A class of prime fusion categories of dimension $2^N$

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Abstract. We study a class of strictly weakly integral fusion categories $\mathcal{I}_{N,\zeta}$, where $N \geq 1$ is a natural number and $\zeta$ is a $2^N$th root of unity, that we call $N$-Ising fusion categories. An $N$-Ising fusion category has Frobenius-Perron dimension $2^{N+1}$ and is a graded extension of a pointed fusion category of rank 2 by the cyclic group of order $\mathbb{Z}_{2^N}$. We show that every braided $N$-Ising fusion category is prime and also that there exists a slightly degenerate $N$-Ising braided fusion category for all $N > 2$. We also prove a structure result for braided extensions of a rank 2 pointed fusion category in terms of braided $N$-Ising fusion categories.

1. Introduction

Among the most basic examples of fusion categories, the pointed fusion categories are those whose simple objects are invertible. A pointed fusion category is determined by its group of invertible objects $G$ and the cohomology class of a 3-cocycle $\omega$ on $G$, who is responsible for the associativity constraint. We denote by $\text{Vec}_{G}^{\omega}$ the pointed fusion category associated to the pair $(G, \omega)$.

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Let $G$ be a finite group. A fusion category $C$ is called a $G$-extension of a fusion category $D$ if it admits a faithful grading by the group $G$,

$$C = \oplus_{g \in G} C_g,$$

such that $C_g \otimes C_h \subseteq C_{gh}$, for all $g, h \in G$, and the trivial homogeneous component is equivalent to $D$ \([10]\). Thus, a fusion category $C$ is pointed if and only if $C$ is a $G$-extension of the fusion category $\text{Vec}$ of finite dimensional vector spaces, for some finite group $G$.

An Ising category is a fusion category of Frobenius-Perron dimension 4 which is not pointed. Ising categories appear in Conformal Field Theory related to 2-dimensional Ising models.

Every Ising fusion category is a $\mathbb{Z}_2$-extension of the rank 2 pointed fusion category $\text{Vec}_{\mathbb{Z}_2}$ and it belongs to the class of fusion categories classified by Tambara and Yamagami in \([20]\); in particular there exist exactly 2 Ising fusion categories up to equivalence, and they are a 3-cocycle twist of each other.

By the main result of \([19]\), every Ising fusion category admits exactly 4 non-equivalent braidings. In particular all such braidings are non-degenerate. Several properties of Ising fusion categories are studied in \([4, \text{Appendix B}]\). See Subsection 2.4.

In this paper we study a family of examples of fusion categories that are obtained from Ising fusion categories and share some features with them. We call them $N$-Ising fusion categories. They are special instances of the cyclic extensions of adjoint categories of ADE type classified in \([5]\) and are defined as follows: Let $\mathcal{J}$ be the semisimplification of the representation category of $U_{-q}(\mathfrak{sl}_2)$, with $q = \exp(i\pi/4)$. Then $\mathcal{J}$ is an Ising fusion category. Let $Z$ be the non-invertible simple object of $\mathcal{J}$. Then an $N$-Ising category is defined as a 3-cocycle twist of the fusion subcategory of $\mathcal{J} \boxtimes \text{Vec}_{\mathbb{Z}_2^N}$ generated by the simple object $Z \boxtimes 1$; c.f. Section 4. (The definition of a 3-cocycle twist of a group-graded fusion category is recalled in Subsection 2.2.)

A 1-Ising fusion category is thus an Ising fusion category. For every $N \geq 1$, an $N$-Ising fusion category has Frobenius-Perron dimension $2^{N+1}$ and is a graded extension of a pointed fusion category of rank 2 by the cyclic group of order $\mathbb{Z}_{2^N}$. In addition every $N$-Ising fusion category is strictly weakly integral. Its group of invertible objects is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{N-1}}$ and it has $2^{N-1}$ simple objects of Frobenius-Perron dimension $\sqrt{2}$, none of which is self-dual except in the case $N = 1$.

