to $C^*(G_A^0)$.

(ii) The fixed point algebra $(\tilde{\mathcal{R}}_A)^{\delta^A}$ of $\tilde{\mathcal{R}}_A$ under $\delta^A$ is isomorphic to $\mathcal{R}_A$, so that the fixed point algebra $(\tilde{\mathcal{R}}_A)^{\gamma^A}$ of $\tilde{\mathcal{R}}_A$ under $\gamma^A$ is isomorphic to $C^*(G_A^0)$.

2.3. Suspension and flow equivalence. We will briefly review discrete suspensions of topological Markov shifts. Let $f : \tilde{X}_A \rightarrow \mathbb{N}$ be a continuous function on the shift space $\tilde{X}_A$ with values in the positive integers. Let $f(\tilde{X}_A) = \{1, 2, \ldots, L\}$. Put $X_j = \{x \in \tilde{X}_A \mid f(x) = j\}, j = 1, \ldots, L$. Define the suspension space $\tilde{X}_{A,f} = \bigcup_{j=1}^L X_j \times \{0, 1, \ldots, j - 1\}$ with transformation $\tilde{\sigma}_{A,f}$ on $\tilde{X}_{A,f}$ by

$$
\tilde{\sigma}_{A,f}([x, k]) = \begin{cases} 
[x, k + 1] & \text{if } 0 \leq k \leq j - 2, \\
[\tilde{\sigma}_{A}(x), 0] & \text{if } k = j - 1
\end{cases}
$$

for $[x, k] \in X_j \times \{0, 1, \ldots, j - 1\}$. The resulting topological dynamical system $(\tilde{X}_{A,f}, \tilde{\sigma}_{A,f})$ is called the discrete suspension of $(\tilde{X}_A, \tilde{\sigma}_A)$ by the ceiling function $f$, which is homeomorphic to a topological Markov shift. If, in particular, the function $f : \tilde{X}_A \rightarrow \mathbb{N}$ depends only on the 0th coordinate of $\tilde{X}_A$, then $f$ is written $f = \sum_{j=1}^N f_j \chi_{U_j(0)}$ for some integers $f_j \in \mathbb{N}$, where $\chi_{U_j(0)}$ is the characteristic function of the cylinder set

$$
U_j(0) = \{(x_n)_{n \in \mathbb{N}} \in \tilde{X}_A \mid x_0 = j\}, \quad j = 1, \ldots, N.
$$

Put $m_j = f_j - 1$ for $j = 1, \ldots, N$. Let $\mathcal{G} = (V, E)$ be the directed graph defined by the matrix $A$ with the vertex set $V = \{1, 2, \ldots, N\}$. An edge of $\mathcal{G}$ is defined by a pair $(i, j)$ of vertices $i, j = 1, \ldots, N$ such that $A(i, j) = 1$, whose source is $i$ and the terminal is $j$. The set of such pairs $(i, j)$ is the edge set $E$. Construct a new graph $\mathcal{G}_f = (V_f, E_f)$ with its transition matrix $A_f$ from the graph $\mathcal{G} = (V, E)$ and the function $f$ such that $V_f = \bigcup_{j=1}^N \{j_0, j_1, j_2, \ldots, j_{m_j}\}$ and if $A(j, k) = 1$, then

$$
A_f(j_0, j_1) = A_f(j_1, j_2) = \cdots = A_f(j_{m_j-1}, j_{m_j}) = A_f(j_{m_j}, k_0) = 1. \quad (2.3)
$$

For other pairs $(j_i, j_{i'}) \in V_f \times V_f$, we define $A_f(j_i, j_{i'}) = 0$. Hence the size of the matrix $A_f$ is $(f_1 + f_2 + \cdots + f_N) \times (f_1 + f_2 + \cdots + f_N)$. Then the discrete
suspension \((\mathcal{X}_{A,f}, \sigma_{A,f})\) is nothing but the topological Markov shift \((\mathcal{X}_{A,f}, \sigma_{A,f})\) defined by the matrix \(A_f\).

Two topological Markov shifts are said to be flow equivalent if they are realized as cross sections with their first return maps of a common one-dimensional flow space. Parry–Sullivan in [22] proved that \((\mathcal{X}_A, \sigma_A)\) and \((\mathcal{X}_B, \sigma_B)\) are flow equivalent if and only if there exist another topological Markov shift \((\mathcal{X}_C, \sigma_C)\) for some matrix \(C\) and continuous maps \(f_A, f_B : \mathcal{X}_C \to \mathbb{N}\) such that \((\mathcal{X}_A, \sigma_A)\) is topologically conjugate to the discrete suspension \((\mathcal{X}_{C,f_A}, \sigma_{C,f_A})\) and \((\mathcal{X}_B, \sigma_B)\) is topologically conjugate to the discrete suspension \((\mathcal{X}_{C,f_B}, \sigma_{C,f_B})\).

Cuntz and Krieger were the first to find interesting relations between flow equivalence of topological Markov shifts and Cuntz–Krieger algebras in [7]. Recall that \(\mathcal{K}\) and \(\mathcal{C}^\ast\) are the \(C^\ast\)-algebra of compact operators on the separable infinite dimensional Hilbert space \(\ell^2(\mathbb{N})\) and its commutative \(C^\ast\)-subalgebra of diagonal operators on \(\ell^2(\mathbb{N})\). Cuntz and Krieger proved that for irreducible non-permutation matrices \(A\) and \(B\), if \((\mathcal{X}_A, \sigma_A)\) and \((\mathcal{X}_B, \sigma_B)\) are flow equivalent, then there exists an isomorphism \(\Phi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}\) of \(C^\ast\)-algebras such that \(\Phi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}\). Its converse implication holds by [21] (for more general matrices a similar assertion is shown in [5]).

In this paper, we will study flow equivalence of topological Markov shifts in terms of the extended Ruelle algebras with its action \(p^A\) of \(\mathbb{T}^2\).

3. Bilateral dimension groups

We keep an irreducible, non-permutation matrix \(A = [A(i, j)]_{i,j=1}^N\) with entries in \(\{0, 1\}\). Following W. Krieger [16] (cf. [14], [15], [8], etc.), the dimension group \((\Delta_A, \Delta_A^+)\) for the matrix \(A\) are defined as an ordered group by the inductive limits

\[
\Delta_A = Z^N \xrightarrow{A^I} Z^N \xrightarrow{A^I} \cdots, \quad \Delta_A^+ = Z_+^N \xrightarrow{A^I} Z_+^N \xrightarrow{A^I} \cdots.
\]

The group \(\Delta_A\) is identified with the equivalence classes of \(\cup_{n=0}^\infty \{(v, n) \mid v \in Z^N, n \in Z_+\}\) by the equivalence relation generated by \((v, n) \sim (A^I v, n + 1)\). The equivalence class of \((v, n)\) is denoted by \([v, n]\). The dimension drop automorphism \(\delta_A\) on \((\Delta_A, \Delta_A^+)\) is defined by \(\delta_A([v, n]) = [(v, n + 1)]\) for \([v, n] \in \Delta_A\).

The triplet \((\Delta_A, \Delta_A^+, \delta_A)\) is called the (future) dimension triplet for the topological Markov shift \((\mathcal{X}_A, \sigma_A)\). We similarly have the (future) dimension triplet \((\Delta_{A^I}, \Delta_{A^I}^+, \delta_{A^I})\) for the topological Markov shift \((\mathcal{X}_{A^I}, \sigma_{A^I})\) for the matrix \(A^I\), which is called the (past) dimension triplet for \((\mathcal{X}_A, \sigma_A)\). Hence we have two dimension triplets \((\Delta_A, \Delta_A^+, \delta_A)\) and \((\Delta_{A^I}, \Delta_{A^I}^+, \delta_{A^I})\) for the matrix \(A\).

Let \(e_i \in Z^N\) be the vector of \(Z^N\) whose \(i\)th component is 1, other components are zeros. We will define a specific element \(\tilde{u}_A\) in \(\Delta_{A^I} \otimes \Delta_A\) by setting

\[
\tilde{u}_A := \sum_{i,j=1}^N [e_j, 1] \otimes A(j, i)[e_i, 1] \in \Delta_{A^I} \otimes \Delta_A.
\]
We then see that
\[
\bar{u}_A = \sum_{j=1}^{N} \left( [e_j, 1] \otimes \left( \sum_{i=1}^{N} A(j, i)e_i, 1 \right) \right) = \sum_{j=1}^{N} [e_j, 1] \otimes [A^t e_j, 1] 
\]
\[
= (\text{id} \otimes \delta_A^{-1}) \sum_{j=1}^{N} ([e_j, 1] \otimes [e_j, 1]) 
\]
and
\[
\bar{u}_A = \sum_{i=1}^{N} \left( \left( \sum_{j=1}^{N} A(j, i)e_j, 1 \right) \otimes [e_i, 1] \right) = \sum_{i=1}^{N} [A e_i, 1] \otimes [e_i, 1] 
\]
\[
= (\delta_A^{-1} \otimes \text{id}) \sum_{i=1}^{N} ([e_i, 1] \otimes [e_i, 1]). 
\]

Define an automorphism \( \delta_A : \Delta_A^+ \otimes \Delta_A \rightarrow \Delta_A^+ \otimes \Delta_A \) by \( \delta_A = \delta_A^{-1} \otimes \delta_A \). It satisfies
\[
\delta_A([u, n] \otimes [v, m]) = [Au, n] \otimes [v, m + 1], \quad [u, n] \otimes [v, m] \in \Delta_A^+ \otimes \Delta_A.
\]
We set the abelian group \( \tilde{\Delta}_A = \Delta_A^+ \otimes \Delta_A \) with its positive cone \( \tilde{\Delta}_A^+ = \Delta_A^+ \otimes \Delta_A^+ \).

**Definition 3.1.** The quadruplet \( (\tilde{\Delta}_A, \tilde{\Delta}_A^+, \delta_A, \bar{u}_A) \) is called the *dimension quadruplet* for the two-sided topological Markov shift \((X_A, \sigma_A)\).

We note that a bilateral version of the dimension groups first appeared in Krieger’s paper [14] (cf. [15], [16]).

**Lemma 3.2.** \( \tilde{\delta}_A(\bar{u}_A) = \bar{u}_A \).

**Proof.** Since
\[
\bar{u}_A = (\text{id} \otimes \delta_A^{-1}) \sum_{j=1}^{N} ([e_j, 1] \otimes [e_j, 1]) = (\delta_A^{-1} \otimes \text{id}) \sum_{i=1}^{N} ([e_i, 1] \otimes [e_i, 1]),
\]
and \( \tilde{\delta}_A = \delta_A^{-1} \otimes \delta_A \), the assertion is immediate. \( \square \)

We will next show that the dimension quadruplet \( (\tilde{\Delta}_A, \tilde{\Delta}_A^+, \delta_A, \bar{u}_A) \) is invariant under shift equivalence of the underlying matrices \( A \). The notion of shift equivalence in square matrices with entries in nonnegative integers has been introduced by W. F. Williams [34]. Two matrices \( A \) and \( B \) are said to be shift equivalent if there exist rectangular matrices \( H, K \) with entries in nonnegative integers and a positive integer \( \ell \) such that
\[
A^\ell = HK, \quad B^\ell = KH, \quad AH = HB, \quad KA = BK. \quad (3.1)
\]
W. Krieger has proved in [16] that two matrices \( A \) and \( B \) are shift equivalent if and only if their dimension triplet \( (\Delta_A, \Delta_A^+, \delta_A) \) and \( (\Delta_B, \Delta_B^+, \delta_B) \) are isomorphic. The following result has been already proved by C. G. Holton [10, Proposition...
6.7] for primitive matrices by using Rohlin property of automorphisms on the AF-algebras $C^*(G_n^2)$. The proof given below does not use $C^*$-algebra theory, nor does it assume that the matrices are primitive.

**Proposition 3.3** (C. G. Holton [10, Proposition 6.7]). Suppose that $A$ and $B$ are shift equivalent. Then there exists an isomorphism $\Phi : \hat{\Delta}_A \to \hat{\Delta}_B$ which yields an isomorphism between the dimension quadruplets $(\hat{\Delta}_A, \hat{\Delta}_A^+, \delta_A, \tilde{u}_A)$ and $(\hat{\Delta}_B, \hat{\Delta}_B^+, \delta_B, \tilde{u}_B)$.

**Proof.** Let $A$ and $B$ be $N \times N$ matrix and $M \times M$ matrix, respectively. Assume that there exist rectangular matrices $H, K$ with entries in nonnegative integers and a positive integer $\ell$ satisfying (3.1). Define

$$
\Phi_+ : \Delta_A \to \Delta_B \quad \text{by} \quad \Phi_+([v, k]) = [H^t v, k],
$$

$$
\Phi_- : \Delta_{A^t} \to \Delta_{B^t} \quad \text{by} \quad \Phi_-([v, k]) = [K v, k + \ell],
$$

so that

$$
\Phi_+^{-1} : \Delta_B \to \Delta_A \quad \text{satisfies} \quad \Phi_+^{-1}([u, j]) = [K^t u, j + \ell],
$$

$$
\Phi_-^{-1} : \Delta_{B^t} \to \Delta_{A^t} \quad \text{satisfies} \quad \Phi_-^{-1}([u, j]) = [H u, j].
$$

As in [16], $\Phi_+ : \Delta_A \to \Delta_B$ and $\Phi_- : \Delta_{A^t} \to \Delta_{B^t}$ yield isomorphisms for each such that

$$
\Phi_+(\Delta_A^+) = \Delta_B^+,
\quad \Phi_+ \circ \delta_A = \delta_B \circ \Phi_+,
$$

$$
\Phi_-(\Delta_{A^t}^+) = \Delta_{B^t}^+,
\quad \Phi_- \circ \delta_{A^t} = \delta_{B^t} \circ \Phi_-.
$$

Hence they induce isomorphisms

$$
\Phi_+ : (\Delta_A, \Delta_A^+, \delta_A) \to (\Delta_B, \Delta_B^+, \delta_B),
$$

$$
\Phi_- : (\Delta_{A^t}, \Delta_{A^t}^+, \delta_{A^t}) \to (\Delta_{B^t}, \Delta_{B^t}^+, \delta_{B^t}).
$$
We define $\Phi = \Phi_+ \otimes \Phi_-$, $\Delta_A \longrightarrow \Delta_B$. Let $f_1 \in \mathbb{Z}^M$ be the vector whose $l$th component is 1, and other components are zeros. It then follows that

$$
\Phi(\tilde{u}_A) = \sum_{i,j=1}^N \Phi_+(\{e_j, 1\}) \otimes \Phi_+(A(j, l)[e_i, 1])
$$

$$
= \sum_{i,j=1}^N [Ke_j, 1 + \ell] \otimes A(j, i)[H^l e_i, 1]
$$

$$
= \sum_{j=1}^N [Ke_j, 1 + \ell] \otimes [(AH)^l e_j, 1]
$$

$$
= \sum_{j=1}^N [Ke_j, 1 + \ell] \otimes [\sum_{l=1}^M (AH)(j, l)f_1, 1]
$$

$$
= \sum_{l=1}^M \sum_{j=1}^N \begin{bmatrix}
K(1, j)(AH)(j, l) \\
K(2, j)(AH)(j, l) \\
\vdots \\
K(M, j)(AH)(j, l)
\end{bmatrix}, 1 + \ell \otimes [f_l, 1]
$$

$$
= \sum_{l=1}^M (KAH)f_1, 1 + \ell \otimes [f_l, 1]
$$

$$
= \sum_{l=1}^M (BKH)f_1, 1 + \ell \otimes [f_l, 1]
$$

$$
= \sum_{l=1}^M B^{\ell+1}f_1, 1 + \ell \otimes [f_l, 1]
$$

$$
= \sum_{l=1}^M [Bf_1, 1] \otimes [f_l, 1] = \tilde{u}_B.
$$

R. F. Williams characterized topological conjugacy of two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ in terms of an equivalence relation of its underlying matrices, called strong shift equivalence ([34]). Two square matrices $A$ and $B$ with entries in nonnegative integers are said to be elementary equivalent if there exist rectangular matrices $C, D$ with entries in nonnegative integers such that $A = CD, B = DC$. If two matrices are connected by a finite chain of elementary equivalences, they are said to be strong shift equivalent. Williams proved that two-sided topological Markov shift $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are topologically conjugate if and only if the matrices $A$ and $B$ are strong shift equivalent ([34]). Since shift equivalence is weaker than strong shift equivalence, by virtue of the Williams' result, we have
Proposition 3.4. The dimension quadruplet $(\tilde{\Delta}_A, \tilde{\Delta}_A^+, \tilde{\delta}_A, \tilde{u}_A)$ is invariant under topological conjugacy of the two-sided topological Markov shift $(\tilde{X}_A, \tilde{\sigma}_A)$.

4. Dimension quadruplets and AF-algebras

In this section, we will study the dimension quadruplet $(\tilde{\Delta}_A, \tilde{\Delta}_A^+, \tilde{\delta}_A, \tilde{u}_A)$ by using K-theory for $C^*$-algebras. D. B. Killough and I. F. Putnam in [13] have deeply studied ring and module structure of the AF-algebras $C^*(G^n_A)$ as well as $C^*(G_2^n_A)$ from a different viewpoint from ours below. Recall that $\mathcal{K}$ denotes the $C^*$-algebra of compact operators on the separable infinite dimensional Hilbert space $H = \ell^2(\mathbb{N})$.

Lemma 4.1. Let $A$ be an irreducible, non-permutation matrix with entries in \{0, 1\}.

(i) There exists a projection $p_0$ in the crossed product $\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2$ of $\tilde{\mathcal{R}}_A$ by $\gamma A$ such that $p_0(\mathcal{R}_A \rtimes_{\gamma A} \mathbb{T}^2)p_0$ is isomorphic to $C^*(G_2^n_A)$. Hence $\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2$ is stably isomorphic to the AF-algebra $C^*(G_2^n_A)$.

(ii) The inclusion $\iota_A : p_0(\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2)p_0 \hookrightarrow \tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2 \mathbb{T}^2$ induces an isomorphism

$$\iota_{A*} : K_0(C^*(G_2^n_A)) \longrightarrow K_0(\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2)$$

on K-theory where $C^*(G_2^n_A)$ is identified with $p_0(\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2)p_0$.

Proof. (i) The fixed point algebra $(\tilde{\mathcal{R}}_A)_{\gamma A}$ of $\tilde{\mathcal{R}}_A$ under $\gamma A$ coincides with the fixed point algebra $(E_A(O_A \otimes O_A)E_A)^{\gamma A \otimes \gamma A}$ which is nothing but $E_A(\mathcal{F}_A \otimes \mathcal{F}_A)E_A$. Hence $\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2$ is isomorphic to $C^*(G_2^n_A)$. Let $p_0$ be the projection in $L^1(\mathbb{T}^2, \tilde{\mathcal{R}}_A)$ defined by $p_0(r, s) = 1$ for all $(r, s) \in \mathbb{T}^2$. We know that $p_0$ is a full projection in $\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2$ and

$$p_0(\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2)p_0 = (\tilde{\mathcal{R}}_A)_{\gamma A} = C^*(G_2^n_A)$$

by [29] or a manner similar to [18]. This shows that the algebra $\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2$ is stably isomorphic to the AF-algebra $C^*(G_2^n_A)$ by [4].

(ii) By [4], there exists a partial isometry $\nu_A$ in the multiplier algebra $M(\mathcal{R}_A \rtimes_{\gamma A} \mathbb{T}^2 \otimes \mathcal{K})$ of $\mathcal{R}_A \rtimes_{\gamma A} \mathbb{T}^2 \otimes \mathcal{K}$ such that $\nu_A^* \nu_A = p_0$, $\nu_A \nu_A^* = 1$. Put $\psi_A = \text{Ad}(\nu_A) : p_0(\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2)p_0 \otimes \mathcal{K} \longrightarrow \tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2 \otimes \mathcal{K}$, which is an isomorphism of $C^*$-algebras. We then have for a projection $p_0f p_0 \otimes q \in \mathcal{R}_A \rtimes_{\gamma A} \mathbb{T}^2 \otimes \mathcal{K}$,

$$(\iota_A \otimes \text{id})_*([p_0f p_0 \otimes q]) = [p_0f p_0 \otimes q]
= [\nu_A^* \nu_A(p_0f p_0 \otimes q) \nu_A^* \nu_A]
= [\nu_A(p_0f p_0 \otimes q) \nu_A^*]
= \psi_A^*([p_0f p_0 \otimes q]).$$

Hence $\iota_{A*} = \psi_{A*} : K_0(C^*(G_2^n_A)) \longrightarrow K_0(\tilde{\mathcal{R}}_A \rtimes_{\gamma A} \mathbb{T}^2)$ is an isomorphism. \qed
Let us denote by $\hat{\rho}^A$ the dual action of the crossed product $\mathbb{R}_A \rtimes \mathbb{R}^2$. Under the identifications

\[ C^*(G^a_A) = (\mathbb{R}_A)^\vee = p_0(\mathbb{R}_A \rtimes \mathbb{R}^2)p_0, \]

we define an action $\beta$ of $\mathbb{Z}^2$ on $K_0(C^*(G^a_A))$ by

\[ \beta_{(m,n)} := t_{A^*}^{-1} \circ \phi^A_{(m,n)} \circ t_{A^*} : K_0(C^*(G^a_A)) \to K_0(C^*(G^a_A)), \quad (m, n) \in \mathbb{Z}^2 \]

such that the diagram

\[ \begin{array}{ccc} K_0(\mathbb{R}_A \rtimes \mathbb{R}^2) & \xrightarrow{\hat{\rho}^A} & K_0(\mathbb{R}_A \rtimes \mathbb{R}^2) \\ \downarrow t_{A^*} & & \downarrow t_{A^*} \\ K_0(C^*(G^a_A)) & \xrightarrow{\beta_{(m,n)}} & K_0(C^*(G^a_A)) \end{array} \]

is commutative.

Let $U_A = \sum_{i=1}^{N} T_i \otimes S_i$ in $\mathcal{O}_{A^*} \otimes \mathcal{O}_A$. As in [19], $U_A$ is a unitary in $\mathcal{R}_A$ and hence in $\mathbb{R}_A$, so that $U_A U_A^* = U_A^* U_A = E_A$. We denote by $1_{C^*(G^a_A)}$ the unit of the $C^*$-algebra $C^*(G^a_A)$. By [6] and [7] (see also [8], [15], [16]), the ordered group $\Delta_A$ is naturally identified with the K-group $K_0(\mathcal{F}_A)$.

**Lemma 4.2.** There exists an isomorphism $\varphi_A : C^*(G^a_A) \otimes \mathcal{K} \to \mathcal{F}_A \otimes \mathcal{F}_A \otimes \mathcal{K}$ of $C^*$-algebras such that the induced isomorphism

\[ \varphi_{A^*} : K_0(C^*(G^a_A)) \to K_0(\mathcal{F}_A) \otimes K_0(\mathcal{F}_A) \]

satisfies

\[ \varphi_{A^*}([1_{C^*(G^a_A)}]) = [E_A], \quad \varphi_{A^*} \circ \text{Ad}(U_A)_* = \hat{\delta}_A \circ \varphi_{A^*}, \]

\[ \varphi_{A^*} \circ \beta_{(m,n)} \circ \varphi_{A^*}^{-1} = \delta_{A^*}^m \otimes \delta_{A^*}^n, \quad (m, n) \in \mathbb{Z}^2. \]

Hence the diagrams

\[ \begin{array}{ccc} K_0(\mathbb{R}_A \rtimes \mathbb{R}^2) & \xrightarrow{\hat{\rho}^A} & K_0(\mathbb{R}_A \rtimes \mathbb{R}^2) \\ \downarrow t_{A^*} & & \downarrow t_{A^*} \\ K_0(C^*(G^a_A)) & \xrightarrow{\beta_{(m,n)}} & K_0(C^*(G^a_A)) \\ \downarrow \varphi_{A^*} & & \downarrow \varphi_{A^*} \\ K_0(\mathcal{F}_A) \otimes K_0(\mathcal{F}_A) & \xrightarrow{\delta_{A^*}^m \otimes \delta_{A^*}^n} & K_0(\mathcal{F}_A) \otimes K_0(\mathcal{F}_A) \end{array} \]

are commutative.

**Proof.** Since the $C^*$-algebra $\mathcal{F}_A \otimes \mathcal{F}_A$ is simple, the projection $E_A$ is full in $\mathcal{F}_A \otimes \mathcal{F}_A$. By using Brown’s theorem [4], there exists an isometry $u_A$ in the
multiplier algebra $M((\mathcal{F}_{A'} \otimes \mathcal{F}_{A}) \otimes \mathcal{K})$ of $(\mathcal{F}_{A'} \otimes \mathcal{F}_{A}) \otimes \mathcal{K}$ such that $u^*_A u_A = 1$, $u_A u^*_A = E_A \otimes 1_H$. Define an isomorphism

$$\varphi_A = \text{Ad}(u^*_A): C^*(G^\alpha_A) \otimes \mathcal{K} = E_A(\mathcal{F}_{A'} \otimes \mathcal{F}_{A})E_A \otimes \mathcal{K} \longrightarrow \mathcal{F}_{A'} \otimes \mathcal{F}_{A} \otimes \mathcal{K}.$$ 

Let $p_1$ be a rank one projection in $\mathcal{K}$. We then have

$$\varphi_{A^*}(1_{C^*(G^\alpha_A)}) = \varphi_{A^*}([E_A \otimes p_1]) = [u^*_A (E_A \otimes p_1) u_A] = [(E_A \otimes p_1) u_A ((E_A \otimes p_1) u_A]^* = [E_A \otimes p_1] = [E_A].$$

We will next see that $\varphi_A \circ \text{Ad}(U_A) = \delta_A \circ \varphi_A$. We note that $K_0(C^*(G^\alpha_A))$ is generated by the classes of projections of the form $T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'}$ where $\mu = (\mu_1, \ldots, \mu_m) \in B_k(\mathcal{X}_A)$, $\bar{\xi} = (\xi_k, \ldots, \xi_1) \in B_k(\mathcal{X}_{A'})$ with $A(\bar{\xi}_k, \mu_1) = 1$. We then have

$$(\varphi_A \circ \text{Ad}(U_A))(T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'}) = [\varphi_A(T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'})] = [u^*_A (T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'}) u_A] = [T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'}].$$

On the other hand,

$$\delta_A \circ \varphi_A((T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'})) = \delta_A \circ \varphi_A(T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'}) = \delta_A \circ \varphi_A(T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'}).$$

As in [18, Lemma 4.5], we know

$$\delta_A^{-1}(T^*_\xi T^*_\xi) = [T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'}],$$

and $\delta_A([S_{\mu'} S^*_{\mu'}]) = [S_{\xi_k} S^*_{\xi_k}]$. Hence we have

$$(\varphi_A \circ \text{Ad}(U_A))(T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'}) = \delta_A \circ \varphi_A((T^*_\xi T^*_\xi \otimes S_{\mu'} S^*_{\mu'})).$$

We note that the K-theoretic class $[E_A]$ of the projection $E_A$ has appeared in studying of K-theoretic duality by J. Kaminker–I. F. Putnam [11].

**Lemma 4.3.** Let $A = [A(i, j)]_{i, j=1}^N$ and $B = [B(i, j)]_{i, j=1}^M$ be irreducible, non-permutation matrices with entries in $\{0, 1\}$. Suppose that there exists an isomorphism $\Phi: \mathcal{R}_A \otimes \mathcal{K} \longrightarrow \mathcal{R}_B \otimes \mathcal{K}$ of $C^*$-algebras such that

$$\Phi(\gamma^r S_{(r, s)} \otimes \text{id}) = (\gamma^r S_{(r, s)} \otimes \text{id}) \circ \Phi, \quad (r, s) \in \mathbb{T}^2.$$ (4.1)
(i) Then \( \Phi \) induces an isomorphism
\[
\Phi_{0*} : K_0(\mathcal{F}_A) \otimes K_0(\mathcal{F}_A) \longrightarrow K_0(\mathcal{F}_B) \otimes K_0(\mathcal{F}_B)
\]
such that
\[
\Phi_{0*} \circ (\delta^m \otimes \delta^n) = (\delta^m \otimes \delta^n) \circ \Phi_{0*}, \quad (m, n) \in \mathbb{Z}^2.
\]
(ii) There exist an \( N^2 \times M^2 \)-matrix \( H \), an \( M^2 \times N^2 \)-matrix \( K \) with entries in nonnegative integers and a natural number \( \ell \) such that
\[
(A' \otimes A) \ell = HK, \quad (B' \otimes B) \ell = KH, \quad (4.2)
\]
\[(1 \otimes A)H = H(1 \otimes B), \quad K(1 \otimes A) = (1 \otimes B)K, \quad (4.3)
\]
\[(A' \otimes 1)H = H(B' \otimes 1), \quad K(A' \otimes 1) = (B' \otimes 1)K. \quad (4.4)
\]

**Proof.** (i) Since \( \Phi : \widetilde{\mathcal{R}}_A \otimes \mathcal{K} \longrightarrow \widetilde{\mathcal{R}}_B \otimes \mathcal{K} \) is an isomorphism of \( C^* \)-algebras satisfying (4.1), it induces an isomorphism
\[
\Phi_1 : (\widetilde{\mathcal{R}}_A \otimes \mathcal{K}) \rtimes_{\gamma_A} \mathbb{T}^2 \longrightarrow (\widetilde{\mathcal{R}}_B \otimes \mathcal{K}) \rtimes_{\gamma_B} \mathbb{T}^2
\]
of \( C^* \)-algebras of the crossed products. Let \( \gamma^A, \gamma^B \) be the dual actions on \( \widetilde{\mathcal{R}}_A \rtimes_{\gamma_A} \mathbb{T}^2, \widetilde{\mathcal{R}}_B \rtimes_{\gamma_B} \mathbb{T}^2 \), respectively. By identifying \( (\mathcal{R}_A \otimes \mathcal{K}) \rtimes_{\gamma_A} \mathbb{T}^2 \) with \( (\widetilde{\mathcal{R}}_A \rtimes_{\gamma_A} \mathbb{T}^2) \otimes \mathcal{K} \), and \( (\widetilde{\mathcal{R}}_B \otimes \mathcal{K}) \rtimes_{\gamma_B} \mathbb{T}^2 \) with \( (\widetilde{\mathcal{R}}_B \rtimes_{\gamma_B} \mathbb{T}^2) \otimes \mathcal{K} \), we see that
\[
\Phi_1 \circ (\gamma^A_{(m,n)} \otimes \text{id}) = (\gamma^B_{(m,n)} \otimes \text{id}) \circ \Phi_1, \quad (m, n) \in \mathbb{Z}^2.
\]
Hence we have an isomorphism
\[
\Phi_{1*} : K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma_A} \mathbb{T}^2) \longrightarrow K_0(\widetilde{\mathcal{R}}_B \rtimes_{\gamma_B} \mathbb{T}^2)
\]
such that
\[
\Phi_{1*} \circ \gamma^A_{(m,n)*} = \gamma^B_{(m,n)*} \circ \Phi_{1*}, \quad (m, n) \in \mathbb{Z}^2.
\]
We then define \( \Phi_{0*} : K_0(\mathcal{F}_A) \otimes K_0(\mathcal{F}_A) \longrightarrow K_0(\mathcal{F}_B) \otimes K_0(\mathcal{F}_B) \) by setting
\[
\Phi_{0*} = \varphi_{0*} \circ \iota_{A*} \circ \Phi_{1*} \circ \varphi^{-1}_{A*},
\]
where \( \iota_{A*} : K_0(C^*(G_A^2)) = K_0(p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma_A} \mathbb{T}^2)p_0) \longrightarrow K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma_A} \mathbb{T}^2) \) is the isomorphism defined in Lemma 4.1 (ii). Hence the following diagram is commutative:

\[
\begin{array}{ccc}
K_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma_A} \mathbb{T}^2) & \xrightarrow{\Phi_{1*}} & K_0(\widetilde{\mathcal{R}}_B \rtimes_{\gamma_B} \mathbb{T}^2) \\
| \quad \iota_{A*} \quad | & & | \quad \iota_{B*} \quad | \\
K_0(p_0(\widetilde{\mathcal{R}}_A \rtimes_{\gamma_A} \mathbb{T}^2)p_0) \quad & = & \quad K_0(p_0(\widetilde{\mathcal{R}}_B \rtimes_{\gamma_B} \mathbb{T}^2)p_0) \\
| \quad \varphi_{A*} \quad | & & | \quad \varphi_{B*} \quad | \\
K_0(C^*(G_A^2)) \quad & \longrightarrow & \quad K_0(C^*(G_A^2)) \\
| \quad \Phi_{1*} \quad | & & | \quad \Phi_{1*} \quad | \\
K_0(\mathcal{F}_A) \otimes K_0(\mathcal{F}_A) \quad & \longrightarrow & \quad K_0(\mathcal{F}_A) \otimes K_0(\mathcal{F}_A).
\end{array}
\]
We then have by Lemma 4.2

\[ \Phi_{0*} \circ (\delta^n_m \otimes \delta^n_A) = (\varphi_{B*} \circ \Phi_1 \circ \varphi_{A*} \circ \varphi_{-1}) \circ (\varphi_{A*} \circ \varphi_{A*}^{-1} \circ \varphi_{A*}^{-1}) \]

\[ = \varphi_{B*} \circ \Phi_1 \circ \varphi_{A*} \circ \varphi_{A*}^{-1} \circ \varphi_{A*}^{-1} \circ \varphi_{A*}^{-1} \]

\[ = \varphi_{B*} \circ \varphi_{A*}^{-1} \circ \varphi_{A*}^{-1} \circ \varphi_{A*}^{-1} \]

\[ = (\varphi_{B*} \circ \varphi_{A*}^{-1} \circ \varphi_{A*}^{-1} \circ \varphi_{A*}^{-1}) \circ \varphi_{A*} \circ \varphi_{A*}^{-1} \]

\[ = (\delta^n_m \otimes \delta^n_A) \circ \Phi_{0*}. \]