As graded extensions of $\text{Vec}_{\mathbb{Z}_2}$, $N$-Ising fusion categories are parameterized by the integer $N$ and a $2^N$th root of unity $\zeta$. The corresponding category is denoted $\mathcal{J}_{N,\zeta}$. We use the notation $\mathcal{J}_N$ to indicate the category $\mathcal{J}_{N,1}$.
Every $N$-Ising fusion category $\mathcal{J}_{N, \pm 1}$ admits the structure of a braided fusion category. We show that a braided $N$-Ising fusion category is always prime (Corollary 4.8), that is, it does not contain any nontrivial non-degenerate fusion subcategory. We also show that with respect to any possible braiding, an $N$-Ising fusion category is non-degenerate if and only if $N = 1$. In addition, we prove that a slightly degenerate braided $N$-Ising category exists if $N > 2$. See Subsection 4.1. We point out that the classification of slightly degenerate fusion categories of Frobenius-Perron dimension 8 in [21, Proposition 4.6] implies that a 2-Ising fusion category cannot be slightly degenerate.

Observe that, as shown in [5], when $N \geq 2$ there is another family of non-pointed $\mathbb{Z}_2^N$-extensions of $\text{Vec}_{\mathbb{Z}_2}$ which is not equivalent to any $N$-Ising fusion category. However, the fusion categories in this family do not admit any braiding (Theorem 5.3).

Our main result for braided extensions of a rank 2 pointed fusion category is the following theorem:

**Theorem 5.5.** Let $\mathcal{C}$ be a non-pointed braided fusion category and suppose that $\mathcal{C}$ is an extension of a rank 2 pointed fusion category. Then $\mathcal{C}$ is equivalent as a fusion category to $\mathcal{J}_N \boxtimes \mathcal{B}$, for some $N \geq 1$, where $\mathcal{J}_N$ is a braided $N$-Ising fusion category, and $\mathcal{B}$ is a pointed braided fusion category. Furthermore, the categories $\mathcal{J}_N$ and $\mathcal{B}$ projectively centralize each other in $\mathcal{C}$.

The notion of projective centralizer of a fusion subcategory, introduced in [4], is recalled in Subsection 2.2.

Theorem 5.5 is proved in Section 5. Its proof relies on the classification results of [5]. We point out that Theorem 5.5 applies in particular when $\mathcal{C}$ is a slightly degenerate braided fusion category with generalized Tambara-Yamagami fusion rules, that is, when $\mathcal{C}$ is slightly degenerate, not pointed, and the tensor product of two non-invertible simple objects decomposes as a sum of invertible objects.

The paper is organized as follows. In Section 2 we discuss some preliminary notions and results on fusion categories that will be relevant in the rest of the paper. Section 3 contains some basic results on the structure of a general group extension of a rank 2 pointed fusion category and on braided such extensions that will be needed in the sequel. In Section 4 we introduce $N$-Ising categories and study their main properties. In Section 5 we give a proof of our main result on braided extensions of a rank 2 pointed fusion category.

### 2. Preliminaries

We shall work over an algebraically closed field $k$ of characteristic zero. A fusion category over $k$ is a $k$-linear semisimple rigid tensor category with
finitely many isomorphism classes of simple objects, finite-dimensional vector spaces of morphisms and such that the unit object $1$ is simple. We refer the reader to [7], [4] for the main notions on fusion categories and braided fusion categories used throughout.

An object of a fusion category $C$ is called \textit{trivial} if it is isomorphic to $1^\oplus n$ for some natural number $n$.

Let $C$ be a fusion category. The tensor product in $C$ induces a ring structure in the Grothendieck ring $K(C)$ of $C$. By [7, Section 8], there is a unique ring homomorphism $FPdim : K(C) \to \mathbb{R}$ such that $FPdim(X) \geq 1$ for all nonzero $X \in C$. The number $FPdim(X)$ is called the Frobenius-Perron dimension of $X$. The Frobenius-Perron dimension of $C$ is defined by

$$FPdim(C) = \sum_{X \in \text{Irr}(C)} FPdim(X)^2,$$

where $\text{Irr}(C)$ is the set of isomorphism classes of simple objects in $C$.