(ii) By (i) the isomorphism \( \Phi : \overline{R}_A \otimes \mathcal{K} \rightarrow \overline{R}_B \otimes \mathcal{K} \) satisfying (4.1) induces an isomorphism

\[ \Phi_{0*} : K_0(\mathcal{F}_{A'}) \otimes K_0(\mathcal{F}_A) \rightarrow K_0(\mathcal{F}_{B'}) \otimes K_0(\mathcal{F}_B) \]

of ordered groups such that \( \Phi_{0*} \circ (\delta^n_{A'} \otimes \delta^n_A) = (\delta^n_{B'} \otimes \delta^n_A) \circ \Phi_{0*}. \) Now

\[ K_0(\mathcal{F}_{A'}) = \lim \{ \mathbb{Z}^N \xrightarrow{A'} \mathbb{Z}^N \xrightarrow{A'} \cdots \}, \]

\[ K_0(\mathcal{F}_A) = \lim \{ \mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \xrightarrow{A} \cdots \} \]

and the dimension drop automorphisms \( \delta_A : K_0(\mathcal{F}_A) \rightarrow K_0(\mathcal{F}_A) \) and \( \delta_{A'} : K_0(\mathcal{F}_{A'}) \rightarrow K_0(\mathcal{F}_{A'}) \) are defined by \( \delta_A([x, n]) = [x, n + 1] (= [Ax, n]) \) for \( [x, n] \in K_0(\mathcal{F}_A) \) and \( \delta_{A'}([y, n]) = [y, n + 1] (= [A'y, n]) \) for \( [y, n] \in K_0(\mathcal{F}_{A'}) \), respectively ([6], [7]). Since \( K_0(\mathcal{F}_{A'}) \otimes K_0(\mathcal{F}_A) = K_0(\mathcal{F}_{A' \otimes A}) \) and \( \delta_{A'} \otimes \delta_A = \delta_{A' \otimes A} \), we have an isomorphism of dimension triplets

\[ (K_0(\mathcal{F}_{A' \otimes A}), \delta_{A' \otimes A}) \cong (K_0(\mathcal{F}_{B' \otimes B}), \delta_{B' \otimes B}) \]

with dimension drop automorphisms. Hence the two matrices \( A' \otimes A \) and \( B' \otimes B \) are shift equivalent by [16], which means that there exist an \( N^2 \times M^2 \)-matrix \( H \), an \( M^2 \times N^2 \)-matrix \( K \) with entries in nonnegative integers and a natural number \( \ell \) such that

\[ (A' \otimes A)^\ell = HK, \quad (B' \otimes B)^\ell = KH, \]

\[ (A' \otimes A)H = H(B' \otimes B), \quad KA(A' \otimes A) = (B' \otimes B)K. \]

Since

\[ K_0(\mathcal{F}_{A'}) \otimes K_0(\mathcal{F}_A) \]

\[ = \lim \{ \mathbb{Z}^N \xrightarrow{A} \mathbb{Z}^N \xrightarrow{A} \cdots \} \otimes \lim \{ \mathbb{Z}^N \xrightarrow{A'} \mathbb{Z}^N \xrightarrow{A'} \cdots \} \]

\[ \cong \lim \{ \mathbb{Z}^N \otimes \mathbb{Z}^{A \otimes A'} \xrightarrow{A \otimes A'} \mathbb{Z}^N \otimes \mathbb{Z}^{A \otimes A'} \cdots \} \]

and similarly

\[ K_0(\mathcal{F}_{B'}) \otimes K_0(\mathcal{F}_B) \cong \lim \{ \mathbb{Z}^M \otimes \mathbb{Z}^{B \otimes B'} \xrightarrow{B \otimes B'} \mathbb{Z}^M \otimes \mathbb{Z}^{B \otimes B'} \cdots \}, \]
Lemma 4.5. Let $A, B$ be irreducible, non-permutation matrices with entries in \{0, 1\}. Suppose that there exist an $N^2 \times M^2$-matrix $H$, an $M^2 \times N^2$-matrix $K$ with entries in nonnegative integers and a natural number $\ell$ satisfying (4.2), (4.3) and (4.4). Then

$$\text{Sp}_m^x(A^t \otimes A) = \text{Sp}_m^x(B^t \otimes B) \quad \text{and} \quad \text{Sp}^x(A) = \text{Sp}^x(B). \quad (4.7)$$

Proof. We note that $\text{Sp}_m^x(A) = \text{Sp}_m^x(A^t)$. By the equalities (4.3) and (4.4), we see that

$$(A^t \otimes A)H = H(B^t \otimes B), \quad K(A^t \otimes A) = (B^t \otimes B)K. \quad (4.8)$$

The equalities (4.2) together with (4.8) show us that the matrices $A^t \otimes A$ and $B^t \otimes B$ are shift equivalent, so that

$$\text{Sp}_m^x(A^t \otimes A) = \text{Sp}_m^x(B^t \otimes B) \quad (4.9)$$

by [17, Theorem 7.4.10]. For $\lambda \in \text{Sp}^x(A)$, one may take nonzero eigenvectors $u, v \in C^N$ such that $Au = \lambda v$ and $A^t u = \lambda u$. By (4.3), we have

$$(1 \otimes B)K(u \otimes v) = K(1 \otimes A)(u \otimes v) = K(u \otimes \lambda v) = \lambda K(u \otimes v).$$

By (4.2), we have

$$HK(u \otimes v) = (A^t \otimes A)^\ell (u \otimes v) = \lambda^\ell (u \otimes v)$$

so that the vector $K(u \otimes v)$ is a nonzero eigenvector of the matrix $1 \otimes B$ for the eigenvalue $\lambda$. Hence $\lambda \in \text{Sp}^x(1 \otimes B)$. Since $\text{Sp}^x(1 \otimes B) = \text{Sp}^x(B)$, we have $\lambda \in \text{Sp}^x(B)$, so that $\text{Sp}^x(A) \subset \text{Sp}^x(B)$. Similarly the inclusion relation $\text{Sp}^x(B) \subset \text{Sp}^x(A)$ holds and hence $\text{Sp}^x(A) = \text{Sp}^x(B)$. \qed

Lemma 4.5. Suppose that two irreducible, non-permutation matrices $A, B$ with entries in \{0, 1\} satisfy (4.7). Then we have

$$\text{Sp}_m^x(A) = \text{Sp}_m^x(B) \quad \text{and hence} \quad \det(1 - A) = \det(1 - B).$$
Proof. Since both $A, B$ are irreducible, they have its periods as irreducible matrices, which we denote by $p_A, p_B$, respectively. Since $Sp_m^X(A) = Sp_m^X(B)$, their Perron-Frobenius eigenvalues coincide. We denote the common eigenvalue by $\lambda_1$ which is positive. There are exactly $p_A$ eigenvalues $\lambda$ of $Sp_m^X(A)$ such that $|\lambda| = \lambda_1$, so that we have $p_A = p_B$ which we denote by $p$. Let $\omega$ be the $p$th root of unity. By Perron-Frobenius theorem for irreducible matrices, one may find distinct eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_L\} \subset Sp_m^X(A)=Sp_m^X(B)$ such that

$$\lambda_1 > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_L|$$

(4.10) and the set $\{\omega^k \lambda_i \mid k = 0, 1, \ldots, p - 1, i = 1, \ldots, L\}$ is the full list of $Sp_m^X(A) = Sp_m^X(B)$ (cf. [32, Section 1.4]). For each $i = 1, \ldots, L$, the $p$ eigenvalues

$$\omega^k \lambda_i, \quad k = 0, 1, \ldots, p - 1$$

have common multiplicities in $Sp_m^X(A)$ and in $Sp_m^X(B)$, respectively, which we denote by $m_i^A$ and $m_i^B$, respectively. Hence we know that $m_1^A = m_1^B = 1$. We put $\lambda_i(k) = \omega^k \lambda_i$ for $k = 0, 1, \ldots, p - 1, i = 1, \ldots, L$. Let $m_0^A, m_0^B$ be the multiplicities of zero eigenvalues of $A, B$, respectively. Then the characteristic polynomials of the matrices $A^t \otimes A, B^t \otimes B$ are written such that

$$\varphi_{A^t \otimes A}(t) = t^{(m_0^A)^2} \prod_{i,j=1}^{L} \prod_{k,l=0}^{p-1} (t - \lambda_i(k) \lambda_j(l))^{m_i^A m_j^A},$$

$$\varphi_{B^t \otimes B}(t) = t^{(m_0^B)^2} \prod_{i,j=1}^{L} \prod_{k,l=0}^{p-1} (t - \lambda_i(k) \lambda_j(l))^{m_i^B m_j^B}.$$ 

By the assumption $Sp_m^X(A^t \otimes A) = Sp_m^X(B^t \otimes B)$, we have

$$\prod_{i,j=1}^{L} \prod_{k,l=0}^{p-1} (t - \lambda_i(k) \lambda_j(l))^{m_i^A m_j^A} = \prod_{i,j=1}^{L} \prod_{k,l=0}^{p-1} (t - \lambda_i(k) \lambda_j(l))^{m_i^B m_j^B}. \tag{4.11}$$

The above polynomial of the left (resp. right) hand side is denoted by $\phi_A(t)$ (resp. $\phi_B(t)$). Suppose that

$$\lambda_1 \lambda_2 = \lambda_i(k) \lambda_j(l) \quad \text{for some} \ i, j = 1, \ldots, L \text{ and} \ k, l = 0, 1, \ldots, p - 1.$$ 

We may assume $i \leq j$. By the inequalities (4.10) with $|\lambda_i(k)| = |\lambda_i|, |\lambda_j(l)| = |\lambda_j|$, we have $i = 1$, so that $\lambda_i(k) = \omega^k \lambda_1$. Hence we have

$$\lambda_2 = \omega^k \lambda_j(l) = \omega^{k+l} \lambda_j$$

so that $j = 2$ and $k + l \equiv 0 \pmod{p}$. We put $a = \lambda_1 \lambda_2$. The power exponent of $(t - a)$ in the polynomial $\phi_A(t)$ is

$$(m_1^A m_2^A + m_2^A m_1^B) \times \{(k, l) \in \{0, 1, \ldots, p - 1\}^2 \mid k + l \equiv 0 \pmod{p})\} = 2m_2^A p.$$ 

Similarly the power exponent of $(t - a)$ in the polynomial $\phi_B(t)$ is $2m_2^B p$. Hence we have $m_2^A = m_2^B$. Next assume that there exists $2 \leq h \leq L$ such that

$$m_h^A = m_h^B \quad \text{for all} \ n \leq h. \tag{4.12}$$
Suppose that
$$\lambda_i \lambda_{i+1} = \lambda_i(k) \lambda_j(l) \quad \text{for some } i, j = 1, \ldots, L \text{ and } k, l = 0, 1, \ldots, p - 1.$$  
We may assume $i \leq j$. If $i = 1$, then $\lambda_i(k) = \omega^k \lambda_1$. Hence we have
$$\lambda_{h+1} = \omega^k \lambda_j(l) = \omega^{k+l} \lambda_j$$
so that $j = h + 1$ and $k + l \equiv 0 \pmod p$. If $i \neq 1$, we have $j < h + 1$ because of the inequalities (4.10). We put
$$p_1(1, h + 1) = \{(i, j) \in \{2, \ldots, L\}^2 \mid i < j, \lambda_1 \lambda_{h+1} = \lambda_i(k) \lambda_j(l) \quad \text{for some } k, l = 0, 1, \ldots, p - 1\},$$
$$p_0(1, h + 1) = \{i \in \{1, 2, \ldots, L\} \mid \lambda_i \lambda_{h+1} = \lambda_i(k)^2 \quad \text{for some } k = 0, 1, \ldots, p - 1\}.$$  
Both sets $p_1(1, h + 1)$ and $p_0(1, h + 1)$ are possibly empty. We note that $\lambda_i(k) \lambda_j(l) = \omega^{k+l} \lambda_i \lambda_j$ and $[(k, l) \in \{1, \ldots, p\}^2 \mid k + l \equiv 0 \pmod p] = p$. Put $b = \lambda_1 \lambda_{h+1}$.  
Hence the power exponent of $(t - b)$ in the polynomial $\phi_A(t)$ is
$$2m_1^A m_{h+1}^A p + 2 \sum_{(i, j) \in p_1(1, h + 1)} m_i^A m_j^A \cdot p + \epsilon_p \sum_{i \in p_0(1, h + 1)} m_i^A,$$
where $\epsilon_p = 2$ if $p$ is even, and $\epsilon_p = 1$ if $p$ is odd. Similarly the power exponent of $(t - b)$ in the polynomial $\phi_B(t)$ is
$$2m_1^B m_{h+1}^B p + 2 \sum_{(i, j) \in p_1(1, h + 1)} m_i^B m_j^B \cdot p + \epsilon_p \sum_{i \in p_0(1, h + 1)} m_i^B.$$  
Any pair $(i, j) \in p_1(1, h + 1)$ satisfies $i < j < h + 1$ and any element $i \in p_0(1, h + 1)$ satisfies $i < h + 1$. Hence the hypothesis (4.12) ensures that
$$m_{h+1}^A = m_{h+1}^B.$$  
Therefore we obtain that $\text{Sp}_m^X(A) = \text{Sp}_m^X(B)$. Since
$$\det(1 - A) = \prod_{i=1}^L \prod_{k=0}^{p-1} (1 - \lambda_i(k) m_i^A), \quad \det(1 - B) = \prod_{i=1}^L \prod_{k=0}^{p-1} (1 - \lambda_i(k) m_i^B),$$
the equality $\det(1 - A) = \det(1 - B)$ follows from $\text{Sp}_m^X(A) = \text{Sp}_m^X(B)$.  

W. Parry and D. Sullivan in [22] proved that the determinant $\det(1 - A)$ is invariant under flow equivalence of topological Markov shift $(\mathcal{X}_A, \sigma_A)$. There is another crucial invariant of flow equivalence called the Bowen–Franks group written $\text{BF}(A)$, which is defined by the abelian group $\mathbb{Z}^N/(1 - A)\mathbb{Z}^N$ for the $N \times N$ matrix $A$ with entries in $\{0, 1\}$ ([2]). J. Franks in [9] proved that the pair $\det(1 - A)$ and $\text{BF}(A)$ is a complete set of invariants of flow equivalence. We note that the group $\text{BF}(A)$ is isomorphic to the $K_0$-group $K_0(\mathcal{O}_A)$ of the Cuntz–Krieger algebra $\mathcal{O}_A$. We reach the following proposition.
**Proposition 4.6.** Assume that $A$ and $B$ are irreducible, non-permutation matrices with entries in $\{0,1\}$. Suppose that there exists an isomorphism $\Phi : \tilde{\mathcal{R}}_A \otimes \mathcal{K} \rightarrow \tilde{\mathcal{R}}_B \otimes \mathcal{K}$ such that

$$\Phi \circ (\gamma^A_{(r,s)} \otimes \text{id}) = (\gamma^B_{(r,s)} \otimes \text{id}) \circ \Phi, \quad (r, s) \in \mathbb{T}^2. \quad (4.13)$$

Then the two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow equivalent by Franks’s theorem [9].

**Proof.** Suppose that there exists an isomorphism $\Phi : \tilde{\mathcal{R}}_A \otimes \mathcal{K} \rightarrow \tilde{\mathcal{R}}_B \otimes \mathcal{K}$ satisfying (4.13). We then have $K_0(\tilde{\mathcal{R}}_A) = K_0(\tilde{\mathcal{R}}_B)$ so that $K_0(\mathcal{O}_A \otimes \mathcal{O}_B) \cong K_0(\mathcal{O}_B \otimes \mathcal{O}_B)$ and hence $K_0(\mathcal{O}_A) \cong K_0(\mathcal{O}_B)$ by Künneth formulas. This implies that $\text{BF}(A)$ is isomorphic to $\text{BF}(B)$. By Lemma 4.3, Lemma 4.4 and Lemma 4.5, we have $\det(1-A) = \det(1-B)$. Hence we conclude that $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow equivalent by Franks’s theorem [9].

We will use Proposition 4.6 to prove Theorem 6.7 in Section 6.

5. Gauge actions with potentials

In this section, we will define gauge actions $\gamma^A,f$ with potential function $f : \tilde{X}_A \rightarrow \mathbb{Z}$ on the $C^*$-algebra $\tilde{\mathcal{R}}_A$. For a continuous function $f \in C(\tilde{X}_A, \mathbb{Z})$ on $\tilde{X}_A$ and $n \in \mathbb{Z}$, we define a continuous function $f^n \in C(\tilde{X}_A, \mathbb{Z})$ by setting

$$f^n(x) = \begin{cases} \sum_{i=0}^{n-1} f(\tilde{\sigma}^i_A(x)) & \text{for } n \geq 1, \\ 0 & \text{for } n = 0, \\ -\sum_{i=-n}^{-1} f(\tilde{\sigma}^i_A(x)) & \text{for } n \leq -1. \end{cases}$$

It is easy to see that the identities

$$f^{n+m}(x) = f^n(x) + f^m(\tilde{\sigma}^n_A(x)), \quad n, m \in \mathbb{Z}, \ x \in \tilde{X}_A$$

hold. For $f \in C(\tilde{X}_A, \mathbb{Z})$ and $(x, p, q, y) \in G^{s,t}_A \times \mathbb{Z}^2$, define

$$\tilde{f}^+(x, p, q, y) = \lim_{n \rightarrow -\infty} \{f^{n+p}(\tilde{\sigma}^n_A(x)) - f^n(\tilde{\sigma}_A(y))\},$$

$$\tilde{f}^-(x, p, q, y) = \lim_{n \rightarrow -\infty} \{f^{n+q}(x) - f^n(y)\}.$$

**Lemma 5.1.** Both $\tilde{f}^+, \tilde{f}^- : G^{s,t}_A \times \mathbb{Z}^2 \rightarrow \mathbb{Z}$ are continuous groupoid homomorphisms from $G^{s,t}_A \times \mathbb{Z}^2$ to $\mathbb{Z}$.

**Proof.** Take an arbitrary point $(x, p, q, y) \in G^{s,t}_A \times \mathbb{Z}^2$ so that

$$\lim_{n \rightarrow -\infty} d(\tilde{\sigma}^{n+p}_A(x), \tilde{\sigma}^n_A(y)) = 0, \quad \lim_{n \rightarrow -\infty} d(\tilde{\sigma}^{n+q}_A(x), \tilde{\sigma}^n_A(y)) = 0. \quad (5.1)$$

By the first equality above, we may find $N_1 \in \mathbb{N}$ locally such that

$$f(\tilde{\sigma}^n_A(\tilde{\sigma}_A(x))) = f(\tilde{\sigma}^n_A(\tilde{\sigma}_A(y))) \quad \text{for all } n \geq N_1. \quad (5.2)$$
For $n \geq N_1$, we have

$$f^{n+p}(\hat{\sigma}_A(x)) - f^n(\hat{\sigma}_A(y)) = f^n(\hat{\sigma}_A(x)) - f^n(\hat{\sigma}_A(y)) = f^n(\hat{\sigma}_A(x)) + f(\hat{\sigma}_A^0(\hat{\sigma}_A(x))) + f(\hat{\sigma}_A^1(\hat{\sigma}_A(x))) + \cdots + f(\hat{\sigma}_A^{p+N_1-1}(\hat{\sigma}_A(x))) - f(\hat{\sigma}_A(y)) - f(\hat{\sigma}_A^2(y)) - \cdots - f(\hat{\sigma}_A^{N_1-1}(\hat{\sigma}_A(y))) = f^n(\hat{\sigma}_A(x)) + f^{N_1}(\hat{\sigma}_A(\hat{\sigma}_A(x))) - f^{N_1}(\hat{\sigma}_A(y))$$

so that

$$f^+(x, p, q, y) = f^{p+N_1}(\hat{\sigma}_A(x)) - f^{N_1}(\hat{\sigma}_A(y)). \quad (5.3)$$

By the second equality of (5.1), we may similarly find a negative integer $N_2 \in \mathbb{Z}$ such that

$$f^-(x, p, q, y) = f^{q+N_2}(x) - f^{N_2}(y). \quad (5.4)$$

Hence both the values $f^+(x, p, q, y)$, $f^-(x, p, q, y)$ are defined.

For $(x, p, q, y), (x', p', q', y') \in \mathcal{O}_A \times \mathbb{Z}^2$ with $x' = y$, we have

$$f^+(x, p, q, y)(x', p', q', y') = f^+(x, p + p', q + q', y') = \lim_{n \to \infty} \{f^{n+p+p'}(\hat{\sigma}_A(x)) - f^n(\hat{\sigma}_A(y'))\} = \lim_{n \to \infty} \{f^{n+p}(\hat{\sigma}_A(x)) + f^n(\hat{\sigma}_A^p(\hat{\sigma}_A(x))) - f^n(\hat{\sigma}_A(y'))\}. \quad (5.5)$$

On the other hand, we have

$$f^+(x, p, q, y) + f^+(x', p', q', y') = \lim_{n \to \infty} \{f^{n+p}(\hat{\sigma}_A(x)) - f^n(\hat{\sigma}_A(y))\} + \lim_{n \to \infty} \{f^{n+p'}(\hat{\sigma}_A(x')) - f^n(\hat{\sigma}_A(y'))\} = \lim_{n \to \infty} \{f^{n+p}(\hat{\sigma}_A(x)) + f^{n+p'}(\hat{\sigma}_A(y')) - f^n(\hat{\sigma}_A(y))\} = \lim_{n \to \infty} \{f^{n+p}(\hat{\sigma}_A(x)) + f^{p}(\hat{\sigma}_A(\hat{\sigma}_A(x))) - f^n(\hat{\sigma}_A(y'))\}.$$ 

By (5.2), the equality

$$\lim_{n \to \infty} f^{p'}(\hat{\sigma}_A^p(\hat{\sigma}_A(x))) = \lim_{n \to \infty} f^{p'}(\hat{\sigma}_A^p(\hat{\sigma}_A(y)))$$

holds, so that

$$f^+(x, p, q, y)(x', p', q', y') = f^+(x, p, q, y) + f^+(x', p', q', y')$$

and similarly

$$f^-(x, p, q, y)(x', p', q', y') = f^-(x, p, q, y) + f^-(x', p', q', y').$$

The identities

$$f^+(x, p, q, y)^{-1} = -f^+(x, p, q, y), \quad f^-(x, p, q, y)^{-1} = -f^-(x, p, q, y)$$
are easily seen. As the continuity of \( \bar{f}^+ \), \( \bar{f}^- \) follows from the formulas (5.3), (5.4) with the uniform continuity of \( f \), we know that they are continuous groupoid homomorphisms from \( G^{s,u}_A \rtimes \mathbb{Z}^2 \) to \( \mathbb{Z} \). □

Define a continuous groupoid homomorphism \( \bar{f} : G^{s,u}_A \rtimes \mathbb{Z}^2 \to \mathbb{Z} \) by \( \bar{f}(x, p, q, y) = \bar{f}^+(x, p, q, y) - \bar{f}^-(x, p, q, y) \). Recall that the \( C^* \)-algebra \( \tilde{\mathcal{R}}_A \) is represented on the Hilbert \( C^* \)-right module \( \ell^2(G^{s,u}_A \rtimes \mathbb{Z}^2 \otimes \mathbb{C}_0) \) over \( C_0((G^{s,u}_A \rtimes \mathbb{Z}^2) \overline{\rtimes} \mathbb{C}) \) as the reduced groupoid \( C^* \)-algebra. For \( f \in C(\tilde{\mathcal{X}}_A, \mathbb{Z}) \), \( (r, s) \in \mathbb{T}^2 \) and \( \xi \in \ell^2(G^{s,u}_A \rtimes \mathbb{Z}^2) \), we set

\[
[U_r(\bar{f}^+)\xi](x, p, q, y) = \exp\{2\pi\sqrt{-1}\bar{f}^+(x, p, q)\} s \xi(x, p, q, y),
\]
\[
[U_r(\bar{f}^-)\xi](x, p, q, y) = \exp\{2\pi\sqrt{-1}\bar{f}^-(x, p, q)\} r \xi(x, p, q, y),
\]
\[
U_{(r, s)}(\bar{f}) = U_r(\bar{f}^-)U_s(\bar{f}^+).
\]

Since \( \bar{f}^+, \bar{f}^- \) and \( \bar{f} \) are groupoid homomorphisms from \( G^{s,u}_A \rtimes \mathbb{Z}^2 \) to \( \mathbb{Z} \), the operators \( U(\bar{f}^+), U(\bar{f}^-) \) yield unitary representations of \( \mathbb{T} \) and \( U(\bar{f}) \) does a unitary representation of \( \mathbb{T}^2 \).

**Proposition 5.2.** For \( f \in C(\tilde{\mathcal{X}}_A, \mathbb{Z}) \), the correspondence

\[
a \in \tilde{\mathcal{R}}_A \mapsto \text{Ad}(U_{(r, s)}(\bar{f}))(a) = U_{(r, s)}(\bar{f}) a U_{(r, s)}(\bar{f}^*) \in \tilde{\mathcal{R}}_A
\]

defines an automorphism on \( \tilde{\mathcal{R}}_A \) such that \( (r, s) \in \mathbb{T}^2 \to \text{Ad}(U_{(r, s)}) \in \text{Aut}(\tilde{\mathcal{R}}_A) \) gives rise to an action of \( \mathbb{T}^2 \) on \( \tilde{\mathcal{R}}_A \) and its restriction to the subalgebra \( C(\tilde{\mathcal{X}}_A) \) is the identity. 

**Proof.** For \( a \in C_\mathbb{C}(G^{s,u}_A \rtimes \mathbb{Z}^2), \xi \in \ell^2(G^{s,u}_A \rtimes \mathbb{Z}^2), (x, p, q, y) \in G^{s,u}_A \rtimes \mathbb{Z}^2 \), we have

\[
[\text{Ad}(U_{(r, s)}(\bar{f}))(a)\xi](x, p, q, y)
\]
\[
= \exp\{2\pi\sqrt{-1}(\bar{f}^+(x, p, q)s + \bar{f}^-(x, p, q)r)\}[aU_{(r, s)}(\bar{f})^*\xi](x, p, q, y).
\]

Now the equalities

\[
[aU_{(r, s)}(\bar{f})^*\xi](x, p, q, y)
\]
\[
n = \sum_{y', y'(y) = x} a(y)[U_{(r, s)}(-\bar{f})\xi](y^{-1} \cdot (x, p, q, y))
\]
\[
= \sum_{y', y'(y) = x} a(y) \cdot \exp\{2\pi\sqrt{-1}(\bar{f}^+(y)s + \bar{f}^-(y)r)\}
\]
\[
\cdot \exp\{-2\pi\sqrt{-1}(\bar{f}^+(x, p, q)\cdot y s + \bar{f}^-(x, p, q)\cdot y r)\} \xi(y^{-1} \cdot (x, p, q, y))
\]

hold, so that

\[
[\text{Ad}(U_{(r, s)}(\bar{f}))(a)\xi](x, p, q, y)
\]
\[
= \sum_{y', y'(y) = x} a(y) \cdot \exp\{2\pi\sqrt{-1}(\bar{f}^+(y)s + \bar{f}^-(y)r)\} \xi(y^{-1} \cdot (x, p, q, y)).
\]
Let us identify $U_{(r,s)}(\tilde{f})$ with the continuous function on $G_{\mathcal{A}}^{s,u} \rtimes \mathbb{Z}^2$ defined by

$$U_{(r,s)}(\tilde{f})(y) = \exp[2\pi \sqrt{-1}(\tilde{f}^+(y)s + \tilde{f}^-(y)r)], \quad y \in G_{\mathcal{A}}^{s,u} \rtimes \mathbb{Z}^2.$$ 

Hence we have

$$[\text{Ad}(U_{(r,s)}(\tilde{f}))(a)\xi](x, p, q, y) = \sum_{y_0(y) = x} (U_{(r,s)}(\tilde{f}) \cdot a)(y)\xi(y^{-1} \cdot (x, p, q, y))$$

so that

$$\text{Ad}(U_{(r,s)}(\tilde{f}))(a) = U_{(r,s)}(\tilde{f}) \cdot a \quad \text{for } a \in C_c(G_{\mathcal{A}}^{s,u} \rtimes \mathbb{Z}^2) \quad (5.5)$$

where $U_{(r,s)}(\tilde{f}) \cdot a$ is the pointwise product between the two functions $U_{(r,s)}(\tilde{f})$ and $a$. Thus $\text{Ad}(U_{(r,s)}(\tilde{f}))(a)$ belongs to $C_c(G_{\mathcal{A}}^{s,u} \rtimes \mathbb{Z}^2)$, so that $\text{Ad}(U_{(r,s)}(\tilde{f}))$ yields an automorphism of the $C^*$-algebra $\mathcal{R}_{\mathcal{A}}$.

Especially for a continuous function $a \in C(\tilde{X}_{\mathcal{A}})$ on $\tilde{X}_{\mathcal{A}}$, it is regarded as an element of $C_c(G_{\mathcal{A}}^{s,u} \rtimes \mathbb{Z}^2)$ by

$$a(x, p, q, y) = \begin{cases} a(x) & \text{if } x = y, \; p = q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $x = y, \; p = q = 0$, we know that $\tilde{f}^+(x, p, q, y) = \tilde{f}^-(x, p, q, y) = 0$ so that

$$\text{Ad}(U_{(r,s)}(\tilde{f}))(a) = U_{(r,s)}(\tilde{f}) \cdot a = a \quad \text{for } a \in C_c(\tilde{X}_{\mathcal{A}}).$$

We denote by $\gamma_{(r,s)}^{A,f}$ the automorphism $\text{Ad}(U_{(r,s)}(\tilde{f}))$ on $\mathcal{R}_{\mathcal{A}}$, which yields an action called gauge action with potential function $f$, or weighted gauge action. For the constant function $f \equiv 1$, the equalities $\tilde{f}^+(x, p, q, y) = p$ and $\tilde{f}^-(x, p, q, y) = q$ hold so that the action $\gamma_{(r,s)}^{A,f}$ for $f \equiv 1$ coincides with the previously defined action $\gamma_{(r,s)}^{A}$.