A simple object $X \in C$ is called invertible if $X \otimes X^* \cong 1$, where $X^*$ is the dual of $X$. Thus $X$ is invertible if and only if $FPdim(X) = 1$. A fusion category $C$ is called pointed if every simple object of $C$ is invertible. Pointed fusion categories whose group of invertible objects is isomorphic to $G$ are classified by the orbits of the action of the group $\text{Out}(G)$ in $H^3(G, k^\times)$. The pointed fusion category corresponding to the class of a 3-cocycle $\omega$ will be denoted by $\text{Vec}_G^{\omega}$.

The largest pointed subcategory of $C$, denoted $C_{pt}$, is the fusion subcategory generated by all invertible simple objects. The set $G = G(C)$ of isomorphism classes of invertible objects of $C$ is a finite group with multiplication given by tensor product. The inverse of $X \in G$ is its dual $X^*$. The group $G$ acts on the set $\text{Irr}(C)$ by left tensor product multiplication. Let $G[X]$ be the stabilizer of $X \in \text{Irr}(C)$ under this action. Then we have a decomposition

$$X \otimes X^* = \bigoplus_{g \in G[X]} g \oplus \sum_{Y \in \text{Irr}(C) - G[X]} \dim \text{Hom}(Y, X \otimes X^*) Y. \quad (2.1)$$

2.1. Group extensions of fusion categories. Let $G$ be a finite group. A fusion category $C$ is graded by $G$ if $C$ has a direct sum decomposition into full abelian subcategories $C = \bigoplus_{g \in G} C_g$ such that $C_g \otimes C_h \subseteq C_{gh}$, for all $g, h \in G$. If $C_g \neq 0$, for all $g \in G$, then the grading is called faithful. When the grading is faithful, $C$ is called a $G$-extension of the trivial component $C_e$.

If $C = \bigoplus_{g \in G} C_g$ is a faithful grading of $C$, then [7, Proposition 8.20] shows that

$$FPdim(C) = |G| FPdim(C_e), \quad FPdim(C_g) = FPdim(C_h), \quad \forall g, h \in G.$$

It follows from the results of [10] that every fusion category $C$ has a canonical faithful grading $C = \bigoplus_{g \in U(C)} C_g$ with trivial component $C_e = C_{ad}$, where $C_{ad}$ is the adjoint subcategory of $C$, that is, the fusion subcategory generated
by the simple constituents of \( X \otimes X^* \), for all \( X \in \text{Irr}(C) \). This grading is called the universal grading of \( C \), and \( U(C) \) is called the universal grading group of \( C \). Any other faithful grading \( C = \oplus_{g \in G} C_g \) of \( C \) is determined by a surjective group homomorphism \( \pi : U(C) \to G \). Hence the trivial component \( C_e \) contains \( C_{ad} \).

Let \( G \) be a finite group and let \( C \) be a \( G \)-extension of a fusion category \( D \cong C_e \). Let also \( \omega \in Z^3(G, k^\times) \) be a 3-cocycle. We shall denote by \( C^\omega \) the fusion category obtained from \( C \) by twisting the associator with \( \omega \). For \( \omega_1, \omega_2 \in Z^3(G, k^\times) \), the categories \( C^{\omega_1} \) and \( C^{\omega_2} \) are equivalent as \( G \)-extensions of \( D \) if and only if the classes of \( \omega_1 \) and \( \omega_2 \) coincide in \( H^3(G, k^\times) \). See \([8]\).

### 2.2. Braided fusion categories.

A braided fusion category \( C \) is a fusion category admitting a braiding \( c \), that is, a family of natural isomorphisms: \( c_{X,Y} : X \otimes Y \to Y \otimes X, \ X, Y \in C \), obeying the hexagon axioms.