Let $U_j(0), j = 1, 2, \ldots, N$ be the cylinder sets on $\tilde{X}_{\mathcal{A}}$ such that

$$U_j(0) = \{(x_n)_{n \in \mathbb{Z}} \in \tilde{X}_{\mathcal{A}} \mid x_0 = j\}.$$ 

Let $\chi_{U_j(0)}$ be the characteristic function of the cylinder set $U_j(0)$ on $\tilde{X}_{\mathcal{A}}$.

**Lemma 5.3.** Suppose that $f = \sum_{j=1}^N f_j \chi_{U_j(0)}$ for some integers $f_j \in \mathbb{Z}$. Then we have

$$\gamma_{(r,s)}^{A,f} = \alpha_r^{A,f} \otimes \alpha_s^{A,f}, \quad (r,s) \in \mathbb{T}^2 \quad \text{on } \mathcal{R}_{\mathcal{A}} = E_A(\mathcal{O}_{A'} \otimes \mathcal{O}_{A})E_A,$$

where $\alpha_r^{A,f} \in \text{Aut}(\mathcal{O}_{A'})$, $\alpha_s^{A,f} \in \text{Aut}(\mathcal{O}_{A})$ are defined by

$$\alpha_r^{A,f}(T_j) = \exp\left(2\pi \sqrt{-1}f_j r\right)T_j, \quad j = 1, 2, \ldots, N,$$

$$\alpha_s^{A,f}(S_j) = \exp\left(2\pi \sqrt{-1}f_j s\right)S_j, \quad j = 1, 2, \ldots, N.$$
Proof. For $m, n, k, l \in \mathbb{N}$ and
\begin{align*}
\mu = (\mu_1, \ldots, \mu_m), & \quad \nu = (\nu_1, \ldots, \nu_n) \in B_{s}(X_{A}), \\
\xi = (\xi_k, \ldots, \xi_1), & \quad \eta = (\eta_1, \ldots, \eta_l) \in B_{s}(X_{A}^{l})
\end{align*}
satisfying $A(\xi_k, \mu_1) = A(\eta_1, \nu_1) = 1$, we write
\begin{align*}
U_{\xi, \mu, \nu} = \{(x, m - n, l - k, y) \in G_{u, u}^s \times \mathbb{Z}^2 | \\
(\bar{\sigma}^m_{A}(x), \bar{\sigma}^n_{A}(y)) \in G_{u, u}^s, & \quad (\bar{\sigma}^k_{A}(x), \bar{\sigma}^l_{A}(y)) \in G_{u, u}^s, \\
x_{[1, m]} = \mu, & \quad y_{[1, n]} = \nu, \quad x_{[-k + 1, 0]} = \xi, \quad y_{[-l + 1, 0]} = \eta
\end{align*}
where
\begin{align*}
G_{u, u}^s & = \{(x, y) \in \tilde{X}_{A} \times \tilde{X}_{A} | x_i = y_i \text{ for all } i \in \mathbb{Z}_+\}, \\
G_{u, u}^s & = \{(x, y) \in \tilde{X}_{A} \times \tilde{X}_{A} | x_{-i} = y_{-i} \text{ for all } i \in \mathbb{Z}_+\}.
\end{align*}

As in [19, Section 9], the correspondence $\chi_{U_{\xi, \mu, \nu}} \leftrightarrow T_{\xi}T_{\eta}^{*} \otimes S_{\mu}S_{\nu}^{*}$ gives rise to an isomorphism between the groupoid $C^{*}$-algebra $C^{*}(G_{u, u}^s \times \mathbb{Z}^2)$ and the algebra $\tilde{A}$. For $(x, p, q, y) \in U_{\xi, \mu, \nu}$ with $p = m - n, q = l - k$, one may take $N_{1} = n$ so that
\begin{align*}
\tilde{f}(x, p, q, y) & = \tilde{f}(x, p, q, y) + \tilde{f}(\bar{\sigma}^{-1}_{A}(x)) + \tilde{f}(\bar{\sigma}^{-2}_{A}(x)) + \cdots + \tilde{f}(-\bar{\sigma}^{-N_{1}}_{A}(x)) \\
& - \tilde{f}(\bar{\sigma}^{0}_{A}(y)) - \tilde{f}(\bar{\sigma}^{1}_{A}(y)) - \cdots - f(-\bar{\sigma}^{N_{1}}_{A}(y)) \\
& = \tilde{f}(\bar{\sigma}^{m}_{A}(x)) + \tilde{f}(\bar{\sigma}^{n}_{A}(y)) + \cdots + f(-\bar{\sigma}^{-N_{1}}_{A}(y)) \\
& - \tilde{f}(\bar{\sigma}^{m}_{A}(x)) - \tilde{f}(\bar{\sigma}^{n}_{A}(y)) - \cdots - f(-\bar{\sigma}^{n}_{A}(y)) \\
& = (\mu_{1} + \mu_{2} + \cdots + \mu_{m}) - (\nu_{1} + \nu_{2} + \cdots + \nu_{n})
\end{align*}
because $\bar{\sigma}(x)_{[0, m-1]} = \mu, \bar{\sigma}(y)_{[0, n-1]} = \nu$, and similarly
\begin{align*}
\tilde{f}(x, p, q, y) & = \tilde{f}(x) + \tilde{f}(\bar{\sigma}^{-1}(x)) + \tilde{f}(\bar{\sigma}^{-2}(x)) + \cdots + f(-\bar{\sigma}^{-N_{1}}(x)) \\
& - \tilde{f}(\bar{\sigma}^{0}(y)) - \tilde{f}(\bar{\sigma}^{1}(y)) - \cdots - f(-\bar{\sigma}^{N_{1}}(y)) \\
& = (\xi_{1} + \xi_{2} + \cdots + \xi_{1}) - (\eta_{1} + \eta_{2} + \cdots + \eta_{l}),
\end{align*}
It then follows that by (5.5)
\begin{align*}
[\text{Ad}(U_{(r,s)}(\tilde{f}))(\chi_{U_{\xi, \mu, \nu}})](x, p, q, y) & = [U_{(r,s)}(\tilde{f}) \cdot \chi_{U_{\xi, \mu, \nu}}](x, p, q, y) \\
& = \exp\{2\pi i(\tilde{f}^{+}(x, p, q, y) + \tilde{f}^{-}(x, p, q, y)r)\} \chi_{U_{\xi, \mu, \nu}}(x, p, q, y)
\end{align*}
so that
\[
\text{Ad}(U_{(x,s)}(\tilde{f}))(\chi_{U_{(x,s),\nu}}) = \exp[2\pi \sqrt{-1}\sum_{\gamma_1} f_{\gamma_1} + \sum_{\gamma_2} f_{\gamma_2} + \cdots + \sum_{\gamma_m} f_{\gamma_m}] - (f_{\gamma_1} + f_{\gamma_2} + \cdots + f_{\gamma_m}) \cdot \chi_{U_{(x,s),\nu}},
\]

proving that the equality
\[
\gamma_{r,s}^A(x_{U_{(x,s),\nu}}) = \alpha_{r,s}^A f(T_x T_n^{-1}) \otimes \alpha_{r,s}^A (S_{\mu} S_{\nu}^*).
\]

\[
\Phi(C(X_A)) = C(X_B), \quad \Phi \gamma_{r,s}^A f = \gamma_{r,s}^B g \circ \Phi, \quad (r,s) \in \mathbb{T}^2.
\]

**Proof.** The topological conjugacy \( \varphi : X_A \to X_B \) induces an isomorphism \( \phi : G_A^{s,u} \times \mathbb{Z}^2 \to G_B^{s,u} \times \mathbb{Z}^2 \) of étale groupoids such that \( \phi(x, p, q, y) = (\varphi(x), p, q, \varphi(y)) \) for \((x, p, q, y) \in G_A^{s,u} \times \mathbb{Z}^2\). It gives rise to a unitary written \( V_\varphi : \ell^2(G_B^{s,u} \times \mathbb{Z}^2) \to \ell^2(G_A^{s,u} \times \mathbb{Z}^2) \) satisfying \( V_\varphi(\xi) = \xi \circ \phi \) for \( \xi \in \ell^2(G_B^{s,u} \times \mathbb{Z}^2) \). Assume that the C*-algebras \( \mathcal{R}_A \) and \( \mathcal{R}_B \) are represented on \( \ell^2(G_A^{s,u} \times \mathbb{Z}^2) \) and \( \ell^2(G_B^{s,u} \times \mathbb{Z}^2) \) as reduced groupoid C*-algebras. Since \( V_\varphi^* a V_\varphi = a \circ \phi^{-1} \in C_c(G_B^{s,u} \times \mathbb{Z}^2) \) for \( a \in C_c(G_A^{s,u} \times \mathbb{Z}^2) \), we know that \( \Phi(a) = V_\varphi^* a V_\varphi, a \in \mathcal{R}_A \) gives rise to an isomorphism \( \mathcal{R}_A \to \mathcal{R}_B \) of C*-algebras. Since \( \varphi : X_A \to X_B \) is a topological conjugacy, we know that

\[
\tilde{g}^+(\varphi(x), p, q, \varphi(y)) = \tilde{f}^+(x, p, q, y), \quad \tilde{g}^-(\varphi(x), p, q, \varphi(y)) = \tilde{f}^-(x, p, q, y)
\]

for \((x, p, q, y) \in G_A^{s,u} \times \mathbb{Z}^2\) so that for \( \xi \in \ell^2(G_A^{s,u} \times \mathbb{Z}^2) \), \( (x, p, q, y) \in G_A^{s,u} \times \mathbb{Z}^2 \), we have

\[
[V_\varphi U_{(x,s)}(\tilde{g})] V_\varphi^* \xi (x, p, q, y) = [U_{(x,s)}(\tilde{f})] V_\varphi^* \xi (x, p, q, y)
\]

for all \((x, p, q, y) \in G_A^{s,u} \times \mathbb{Z}^2\). Hence we have \( V_\varphi U_{(x,s)}(\tilde{g}) = U_{(x,s)}(\tilde{f}) \), so that the equality \( \Phi \gamma_{r,s}^A f = \gamma_{r,s}^B g \circ \Phi \) holds. Since \( a \circ \phi^{-1} \in C_c(G_B^{s,u} \times \mathbb{Z}^2) \) for \( a \in C_c(G_A^{s,u} \times \mathbb{Z}^2) \), the equality \( \Phi(C(X_A)) = C(X_B) \) is obvious (cf. [20, Theorem 1.1]).
Corollary 5.5. Let $B$ be an $M \times M$ irreducible non-permutation matrix with entries in $\{0, 1\}$. For any continuous function $g \in C(\hat{X}_B, \mathbb{Z})$ on $\hat{X}_B$, there exist an $N \times N$ irreducible non-permutation matrix $A$ with entries in $\{0, 1\}$ and a continuous function $f = \sum_{j=1}^{N} f_j \mathcal{X}_{U_j(0)}$ for some integers $f_j \in \mathbb{Z}$ such that $(\hat{X}_A, \tilde{\sigma}_A)$ is topologically conjugate to $(\hat{X}_B, \sigma_B)$ and there exists an isomorphism $\Phi : \mathcal{R}_A \rightarrow \mathcal{R}_B$ such that
\[
\Phi(C(\hat{X}_A)) = C(\hat{X}_B), \quad \Phi \circ \gamma^A_{(r,s)} = \gamma^B_{(r,s)} \circ \Phi, \quad (r,s) \in \mathbb{T}^2.
\]

Proof. There exists $K \in \mathbb{N}$ such that $g = \sum_{\mu \in B_K(\hat{X}_B)} g_{\mu} \mathcal{X}_{U_{\mu}}$ for some $g_{\mu} \in \mathbb{Z}$ where $U_{\mu}$ is the cylinder set of $\hat{X}_B$ for a word $\mu \in B_K(\hat{X}_B)$. By taking $K$-higher block representation of $\hat{X}_B$ and its $K$ high block matrix of $B$ as $A$ (see [17, 1.4]), and shifting $g$, one may have a topological conjugacy $\varphi : \hat{X}_A \rightarrow \hat{X}_B$ and a continuous function $f = \sum_{j=1}^{N} f_j \mathcal{X}_{U_j(0)}$ for some integers $f_j \in \mathbb{Z}$ such that $f = g \circ \varphi$. Hence we get the desired assertion by Proposition 5.4. \qed

6. Flow equivalence

We fix an irreducible, non-permutation matrix $A$. Let $f : \hat{X}_A \rightarrow \mathbb{N}$ be a continuous function on $\hat{X}_A$ such that $f = \sum_{j=1}^{N} f_j \mathcal{X}_{U_j(0)}$ for some positive integers $f_j \in \mathbb{N}$. Put $m_j = f_j - 1$ for $j = 1, \ldots, N$. Consider the new graph $\mathcal{G}_f = (\mathcal{V}_f, \mathcal{E}_f)$ with its transition matrix $A_f$ from the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ for the matrix $A$ defined in (2.3) in Section 2. The vertex set $\mathcal{V}_f$ is
\[
\bigcup_{j=1}^{N} \{j_0, j_1, j_2, \ldots, j_{m_j}\},
\]
which is denoted by $\tilde{\Sigma}$, and if $A(j, k) = 1$, then
\[
A_f(j_0, j_1) = A_f(j_1, j_2) = \cdots = A_f(j_{m_j-1}, j_{m_j}) = A_f(j_{m_j}, k_0) = 1.
\]

Let us denote by
\[
\tilde{S}_{j_0}, \tilde{S}_{j_1}, \ldots, \tilde{S}_{j_{m_j}} \quad \text{and} \quad \tilde{T}_{j_0}, \tilde{T}_{j_1}, \ldots, \tilde{T}_{j_{m_j}}
\]
the canonical generating partial isometries of $\mathcal{O}_{A_f}$ and $\mathcal{O}_{(A_f)^{\gamma}}$ respectively which satisfy
\[
\sum_{j=1}^{N} (\tilde{S}_{j_0} \tilde{S}_{j_1}^* + \ldots + \tilde{S}_{j_{m_j}} \tilde{S}_{j_{m_j}}^*) = 1, \quad (6.1)
\]
\[
\tilde{S}_{j_0}^* \tilde{S}_{j_n} = \tilde{S}_{j_{n+1}} \tilde{S}_{j_{n+1}}^*, \quad n = 0, 1, \ldots, m_j - 1, \quad (6.2)
\]
\[
\tilde{S}_{j_{m_j}}^* \tilde{S}_{j_{m_j}} = \sum_{k=1}^{N} A(j, k) \tilde{S}_{k_0} \tilde{S}_{k_0}^*, \quad j = 1, 2, \ldots, N \quad (6.3)
\]
and
\[\sum_{j=1}^{N} (\bar{T}_j \bar{T}_j + \bar{T}_j \bar{T}_j^* + \cdots + \bar{T}_j \bar{T}_j^*) = 1, \quad (6.4)\]
\[\bar{T}_{j_{n+1}} \bar{T}_{j_{n+1}} = \bar{T}_{j_n} \bar{T}_{j_n}^*, \quad n = 0, 1, \ldots, m_j - 1, \quad (6.5)\]
\[\bar{T}_{j_0} \bar{T}_{j_0} = \sum_{k=1}^{N} A^j(k) \bar{T}_k \bar{T}_k^*, \quad j = 1, 2, \ldots, N. \quad (6.6)\]

We set
\[S_j = \bar{S}_{j_0} \bar{S}_{j_1} \cdots \bar{S}_{j_m}, \quad T_j = \bar{T}_{j_m} \cdots \bar{T}_{j_1} \bar{T}_{j_0} \quad \text{for} \quad j = 1, \ldots, N.\]

Define the projections
\[P_A = \sum_{j=1}^{N} S_j S_j^* \quad \text{and} \quad P_{A'} = \sum_{j=1}^{N} \bar{T}_j \bar{T}_j^*.\]

We denote by \(C^*(S_1, \ldots, S_N)\) (resp. \(C^*(T_1, \ldots, T_N)\)) the \(C^*\)-subalgebra of \(\mathcal{O}_{A_f}\) (resp. \(\mathcal{O}_{(A_f)'})\) generated by \(S_1, \ldots, S_N\) (resp. \(T_1, \ldots, T_N\)).