Let \( C \) be a braided fusion category. Two objects \( X, Y \in C \) are said to centralize each other if \( c_{Y,X} c_{X,Y} = \text{id}_{X \otimes Y} \). The centralizer \( D' \) of a fusion subcategory \( D \subseteq C \) is the full subcategory of objects which centralize every object of \( D \), that is

\[
D' = \{ X \in C \mid c_{Y,X} c_{X,Y} = \text{id}_{X \otimes Y}, \forall Y \in D \}.
\]

The Müger center \( Z_2(C) \) of a braided fusion category \( C \) is the centralizer \( C' \) of \( C \) itself. A braided fusion category \( C \) is called non-degenerate if \( Z_2(C) \) is equivalent to the category \( \text{Vec} \) of finite-dimensional vector spaces. A braided fusion category \( C \) is called slightly degenerate if \( Z_2(C) \) is equivalent to the category \( \text{sVec} \) of finite-dimensional super-vector spaces.

Two full subcategories \( D \) and \( \tilde{D} \) of \( C \) are said to projectively centralize each other if for all simple objects \( X \in D \) and \( Y \in \tilde{D} \), the squared braiding \( c_{Y,X} c_{X,Y} \) is a scalar multiple of the identity \( \text{id}_{X \otimes Y} \). See \([4, \text{Subsection 3.3}]\).

Suppose that \( D \) and \( \tilde{D} \) are fusion subcategories of \( C \) that projectively centralize each other. Then \([4, \text{Proposition 3.32}]\) shows that there exist finite groups \( G \) and \( \tilde{G} \) endowed with a non-degenerate pairing \( b : G \times \tilde{G} \to k^\times \) and faithful gradings \( D = \bigoplus_{g \in G} D_g, \tilde{D} = \bigoplus_{g \in G} \tilde{D}_g \), such that \( D_0 = D \cap \tilde{D}, \tilde{D}_0 = D' \cap \tilde{D} \), and for all homogeneous simple objects \( X \in D_g, Y \in \tilde{D}_h, \ g \in G, h \in \tilde{G} \), the squared braiding \( c_{Y,X} c_{X,Y} \) is given by

\[
c_{Y,X} c_{X,Y} = b(g, h) \text{id}_{X \otimes Y}.
\]

A braided fusion category \( C \) is called symmetric if \( Z_2(C) = C \). Hence the Müger center of a braided fusion category is a symmetric fusion category.

A symmetric fusion category \( C \) is called Tannakian if it is equivalent to the category \( \text{Rep}(G) \) of finite-dimensional representations of a finite group \( G \), as braided fusion categories.

Let \( C \) be a symmetric fusion category. Deligne proved that there exist a finite group \( G \) and a central element \( u \) of order 2, such that \( C \) is equivalent to
the category \( \text{Rep}(G, u) \) of representations of \( G \) on finite-dimensional super vector spaces, where \( u \) acts as the parity operator [3].

The symmetric category \( \mathcal{C} \) is either Tannakian or a \( \mathbb{Z}_2 \)-extension of a Tannakian subcategory. Therefore, if \( \text{FPdim}(\mathcal{C}) \) is odd, then \( \mathcal{C} \) is Tannakian. Moreover if \( \text{FPdim}(\mathcal{C}) \) is bigger than 2 then \( \mathcal{C} \) necessarily contains a Tannakian subcategory. Also, a non-Tannakian symmetric fusion category of Frobenius-Perron dimension 2 is equivalent to the category \( \text{sVec} \). See [4, Subsection 2.12].

The following proposition is a special case of Corollary 3.26 of [4].

**Proposition 2.1.** Let \( \mathcal{C} \) be a braided fusion category. Then \( \mathcal{C}_{ad} \subseteq (\mathcal{C}_{pt})' \).

The following theorem is due to Drinfeld et al. In the case when \( \mathcal{C} \) is modular, it is due to Müger [16, Theorem 4.2].

**Theorem 2.2.** [4, Theorem 3.13] Let \( \mathcal{C} \) be a braided fusion category and let \( \mathcal{D} \) be a non-degenerate subcategory of \( \mathcal{C} \). Then \( \mathcal{C} \) is braided equivalent to \( \mathcal{D} \boxtimes \mathcal{D}' \), where \( \mathcal{D}' \) is the centralizer of \( \mathcal{D} \) in \( \mathcal{C} \).

For a pair of fusion subcategories \( \mathcal{A}, \mathcal{B} \) of \( \mathcal{D} \), we use the notation \( \mathcal{A} \triangleright \mathcal{B} \) to indicate the smallest fusion subcategory of \( \mathcal{C} \) containing \( \mathcal{A} \) and \( \mathcal{B} \). The following result will be used frequently.

**Lemma 2.3.** [4, Corollary 3.11] Let \( \mathcal{C} \) be a braided fusion category. If \( \mathcal{D} \) is any fusion subcategory of \( \mathcal{C} \) then \( \mathcal{D}'' = \mathcal{D} \triangleright \mathcal{Z}_2(\mathcal{C}) \).

### 2.3. Pointed braided fusion categories.

We recall in this subsection some facts related to the classification of pointed braided fusion categories. We refer the reader to [12], [18], [4] for a detailed exposition.

Let \( G \) be a finite abelian group. An abelian 3-cocycle on \( G \) with values in \( k^\times \) is a pair \( (\omega, \sigma) \), where \( \omega : G \times G \times G \to k^\times \) is a normalized 3-cocycle and \( \sigma : G \times G \to k^\times \) is a 2-cochain such that

\[
\omega(a, b, c) \omega(b, c, a) \sigma(a, bc) = \omega(b, a, c) \sigma(a, b) \sigma(a, c),
\]

for all \( a, b, c \in G \). Abelian 3-cocycles form an abelian group \( Z^3_{ab}(G, k^\times) \). Let \( B^3_{ab}(G, k^\times) \subseteq Z^3_{ab}(G, k^\times) \) be the subgroup of abelian coboundaries, that is, abelian 3-cocycles of the form \( (du, u(u(21))^{-1}) \) where \( u : G \times G \to k^\times \) is a normalized 2-cochain, \( du(a, b, c) = u(b, c) u(ab, c)^{-1} u(a, bc) u(a, b)^{-1} \), and \( u(21) \) is defined as \( u_{21}(a, b) = u(b, a) \), for all \( a, b, c \in G \).

The quotient \( H^3_{ab}(G, k^\times) = Z^3_{ab}(G, k^\times) / B^3_{ab}(G, k^\times) \) is called the abelian cohomology group of \( G \) with coefficients in \( k^\times \). Every braiding of a pointed fusion category with group \( G \) of invertible objects corresponds to an element of the group \( H^3_{ab}(G, k^\times) \). In particular, given a normalized 3-cocycle \( \omega \) and a 2-cochain \( \sigma \) on \( G \), we have that the rule

\[
\sigma_{a,b} \text{id}_{ab} : a \otimes b \to b \otimes a, \quad a, b \in G,
\]

defines a braiding in the fusion category \( \text{Vec}_{G}^{\omega} \) if and only if \( (\omega, \sigma) \in Z^3_{ab}(G, k^\times) \).
A quadratic form on $G$ with values in $k^\times$ is a map $q : G \to k^\times$ satisfying $q(g) = q(g^{-1})$, for all $g \in G$, and such that the map $\beta : G \times G \to k^\times$ defined by $\beta(a, b) = q(ab)q(a)^{-1}q(b)^{-1}$ is a symmetric bicharacter on $G$. If $q$ is a quadratic form on $G$, then the pair $(G, q)$ is called a pre-metric group.

To every abelian 3-cocycle $(\omega, \sigma)$ on $G$ one can associate a quadratic form on $G$ defined by
\[ q(g) = \sigma(g, g), \quad g \in G. \]  
(2.2)

A result of Eilenberg and Mac Lane states that this correspondence defines a group isomorphism between the abelian cohomology group $H^3_{ab}(G, k^\times)$ and the abelian group of quadratic forms on $G$.