**Lemma 6.1.** Keep the above notation. We have

(i)
\[\sum_{j=1}^{N} S_j S_j^* = P_A, \quad S_j S_j^* = \sum_{k=1}^{N} A(j, k) S_k S_k^* \quad \text{for} \quad j = 1, \ldots, N,\]

and the \(C^*\)-algebra \(P_A \mathcal{O}_{A_f} P_A\) coincides with \(C^*(S_1, \ldots, S_N)\) that is isomorphic to \(\mathcal{O}_A\).

(ii)
\[\sum_{j=1}^{N} T_j T_j^* = P_{A'}, \quad T_j T_j^* = \sum_{k=1}^{N} A^j(k) T_k T_k^* \quad \text{for} \quad j = 1, \ldots, N,\]

and the \(C^*\)-algebra \(P_{A'} \mathcal{O}_{(A_f)'} P_{A'}\) coincides with \(C^*(T_1, \ldots, T_N)\) that is isomorphic to \(\mathcal{O}_{A'}\).

**Proof.** We will prove (i). By (6.2), we have the following equalities
\[S_j^* S_j = (S_j^* \bar{S}_{j_1} \cdots \bar{S}_{j_m})^* (S_j^* \bar{S}_{j_1} \cdots \bar{S}_{j_m}) = \bar{S}_{j_{m+1}} \bar{S}_{j_{m+1}} \cdots \bar{S}_{j_{m+1}} \bar{S}_{j_{m+1}} \cdots \bar{S}_{j_{m+1}} .\]
By continuing this procedure, the last term above goes to $S_j^* S_{jm_j}^*$ so that $S_j^* S_j = S_{jm_j}^* S_{jm_j}^*$. We also have

\[ S_j^* S_j = (S_{j_0}^* S_{j_1}^* S_{j_2}^* \cdots S_{jm_j}^*) (S_{j_0}^* S_{j_1}^* S_{j_2}^* \cdots S_{jm_j}^*)^* \\
= S_{j_0}^* S_{j_1}^* S_{j_2}^* \cdots S_{jm_j}^* S_{jm_j}^* S_{jm_j-1}^* \cdots S_{j_1}^* S_{j_0}^* \\
= S_{j_0}^* S_{j_1}^* S_{j_2}^* \cdots S_{jm_j-1}^* S_{jm_j}^* S_{jm_j-1}^* \cdots S_{j_1}^* S_{j_0}^* \\
= S_{j_0}^* S_{j_1}^* S_{j_2}^* \cdots S_{jm_j-1}^* \cdots S_{j_1}^* S_{j_0}^*.
\]

By continuing this procedure, the last term above goes to $S_{j_0}^* S_{j_0}^*$. Hence we have

\[ S_j^* S_j = S_{jm_j}^* S_{jm_j}, \quad S_j^* S_j = S_{j_0}^* S_{j_0}^*, \quad j = 1, \ldots, N \tag{6.7} \]

so that

\[ \sum_{j=1}^{N} S_j^* S_j = P_A \quad \text{and} \quad S_j^* S_j = \sum_{k=1}^{N} A(j, k) S_k^* S_k^*, \quad j = 1, \ldots, N. \]

Similarly we have

\[ T_j^* T_j = T_{j_0}^* T_{j_0}, \quad T_j^* T_j = T_{jm_j}^* T_{jm_j}, \quad j = 1, \ldots, N \tag{6.8} \]

so that

\[ \sum_{j=1}^{N} T_j^* T_j = P_{A^t} \quad \text{and} \quad T_j^* T_j = \sum_{k=1}^{N} A^t(j, k) T_k^* T_k^*, \quad j = 1, \ldots, N. \]

As $S_j = P_A S_{j_0}^* S_{j_1}^* S_{j_2}^* \cdots S_{jm_j}^* P_A$, one sees that $S_j \in P_A O A_f P_A$ for $j = 1, \ldots, N$. Hence we have $C^* (S_1, \ldots, S_N) \subset P_A O A_f P_A$. We will show the converse inclusion relation. We note that for $j_k, j'_k \in \Sigma$, the equality $A_f (j_k, j'_k) = 1$ holds if and only if either of the following two cases occurs

(1) $j' = j$ and $k' = k + 1$
(2) $A(j, j') = 1$ and $k = m_j, k' = 0$.

For $\bar{\mu} = (\bar{\mu}_1, \ldots, \bar{\mu}_m), \bar{\nu} = (\bar{\nu}_1, \ldots, \bar{\nu}_n) \in B_\sigma (X_{A_f})$, suppose that $P_A S_{\bar{\mu}}^* S_{\bar{\nu}}^* P_A \neq 0$.

We first see that $\bar{\mu}_1 = m(1)_0$ and $\bar{\nu}_1 = n(1)_0$ for some $m(1), n(1) = 1, \ldots, N$. By the conditions (1), (2), we know that

\[ S_{\bar{\mu}} = S_{m(1)} \cdots S_{m(p)} S_{j_0} \cdots S_{j_k}, \quad S_{\bar{\nu}} = S_{n(1)} \cdots S_{n(q)} S_{i_0} \cdots S_{i_l} \]

for some $m(1), \ldots, m(p), n(1), \ldots, n(q), j, i \in \{1, \ldots, N\}$ and $0 \leq k \leq m_j, 0 \leq l \leq m_i$. Hence we may assume that $\bar{\mu} = (j_0, j_1, \ldots, j_k), \bar{\nu} = (i_0, i_1, \ldots, i_l)$ with $0 \leq k \leq m_j, 0 \leq l \leq m_i$, so that

\[ S_{\bar{\mu}}^* S_{\bar{\nu}}^* = S_{j_0}^* S_{j_1} \cdots S_{j_k}^* S_{i_0}^* \cdots S_{i_l}^* \neq 0. \]

Since

\[ S_{j_k}^* = S_{j_k}^* S_{j_k}^* S_{j_k}^* = S_{j_k}^* S_{j_{k+1}}^* S_{j_{k+1}}^*, \quad S_{i_l}^* = S_{i_l}^* S_{i_l}^* S_{i_l}^* = S_{i_{l+1}}^* S_{i_{l+1}}^* S_{i_{l+1}}^*. \]
we have
\[ S_{\mu}S_{\nu} = S_{j_0}S_{i_1} \cdots S_{j_{k-1}}S_{j_k}S_{j_{k+1}}^* S_{i_{k+1}}^* S_{i_k}^* \cdots S_{i_1}^* S_{j_0}^*. \]

The condition \( S_{\mu}S_{\nu} \neq 0 \) leads \( j_{k+1} = i_{k+1} \), so that we have \( j_{k} = i_{k}, \ldots, j_1 = i_1, j_0 = i_0 \). Hence
\[
S_{j_0}S_{i_1} \cdots S_{j_{k-1}}S_{j_k}S_{j_{k+1}}^* S_{i_{k+1}}^* S_{i_k}^* \cdots S_{i_1}^* S_{j_0}^* = S_{j_0}S_{i_1} \cdots S_{j_{k-1}}S_{j_k}S_{j_{k+1}}^* S_{j_{k+1}}^* \cdots S_{j_0}^*. 
\]

As \( S_{j_{k+1}}S_{j_{k+1}}^* = S_{j_{k+1}^*}S_{j_{k+1}^*} \), by continuing this procedure we know that
\[
S_{\mu}S_{\nu} = S_{j_0}S_{i_1} \cdots S_{j_{k-1}}S_{j_k}S_{j_{k+1}}^* \cdots S_{j_0}^* = S_jS_j^*. 
\]

This shows that the element \( P_A S_{\mu}S_{\nu} P_A \) belongs to \( C^* (S_1, \ldots, S_N) \), so that
\[ P_A \mathcal{O}_{A_f^*} P_A = C^* (S_1, \ldots, S_N) \]
and similarly
\[ P_{A^*} \mathcal{O}_{(A_f)^*} P_{A^*} = C^* (T_1, \ldots, T_N). \]

Recall that the vertex set \( \mathcal{V}_f \) defined by \( \bigcup_{j=1}^N \{ i_0, i_1, \ldots, i_{m_j} \} \) of the graph \( \mathcal{G}_f \) is denoted by \( \Sigma \). We set
\[ \bar{U}_{j_k} = T_{j_k}^* \otimes S_{j_k}, \quad j_k \in \Sigma. \]

The partial isometries \( \bar{U}_{j_k}, j_k \in \Sigma \) belong to the Ruelle algebra \( \mathcal{R}_{A_f} \) for the matrix \( A_f \). The following lemma is direct from the identities (6.2), (6.5).

**Lemma 6.2.** For \( j = 1, \ldots, N \), \( k = 0, 1, \ldots, m_j \), we have
\[
(U_{j_k} U_{j_{k+1}} \cdots U_{j_{m_j}}) (U_{j_k} U_{j_{k+1}} \cdots U_{j_{m_j}})^* = U_{j_k} U_{j_k}^* = T_{j_k}^* T_{j_k} \otimes S_{j_k}^* S_{j_k},
\]
\[
(U_{j_k} U_{j_{k+1}} \cdots U_{j_{m_j}})^* (U_{j_k} U_{j_{k+1}} \cdots U_{j_{m_j}}) = U_{j_k}^* U_{j_k} T_{j_k}^* \otimes S_{j_k}^* S_{j_k}. 
\]

Hence we have
\[
E_{A_f} = \sum_{j=1}^N \sum_{k=0}^{m_j} \bar{T}_{j_k} \bar{T}_{j_k}^* \bar{S}_{j_k}^* \bar{S}_{j_k} = \sum_{j=1}^N \sum_{k=0}^{m_j} \bar{U}_{j_k} \bar{U}_{j_k}^* \quad \text{in} \quad \mathcal{R}_{A_f},
\]
\[
E_A = \sum_{j=1}^N T_j T_j^* \otimes S_j^* S_j = \sum_{j=1}^N U_{j_{m_j}} U_{j_{m_j}} \quad \text{in} \quad \mathcal{R}_A. 
\]

Let \( H \) be the separable infinite dimensional Hilbert space \( \ell^2 (\mathbb{N}) \). Take isometries \( s_{j_k}, j_k \in \Sigma \) on \( H \) such that
\[
\sum_{k=0}^{m_j} s_{j_k} s_{j_k}^* = 1_H, \quad j = 1, \ldots, N. \quad (6.9)
\]
We define a partial isometry $V_f$ in the tensor product $C^*$-algebra $\mathcal{R}_{A_f} \otimes B(H)$ by setting

$$V_f = \sum_{j=1}^N \sum_{k=0}^{m_j} U_{j_k} U_{j_{k+1}} \cdots U_{j_{m_j}} \otimes s_{j_k}^*.$$  

(6.10)

Lemma 6.3. $V_f V_f^* = E_{A_f} \otimes 1_H$ and $V_f^* V_f = E_A \otimes 1_H$.

Proof. By Lemma 6.2 we have the following equalities.

$$V_f V_f^* = \sum_{j=1}^N \sum_{k=0}^{m_j} (U_{j_k} U_{j_{k+1}} \cdots U_{j_{m_j}})^* (U_{j_k} U_{j_{k+1}} \cdots U_{j_{m_j}}) \otimes s_{j_k}^* s_{j_k}^*$$

$$= \sum_{j=1}^N (T_j T_j^* \otimes S_j^* S_j) \otimes \left( \sum_{k=0}^{m_j} s_{j_k} s_{j_k}^* \right)$$

$$= \sum_{j=1}^N T_j T_j^* \otimes S_j^* S_j \otimes 1_H = E_{A_f} \otimes 1_H$$

and

$$V_f^* V_f = \sum_{j=1}^N \sum_{k=0}^{m_j} (U_{j_k} U_{j_{k+1}} \cdots U_{j_{m_j}})^* (U_{j_k} U_{j_{k+1}} \cdots U_{j_{m_j}}) \otimes s_{j_k}^* s_{j_k}^*$$

$$= \sum_{j=1}^N \sum_{k=0}^{m_j} U_{j_k} U_{j_k}^* \otimes 1_H = E_A \otimes 1_H.$$

Recall that $\mathcal{K}$ denotes the $C^*$-algebra $\mathcal{K}(H)$ of compact operators on the Hilbert space $H$. As $\bar{U}_{j_k}$ belongs to $\mathcal{R}_{A_f}$ and $\mathcal{R}_{A_f} \otimes B(H)$ is contained in the multiplier algebra $M(\mathcal{R}_{A_f} \otimes \mathcal{K})$ of $\mathcal{R}_{A_f} \otimes \mathcal{K}$, the partial isometry $V_f$ belongs to $M(\mathcal{R}_{A_f} \otimes \mathcal{K})$.

Lemma 6.4. $E_{A_f} (P_{A'} \otimes P_A) = (P_{A'} \otimes P_A) E_{A_f} = E_A$.

Proof. We have

$$E_{A_f} (P_{A'} \otimes P_A) = \sum_{j=1}^N \sum_{k=0}^{m_j} \bar{T}_{j_k} \bar{T}_{j_k}^* \otimes \bar{S}_{j_k} \bar{S}_{j_k}^* \cdot (\sum_{j=1}^N \bar{T}_{j_{m_j}} \bar{T}_{j_{m_j}}^* \otimes \sum_{i=1}^N \bar{S}_{i_0} \bar{S}_{i_0}^*)$$

$$= \sum_{j=1}^N \{ T_{j_{m_j}} \bar{T}_{j_{m_j}}^* \otimes (\sum_{i=1}^N S_{j_{m_j}} \bar{S}_{j_{m_j}}^* \cdot S_{i_0} \bar{S}_{i_0}^*) \}.$$  

By the identities (6.7) and (6.8), we know

$$\bar{S}_{j_{m_j}} \bar{S}_{j_{m_j}} = S_{j} S_{j}, \quad \bar{S}_{i_0} \bar{S}_{i_0} = S_{i_0} S_{i_0}^*, \quad \bar{T}_{j_{m_j}} \bar{T}_{j_{m_j}}^* = T_j T_j^*.$$
so that we have
\[ E_{A_j}(P_{A'} \otimes P_A) = \sum_{j=1}^{N} \{ T_j T_j^* \otimes S_j^* S_j \} = E_A, \]
and hence \((P_{A'} \otimes P_A)E_{A_j} = E_A.\]

**Theorem 6.5.** Let \(A\) be an irreducible, non-permutation matrix with entries in \(\{0,1\}\). For a continuous function \(f : \mathcal{X}_A \rightarrow \mathcal{K}\), there exists an isomorphism \(\Phi_f : \tilde{\mathcal{R}}_{A_j} \otimes \mathcal{K} \rightarrow \tilde{\mathcal{R}}_A \otimes \mathcal{K}\) of \(C^*\)-algebras such that

\[ \Phi_f \circ (\gamma_{(r,s)}^{A_j} \otimes \text{id}) = (\gamma_{(r,s)}^{A_j} \otimes \text{id}) \circ \Phi_f, \quad (r, s) \in \mathbb{T}^2. \]

**Proof.** By Corollary 5.5, we may assume that \(f\) is of the form \(\sum_{j=1}^{N} f_j \chi_{U_j(0)}\).

Define \(\Phi_f : \tilde{\mathcal{R}}_{A_j} \otimes \mathcal{K} \rightarrow \tilde{\mathcal{R}}_A \otimes \mathcal{K}\) by setting \(\Phi_f(x \otimes K) = \tilde{\mathcal{V}}_f^*(x \otimes K)\tilde{\mathcal{V}}_f\) for \(x \otimes K \in \tilde{\mathcal{R}}_{A_j} \otimes \mathcal{K}\). We note that

\[ \tilde{\mathcal{R}}_{A_j} = E_{A_j}(\mathcal{O}_{(A_j)^c} \otimes \mathcal{O}_{A_j})E_{A_j}, \quad \tilde{\mathcal{R}}_A = E_A(\mathcal{O}_{A'} \otimes \mathcal{O}_A)E_A, \]

\[ \tilde{\mathcal{V}}_f \tilde{\mathcal{V}}_f^* = E_{A_j} \otimes 1_H, \quad \tilde{\mathcal{V}}_f^* \tilde{\mathcal{V}}_f = E_A \otimes 1_H, \]

\[ (P_{A'} \otimes P_A)(\mathcal{O}_{(A_j)^c} \otimes \mathcal{O}_{A_j})(P_{A'} \otimes P_A) = \mathcal{O}_{A'} \otimes \mathcal{O}_A. \]

For \(x \otimes K \in \tilde{\mathcal{R}}_{A_j} \otimes \mathcal{K}\), we have

\[ \tilde{\mathcal{V}}_f^*(x \otimes K)\tilde{\mathcal{V}}_f = (E_A \otimes 1_H)\tilde{\mathcal{V}}_f^*(x \otimes K)\tilde{\mathcal{V}}_f(E_A \otimes 1_H) \]
\[ = (E_A \otimes 1_H)(P_{A'} \otimes P_A) \otimes 1_H)\tilde{\mathcal{V}}_f^*(x \otimes K)\tilde{\mathcal{V}}_f((P_{A'} \otimes P_A) \otimes 1_H)(E_A \otimes 1_H). \]

As \(\tilde{\mathcal{V}}_f \in \mathcal{M}(\tilde{\mathcal{R}}_{A_j} \otimes \mathcal{K})\), the element \(\tilde{\mathcal{V}}_f^*(x \otimes K)\tilde{\mathcal{V}}_f\) belongs to \(\tilde{\mathcal{R}}_{A_j} \otimes \mathcal{K}\) which is \((E_{A_j} \otimes 1_H)((\mathcal{O}_{(A_j)^c} \otimes \mathcal{O}_{A_j}) \otimes \mathcal{K})E_{A_j} \otimes 1_H\). By Lemma 6.4, we have

\[ (E_A \otimes 1_H)((P_{A'} \otimes P_A) \otimes 1_H)(E_{A_j} \otimes 1_H) = (E_A \otimes 1_H)((P_{A'} \otimes P_A) \otimes 1_H), \]

so that \(\tilde{\mathcal{V}}_f^*(x \otimes K)\tilde{\mathcal{V}}_f\) belongs to the algebra

\[ (E_A \otimes 1_H)((P_{A'} \otimes P_A) \otimes 1_H)((\mathcal{O}_{(A_j)^c} \otimes \mathcal{O}_{A_j}) \otimes \mathcal{K})(P_{A'} \otimes P_A) \otimes 1_H)(E_A \otimes 1_H) \]

which is

\[ E_A(\mathcal{O}_{A'} \otimes \mathcal{O}_A)E_A \otimes \mathcal{K} = \tilde{\mathcal{R}}_A \otimes \mathcal{K}. \]

Hence \(\tilde{\mathcal{V}}_f^*(x \otimes K)\tilde{\mathcal{V}}_f\) belongs to the algebra \(\tilde{\mathcal{R}}_A \otimes \mathcal{K}\). This shows that the inclusion relation

\[ \tilde{\mathcal{V}}_f^*(\tilde{\mathcal{R}}_{A_j} \otimes \mathcal{K})\tilde{\mathcal{V}}_f \subset \tilde{\mathcal{R}}_A \otimes \mathcal{K} \quad (6.11) \]
holds. Conversely, for \(y \otimes K \in \tilde{\mathcal{R}}_A \otimes \mathcal{K}\), we have

\[ \tilde{\mathcal{V}}_f(y \otimes K)\tilde{\mathcal{V}}_f^* = (E_A \otimes 1_H)\tilde{\mathcal{V}}_f(y \otimes K)\tilde{\mathcal{V}}_f(E_A \otimes 1_H). \]
As $\mathcal{V}_f \in \mathcal{M}(\mathcal{R}_A \otimes \mathcal{K})$ and $y \otimes K \in \mathcal{R}_A \otimes \mathcal{K} \subset \mathcal{R}_A \otimes \mathcal{K}$, the element $\mathcal{V}_f(y \otimes K)\mathcal{V}_f^*$ belongs to $\mathcal{R}_A \otimes \mathcal{K}$ which is $(E_{A_f} \otimes 1_H)((\mathcal{O}_{A_f})^* \otimes \mathcal{O}_{A_f}) \otimes \mathcal{K})(E_{A_f} \otimes 1_H)$. Hence $\mathcal{V}_f(x \otimes K)\mathcal{V}_f^*$ belongs to the algebra $$(E_{A_f} \otimes 1_H)(E_{A_f} \otimes 1_H)((\mathcal{O}_{A_f})^* \otimes \mathcal{O}_{A_f}) \otimes \mathcal{K})(E_{A_f} \otimes 1_H)$$ which is $$E_{A_f}(\mathcal{O}_{A_f})^* \otimes \mathcal{O}_{A_f})E_{A_f} \otimes \mathcal{K} = \mathcal{R}_A \otimes \mathcal{K}.$$ Hence $\mathcal{V}_f(y \otimes K)\mathcal{V}_f^*$ belongs to the algebra $\mathcal{R}_A \otimes \mathcal{K}$. This shows that the inclusion relation $$\mathcal{V}_f(\mathcal{R}_A \otimes K)\mathcal{V}_f^* \subset \mathcal{R}_A \otimes \mathcal{K}$$ (6.12) holds. Since $\mathcal{V}_f \mathcal{V}_f^* = E_A \otimes 1_H$ and $E_{A_f} \mathcal{R}_A E_{A_f} = \mathcal{R}_A$, the inclusion relation (6.12) implies $$\mathcal{R}_A \otimes K \subset \mathcal{V}_f(\mathcal{R}_A \otimes \mathcal{K})\mathcal{V}_f.$$ By (6.11) and (6.13), we have $\mathcal{V}_f(\mathcal{R}_A \otimes \mathcal{K})\mathcal{V}_f = \mathcal{R}_A \otimes \mathcal{K}$. Therefore we have an isomorphism $\Phi_f = \text{Ad}(\mathcal{V}_f) : \mathcal{R}_A \otimes \mathcal{K} \longrightarrow \mathcal{R}_A \otimes \mathcal{K}$.

Since $\gamma^{A_f}_{(r,s)}(U_{j_k}) = U_{j_k}$ for $j_k \in \hat{\Sigma}$ by Lemma 5.3, we know the equality $(\gamma^{A_f}_{(r,s)} \otimes \text{id})(\mathcal{V}_f) = \mathcal{V}_f$. For $x \otimes K \in \mathcal{R}_A \otimes \mathcal{K}$, we have

$$(\Phi_f \circ (\gamma^{A_f}_{(r,s)} \otimes \text{id}))(x \otimes K) = \mathcal{V}_f(\gamma^{A_f}_{(r,s)}(x) \otimes K)\mathcal{V}_f$$

$$=(\gamma^{A_f}_{(r,s)} \otimes \text{id})(\mathcal{V}_f(x \otimes K)\mathcal{V}_f)$$

$$=(\gamma^{A_f}_{(r,s)} \otimes \text{id})(\Phi_f(x \otimes K)).$$

Now

$$\gamma^{A_f}_{(r,s)}(T_j^* \otimes S_k)$$

$$=\gamma^{A_f}_{(r,s)}((\tilde{T}_{j_1} \cdots \tilde{T}_{j_{m_k}})^* \otimes S_k \otimes \cdots \otimes S_k)$$

$$=\exp(2\pi i (f_k s - f_j r))((\tilde{T}_{j_1} \cdots \tilde{T}_{j_{m_k}})^* \otimes (\tilde{S}_k \otimes \cdots \otimes \tilde{S}_k))$$

$$=\alpha_r^{A_f}(T_j^*) \otimes \alpha_s^{A_f}(S_k).$$

Hence the restriction of $\gamma^{A_f}_{(r,s)} \otimes \text{id}$ to the subalgebra $\mathcal{R}_A \otimes \mathcal{K}$ coincides with $\gamma^{A_f}_{(r,s)} \otimes \text{id}$ so that we conclude that

$$\Phi_f \circ (\gamma^{A_f}_{(r,s)} \otimes \text{id}) = (\gamma^{A_f}_{(r,s)} \otimes \text{id}) \circ \Phi_f.$$

\[ \square \]

**Remark 6.6.** We note that it is not difficult to see that the above isomorphism $\Phi_f : \mathcal{R}_A \otimes \mathcal{K} \longrightarrow \mathcal{R}_A \otimes \mathcal{K}$ satisfies $\Phi_f(C(X_{A_f}) \otimes \mathcal{E}) = C(X_{A_f}) \otimes \mathcal{E}$, where $\mathcal{E}$ is the commutative $C^*$-algebra of diagonal operators on the Hilbert space $H = l^2(\mathbb{N})$. 
By using Proposition 4.6, we have the converse implication of Theorem 6.5 in the following way.

**Theorem 6.7.** Let $A, B$ be irreducible, non-permutation matrices with entries in $\{0, 1\}$. Suppose that there exist a continuous function $f : X_A \to \mathbb{N}$ and an isomorphism $\Phi : \tilde{\mathcal{R}}_B \otimes \mathcal{K} \to \tilde{\mathcal{R}}_A \otimes \mathcal{K}$ such that

$$\Phi(\gamma^b_{(r,s)} \otimes \text{id}) = (\gamma^a_{(r,s)} \otimes \text{id}) \circ \Phi, \quad (r, s) \in \mathbb{T}^2.$$  

Then the two-sided topological Markov shifts $(\tilde{X}_B, \tilde{\sigma}_B)$ and $(\tilde{X}_A, \tilde{\sigma}_A)$ are flow equivalent.

**Proof.** By Theorem 6.5, there exists an isomorphism $\Phi_f : \tilde{\mathcal{R}}_{A_f} \otimes \mathcal{K} \to \tilde{\mathcal{R}}_{A} \otimes \mathcal{K}$ of $C^*$-algebras such that

$$\Phi_f(\gamma^a_{(r,s)} \otimes \text{id}) = (\gamma^a_{(r,s)} \otimes \text{id}) \circ \Phi_f, \quad (r, s) \in \mathbb{T}^2.$$  

We define the isomorphism $\Phi_o = \Phi_{f^{-1}} \circ \Phi : \tilde{\mathcal{R}}_B \otimes \mathcal{K} \to \tilde{\mathcal{R}}_{A_f} \otimes \mathcal{K}$ which satisfies

$$\Phi_o(\gamma^b_{(r,s)} \otimes \text{id}) = (\gamma^a_{(r,s)} \otimes \text{id}) \circ \Phi_o, \quad (r, s) \in \mathbb{T}^2.$$  

By Proposition 4.6, we have the converse implication of Theorem 6.5. Since $(\tilde{X}_{A_f}, \tilde{\sigma}_{A_f})$ is a discrete suspension of $(\tilde{X}_A, \tilde{\sigma}_A)$, they are flow equivalent, so $(\tilde{X}_B, \tilde{\sigma}_B)$ and $(\tilde{X}_A, \tilde{\sigma}_A)$ are flow equivalent.

Therefore we have a characterization of flow equivalence in terms of the $C^*$-algebras $\tilde{\mathcal{R}}_A$ with their gauge actions with potentials.

**Theorem 6.8.** Let $A, B$ be irreducible, non-permutation matrices with entries in $\{0, 1\}$. Two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow equivalent if and only if there exist an irreducible non-permutation matrix $C$ with entries in $\{0, 1\}$ and continuous functions $f_A, f_B : \tilde{X}_C \to \mathbb{N}$ such that there exist isomorphisms $\Phi_A : \tilde{\mathcal{R}}_A \otimes \mathcal{K} \to \tilde{\mathcal{R}}_C \otimes \mathcal{K}$ and $\Phi_B : \tilde{\mathcal{R}}_B \otimes \mathcal{K} \to \tilde{\mathcal{R}}_C \otimes \mathcal{K}$ satisfying

$$\Phi_A(\gamma^A_{(r,s)} \otimes \text{id}) = (\gamma^C_{(r,s)} \otimes \text{id}) \circ \Phi_A \quad (6.14)$$

and

$$\Phi_B(\gamma^B_{(r,s)} \otimes \text{id}) = (\gamma^C_{(r,s)} \otimes \text{id}) \circ \Phi_B. \quad (6.15)$$

**Proof.** Suppose the two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow equivalent. By Parry–Sullivan [22], there exist an irreducible non-permutation matrix $C$ with entries in $\{0, 1\}$ and continuous functions $f_A, f_B : \tilde{X}_C \to \mathbb{N}$ such that $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_{Cf_A}, \tilde{\sigma}_{Cf_A})$ are topologically conjugate, and $(\tilde{X}_B, \tilde{\sigma}_B)$ and $(\tilde{X}_{Cf_B}, \tilde{\sigma}_{Cf_B})$ are topologically conjugate. By [20, Theorem 1.1] and Theorem 6.5, we have isomorphisms $\Phi_A : \tilde{\mathcal{R}}_A \otimes \mathcal{K} \to \tilde{\mathcal{R}}_C \otimes \mathcal{K}$ and $\Phi_B : \tilde{\mathcal{R}}_B \otimes \mathcal{K} \to \tilde{\mathcal{R}}_C \otimes \mathcal{K}$ of $C^*$-algebras satisfying (6.14) and (6.15), respectively.
The converse implication immediately follows from Theorem 6.7.

As a corollary we have

**Corollary 6.9.** Let $A$, $B$ be irreducible, non-permutation matrices with entries in $\{0, 1\}$. Two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exist continuous functions $f_A : \bar{X}_A \to \mathbb{N}$ and $f_B : \bar{X}_B \to \mathbb{N}$ such that there exists an isomorphism $\Phi : \mathcal{R}_A \otimes \mathcal{K} \to \mathcal{R}_B \otimes \mathcal{K}$ of $C^*$-algebras satisfying

$$\Phi(y_{(r,s)}^{A,f_A} \otimes \text{id}) = (y_{(r,s)}^{B,f_B} \otimes \text{id}) \circ \Phi, \quad (r, s) \in \mathbb{T}^2.$$

**Proof.** By Parry–Sullivan [22], $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent if and only if there exist continuous functions $f_A : \bar{X}_A \to \mathbb{N}$ and $f_B : \bar{X}_B \to \mathbb{N}$ such that $(\bar{X}_{A_{f_A}}, \bar{\sigma}_{A_{f_A}})$ and $(\bar{X}_{B_{f_B}}, \bar{\sigma}_{B_{f_B}})$ are topologically conjugate. The assertion follows from [20, Theorem 1.1] and Theorem 6.5, Theorem 6.7.

**7. Flow equivalence and topological conjugacy**

In [20, Theorem 1.1], it was proved that two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are topologically conjugate if and only if there exists an isomorphism $\Phi : \mathcal{R}_A \to \mathcal{R}_B$ of $C^*$-algebras such that $\Phi(C(\bar{X}_A)) = C(\bar{X}_B)$ and $\Phi(y_{(r,s)}^{A}) = y_{(r,s)}^{B} \circ \Phi$, $(r, s) \in \mathbb{T}^2$. Although topological conjugacy is a special case of flow equivalence, Theorem 6.8 and Corollary 6.9 do not refer to the case of topological conjugacy. In this final section, we reformulate both Theorem 6.8 and Corollary 6.9 to give characterizations of flow equivalence in terms of $C^*$-algebras that simultaneously include characterizations of topological conjugacy as a special case. They appear as Theorem 7.2 and Theorem 7.3.

We keep the assumption that $A$ is an irreducible, non-permutation matrix with entries in $\{0, 1\}$, and $f : \bar{X}_A \to \mathbb{N}$ is a continuous function such that $f = \sum_{j=1}^{N} f_j x_{U_j(0)}$ for some positive integers $f_j$. Let $e_n, n \in \mathbb{N}$ be a sequence of vectors of complete orthonormal basis of the separable infinite dimensional Hilbert space $H = \ell^2(\mathbb{N})$. We fix $j = 1, \ldots, N$. For $k = 0, 1, \ldots, m_j$, where $m_j = f_j - 1$, we set

$$N_k = \{ n \in \mathbb{N} \mid n \equiv k \text{ (mod } f_j) \};$$

so that we have a disjoint union $\mathbb{N} = \mathbb{N}_0 \cup \mathbb{N}_1 \cup \cdots \cup \mathbb{N}_{m_j}$. We write

$$\mathbb{N}_k = \{1_k, 2_k, 3_k, \ldots\} \quad \text{where } 1_k < 2_k < 3_k < \ldots.$$

Define an isometry $s_{j_k}$ on $H$ by setting

$$s_{j_k} e_n = e_{n_k}, \quad n \in \mathbb{N}, \quad k = 0, 1, \ldots, m_j.$$

The family $\{s_{j_k}\}_{k=0}^{m_j}$ satisfy (6.9). We may construct the operator $\widetilde{V}_f$ from them by the formula (6.10). As in the proof of Theorem 6.5, define the isomorphism $\Phi_f : \mathcal{R}_A \otimes \mathcal{K} \to \mathcal{R}_B \otimes \mathcal{K}$ of $C^*$-algebras by setting $\Phi_f(x \otimes K) = \widetilde{V}_f(x \otimes k)\widetilde{V}_f$.
is a reformulation of Theorem 6.8 in the following way.

Proposition 7.1. Let $A$ be an irreducible, non-permutation matrix with entries in $\{0, 1\}$. Then there exists an isomorphism $\Phi_f : \tilde{\mathcal{R}}_A \otimes \mathcal{K} \rightarrow \mathcal{R}_A \otimes \mathcal{K}$ of $C^*$-algebras such that

$$\Phi_f(C(\tilde{\mathcal{R}}_A) \otimes \mathcal{E}) = C(\tilde{\mathcal{R}}_A) \otimes \mathcal{E},$$

$$\Phi_f \circ \gamma^A_{(r,s)} \otimes \text{id} = (\gamma^A_{(r,s)} \otimes \text{id}) \circ \Phi_f,$$

$$\Phi_f([E_{A_f}]) = f([E_A]) \text{ in } K_0(C(\tilde{\mathcal{R}}_A)).$$

By virtue of Proposition 7.1, we may finally state two theorems. The first one is a reformulation of Theorem 6.5 in the following way.
**Theorem 7.2.** Let $A, B$ be irreducible, non-permutation matrices with entries in $\{0, 1\}$. Two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow equivalent if and only if there exist an irreducible non-permutation matrix $C$ with entries in $\{0, 1\}$ and continuous functions $f_A, f_B : \tilde{X}_C \to \mathbb{N}$ such that $\tilde{R}_A \otimes \mathcal{K}$ and $\tilde{R}_B \otimes \mathcal{K}$ are isomorphic to $\tilde{R}_C \otimes \mathcal{K}$ via isomorphisms $\Phi_A$ and $\Phi_B$ satisfying

\[
\begin{align*}
\Phi_A(\gamma_{(r,s)}^{A} \otimes \text{id}) &= (\gamma_{(r,s)}^{C \cdot f_A} \otimes \text{id}) \circ \Phi_A, \\
\Phi_A(C(\tilde{X}_A) \otimes \mathcal{C}) &= C(\tilde{X}_C) \otimes \mathcal{C}, \\
\Phi_A([E_A]) &= f_A([E_C]) \quad \text{in } K_0(C(\tilde{X}_C)), \\
\Phi_B(\gamma_{(r,s)}^{B} \otimes \text{id}) &= (\gamma_{(r,s)}^{C \cdot f_B} \otimes \text{id}) \circ \Phi_B, \\
\Phi_B(C(\tilde{X}_B) \otimes \mathcal{C}) &= C(\tilde{X}_C) \otimes \mathcal{C}, \\
\Phi_B([E_B]) &= f_B([E_C]) \quad \text{in } K_0(C(\tilde{X}_C)).
\end{align*}
\] 

(7.4)

In particular, $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are topologically conjugate if and only if the equalities (7.4) and (7.5) hold for $f_A \equiv f_B \equiv 1$.

**Proof.** Assume the two-sided topological Markov shifts $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are flow equivalent. By Parry–Sullivan [22], there exist an irreducible non-permutation matrix $C$ with entries in $\{0, 1\}$ and continuous functions $f_A, f_B : \tilde{X}_C \to \mathbb{N}$ such that $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_{C^{f_A}}, \tilde{\sigma}_{C^{f_A}})$ are topologically conjugate, and $(\tilde{X}_B, \tilde{\sigma}_B)$ and $(\tilde{X}_{C^{f_B}}, \tilde{\sigma}_{C^{f_B}})$ are topologically conjugate. By [20, Theorem 1.1] there exist isomorphisms $\Phi_A : \tilde{R}_A \to \tilde{R}_{C^{f_A}}$ and $\Phi_B : \tilde{R}_B \to \tilde{R}_{C^{f_B}}$ of $C^*$-algebras such that $\Phi_A \circ \gamma_{(r,s)}^{A} = \gamma_{(r,s)}^{C \cdot f_A} \circ \Phi_A$, $\Phi_B \circ \gamma_{(r,s)}^{B} = \gamma_{(r,s)}^{C \cdot f_B} \circ \Phi_B$ and $\Phi_A(C(\tilde{X}_A)) = C(\tilde{X}_{C^{f_A}})$, $\Phi_B(C(\tilde{X}_B)) = C(\tilde{X}_{C^{f_B}})$.

By Proposition 7.1, there exist isomorphisms $\Phi_{f_A} : \tilde{R}_{C^{f_A}} \otimes \mathcal{K} \to \tilde{R}_C \otimes \mathcal{K}$ and $\Phi_{f_B} : \tilde{R}_{C^{f_B}} \otimes \mathcal{K} \to \tilde{R}_C \otimes \mathcal{K}$ satifying the equalities stated as (7.1), (7.2) and (7.3). By putting

\[
\begin{align*}
\Phi_A := \Phi_{f_A} \circ (\Phi_A \otimes \text{id}) : \tilde{R}_A \otimes \mathcal{K} \to \tilde{R}_C \otimes \mathcal{K}, \\
\Phi_B := \Phi_{f_B} \circ (\Phi_B \otimes \text{id}) : \tilde{R}_B \otimes \mathcal{K} \to \tilde{R}_C \otimes \mathcal{K},
\end{align*}
\]

we know the isomorphisms $\Phi_A$ and $\Phi_B$ satisfy the desired properties. The converse implication immediately follows from Theorem 6.8.

It remains to show that if the equalities (7.4) and (7.5) hold for $f_A \equiv f_B \equiv 1$, then $(\tilde{X}_A, \tilde{\sigma}_A)$ and $(\tilde{X}_B, \tilde{\sigma}_B)$ are topologically conjugate. Let $e_n, n \in \mathbb{N}$ be a complete orthonormal system of the Hilbert space $\ell^2(\mathbb{N})$, and $p_1$ be the projection on $\ell^2(\mathbb{N})$ of rank one onto the vector $e_1$. By (7.4) for $f_A \equiv 1$, we have $[\Phi_A([E_A \otimes p_1])] = [E_C \otimes p_1]$ in $K_0(C(\tilde{X}_C))$. Hence we may take a partial isometry $V_A \in C(\tilde{X}_C) \otimes \mathcal{K}$ such that $V_A^* V_A = E_C \otimes p_1, V_A^* V_A = \Phi_A(E_A \otimes p_1)$. Put $\Psi_A = \text{Ad}(V_A) \circ \Phi_A$. Since we see

$$\Psi_A(E_A \otimes p_1) = V_A \Phi_A(E_A \otimes p_1) V_A^* = V_A V_A^* = E_C \otimes p_1,$$
we have an isomorphism \( \Psi_A : \widetilde{\mathcal{R}}_A \otimes \mathbb{C} p_1 \to \widetilde{\mathcal{R}}_C \otimes \mathbb{C} p_1 \). For any \( a \in C(X_A) \), we have

\[
\Psi_A(a \otimes p_1) = V_A \Phi_A((E_A \otimes p_1)(a \otimes p_1)(E_A \otimes p_1))V_A^* \\
= V_A V_A^* V_A \Phi_A(a \otimes p_1)V_A^* V_A \\
= (E_C \otimes p_1)V_A \Phi_A(a \otimes p_1)V_A^* (E_C \otimes p_1),
\]

so that we know that \( \Psi_A(a \otimes p_1) \in C(X_C) \otimes \mathbb{C} p_1 \). We thus have \( \Psi_A(C(X_A)) \otimes \mathbb{C} p_1 = C(X_C) \otimes \mathbb{C} p_1 \). Since \( V_A \) belongs to \( C(X_C) \otimes \mathcal{K} \), one knows that \((y^A_{(r,s)} \otimes \text{id})(V_A) = V_A \) so that

\[
\Psi_A \circ (y^A_{(r,s)} \otimes \text{id}) = (y^A_{r,s} \otimes \text{id}) \circ \Psi_A
\]

for \( f_A \equiv 1 \). By restricting \( \Psi_A \) to \( \widetilde{\mathcal{R}}_A \otimes \mathbb{C} p_1 \), we have an isomorphism \( \Phi_A : \widetilde{\mathcal{R}}_A \to \widetilde{\mathcal{R}}_C \) such that \( \Phi_A \circ y^A_{(r,s)} = y^C_{(r,s)} \circ \Phi_A \) and \( \Phi_A(C(X_A)) = C(X_C) \). This shows that \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_C, \bar{\sigma}_C)\) are topologically conjugate by [20]. We similarly know that \((\bar{X}_B, \bar{\sigma}_B)\) and \((\bar{X}_C, \bar{\sigma}_C)\) are topologically conjugate, so that \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are topologically conjugate.

Converse implication under the condition that \( f_A \equiv 1 \) and \( f_B \equiv 1 \) is obvious by [20, Theorem 1.1].

The second one is a reformulation of Corollary 6.9 in the following way.

**Theorem 7.3.** Let \( A, B \) be irreducible, non-permutation matrices with entries in \( \{0, 1\} \). Two-sided topological Markov shifts \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent if and only if there exist continuous functions \( f_A : \bar{X}_A \to \mathbb{N} \) and \( f_B : \bar{X}_B \to \mathbb{N} \) such that there exists an isomorphism \( \Phi : \mathcal{R}_A \otimes \mathcal{K} \to \mathcal{R}_B \otimes \mathcal{K} \) of \( C^* \)-algebras satisfying

\[
\Phi(y^A_{(r,s)} \otimes \text{id}) = (y^B_{(r,s)} \otimes \text{id}) \circ \Phi, \quad (r, s) \in \mathbb{T}^2, \quad (7.6)
\]

\[
\Phi(C(\bar{X}_A) \otimes \mathcal{C}) = C(\bar{X}_B) \otimes \mathcal{C}, \quad (7.7)
\]

\[
\Phi_*([E_A]) = f_B([E_B]) \quad \text{in } K_0(C(\bar{X}_B)). \quad (7.8)
\]

In particular, \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are topologically conjugate if and only if the equalities \((7.6), (7.7)\) and \((7.8)\) hold for \( f_A \equiv f_B \equiv 1 \).

**Proof.** It suffices to show the only if part. Assume that \((\bar{X}_A, \bar{\sigma}_A)\) and \((\bar{X}_B, \bar{\sigma}_B)\) are flow equivalent. By Parry–Sullivan [22], there exist continuous functions \( f_A : \bar{X}_A \to \mathbb{N} \) and \( f_B : \bar{X}_B \to \mathbb{N} \) such that \((\bar{X}_{A_f}, \bar{\sigma}_{A_f})\) and \((\bar{X}_{B_f}, \bar{\sigma}_{B_f})\) are topologically conjugate. By [20], there exists an isomorphism \( \Phi_0 : \mathcal{R}_{A_f} \to \mathcal{R}_{B_f} \) of \( C^* \)-algebras such that \( \Phi_0(C(\bar{X}_{A_f})) = C(\bar{X}_{B_f}) \) and

\[
\Phi_0 \circ y^A_{(r,s)} = y^B_{(r,s)} \circ \Phi_0, \quad (r, s) \in \mathbb{T}^2.
\]
By Theorem 6.5 and Remark 6.6 with Proposition 7.1, we have an isomorphism
\[ \Phi_{f_A} : \widetilde{R}_{A_f} \otimes \mathcal{K} \rightarrow \widetilde{R}_A \otimes \mathcal{K} \] of C*-algebras satisfying
\[
\Phi_{f_A}(g_{A_f}(r,s)) = (g_{A_f}(r,s) \otimes 1) \circ \Phi_{f_A}, \quad (r,s) \in \mathbb{T}^2,
\]
\[
\Phi_{f_A}(C(X_{A_f}/\tilde{\sigma})) = C(X_A) \otimes \mathcal{C},
\]
\[
\Phi_{f_A*}(\mathcal{E}) = f_A([\mathcal{E}]) \quad \text{in } K_0(C(X_A)).
\]
Similarly we have an isomorphism \( \Phi_{f_B} : \widetilde{R}_{B_f} \otimes \mathcal{K} \rightarrow \widetilde{R}_B \otimes \mathcal{K} \) of C*-algebras satisfying
\[
\Phi_{f_B}(g_{B_f}(r,s)) = (g_{B_f}(r,s) \otimes 1) \circ \Phi_{f_B}, \quad (r,s) \in \mathbb{T}^2,
\]
\[
\Phi_{f_B}(C(X_{B_f}/\tilde{\sigma})) = C(X_B) \otimes \mathcal{C},
\]
\[
\Phi_{f_B*}(\mathcal{E}) = f_B([\mathcal{E}]) \quad \text{in } K_0(C(X_B)).
\]
Put \( \Phi := \Phi_{f_A} \circ (\Phi_0 \otimes 1) \circ \Phi_{f_A}^{-1} : \widetilde{R}_A \otimes \mathcal{K} \rightarrow \widetilde{R}_B \otimes \mathcal{K} \). We then see that the equality \( \Phi(C(X_A) \otimes \mathcal{C}) = C(X_B) \otimes \mathcal{C} \) holds. Since \( E_{A_f}, E_{B_f} \) are the units of \( \widetilde{R}_{A_f}, \widetilde{R}_{B_f} \), respectively, we know that \( (\Phi_0 \otimes 1)(E_{A_f} \otimes p_1) = E_{B_f} \otimes p_1 \), where \( p_1 \) is the projection on \( \mathcal{C}^2(\mathbb{N}) \) of rank one as in the proof of the preceding theorem. We thus have
\[
\Phi_*([E_A]) = \Phi_{f_B*}((\Phi_0 \otimes 1)([E_A]) \otimes p_1)) = \Phi_{f_B*}([E_B]) = f_B([E_B]).
\]
Suppose next that \( f_A \equiv 1 \) and \( f_B \equiv 1 \). The condition (7.8) goes to \( \Phi_*([E_A]) = [E_B] \) in \( K_0(C(X_B)) \). Hence we may take a partial isometry \( V \in C(X_B) \otimes \mathcal{K} \) such that \( \Phi(E_A \otimes p_1) = V^*V \) and \( E_B \otimes p_1 = VV^* \). By a manner similar to the proof of Theorem 7.2, we obtain an isomorphism \( \Phi : \widetilde{R}_A \rightarrow \widetilde{R}_B \) such that \( \Phi(C(X_A)) = C(X_B) \) and \( \Phi g_{A_f} = \gamma_{f_B} \circ \Phi \). Now \( f_A \equiv 1, f_B \equiv 1 \), we conclude that \( X_A, \tilde{\sigma}_A \) and \( X_B, \tilde{\sigma}_B \) are topologically conjugate by [20, Theorem 1.1].

The converse implication under the condition \( f_A \equiv 1, f_B \equiv 1 \) is obvious. \( \square \)

References


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FLOW EQUIVALENCE OF TOPOLOGICAL MARKOV SHIFTS AND RUÉLLE ALGEBRAS


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This paper is available via http://nyjm.albany.edu/j/2021/27-53.html.