Moreover, the functor that associates to every pointed fusion category $\mathcal{C}$ the pre-metric group $(G, q)$, where $G$ is the group of invertible objects of $\mathcal{C}$ and $q$ is the quadratic form (2.2), where $\sigma$ is the braiding of $\mathcal{C}$, defines an equivalence between the category of pointed fusion categories and braided functors up to braided isomorphism and the category of pre-metric groups.

Thus, two braided fusion categories $\mathcal{C}(G, q)$ and $\mathcal{C}(G, q')$ associated to the quadratic forms $q$ and $q'$ on $G$ are equivalent if and only if there exists an automorphism $\varphi$ of $G$ such that $q'(\varphi(g)) = q(g)$, for all $g \in G$.

The squared braiding of the braided fusion category $\mathcal{C}(G, q)$ associated to a quadratic form $q$ is given by the symmetric bilinear form $\beta : G \times G \to k^\times$ associated to $q$.

Let $M$ be a natural number and let $G = \mathbb{Z}_M$ be the cyclic group of order $M$. Let also $\zeta \in k^\times$ be an $M$th root of 1. Then $\zeta$ determines a 3-cocycle $\omega_\zeta$ on $\mathbb{Z}_M$ where, for all $0 \leq i, j, \ell \leq M - 1$,
\[ \omega_\zeta(i, j, \ell) = \begin{cases} 1, & \text{if } j + \ell < M, \\ \zeta^i, & \text{if } j + \ell \geq M. \end{cases} \]  
(2.3)

The assignment $\zeta \mapsto \omega_\zeta$ gives rise to a group isomorphism between the group $G_M$ of $M$th roots of 1 in $k^\times$ and the group $H^3(\mathbb{Z}_M, k^\times)$. In particular $H^3(\mathbb{Z}_M, k^\times) \cong \mathbb{Z}_M$.

We shall denote by $\text{Vec}_{\mathbb{Z}_M}^\zeta$ the pointed fusion category corresponding to the 3-cocycle $\omega_\zeta$. Thus $\text{Vec}_{\mathbb{Z}_M}^1 = \text{Vec}_{\mathbb{Z}_M}$ and, if $M$ is even, $\text{Vec}_{\mathbb{Z}_M}^{-1}$ is the pointed fusion category corresponding to the 3-cocycle $\omega_{-1}$ associated to $\zeta = -1 \in G_M$.

Let $\xi \in k^\times$ such that $\xi^{M^2} = 1 = \xi^{2M}$. Then the pair $(\omega_{\xi M}, \sigma_{\xi})$ is an abelian 3-cocycle on $G$ where, for all $0 \leq i, j, \ell \leq M - 1$,
\[ \sigma_{\xi}(i, j) = \xi^{ij}. \]  
(2.4)

Furthermore, this gives rise to a group isomorphism between $H^3_{ab}(\mathbb{Z}_M, k^\times)$ and the group $G_d$ of $d$th roots of 1 in $k^\times$, where $d = \gcd(M^2, 2M)$. See [12, pp. 49], [18, Subsection 2.5.2].
Thus Vec\(^{\xi}_{Z_M}\) is a braided fusion category whose squared braiding is given by \(\beta_{\xi}(i, j) \, \text{id}_{i+j} : i + j \to i + j\), where \(\beta_{\xi} : \mathbb{Z}_M \times \mathbb{Z}_M \to k^\times\) is the bilinear form defined as

\[
\beta_{\xi}(i, j) = \xi^{2ij}, \quad 0 \leq i, j < M.
\]

The quadratic form \(q : \mathbb{Z}_M \to k^\times\) and the corresponding symmetric bilinear form on \(\mathbb{Z}_M\) associated to the braiding (2.4) are given, respectively, by the formulas

\[
q(j) = \xi^{2j}, \quad \beta(i, j) = \xi^{2ij},
\]

(2.5) for all \(0 \leq i, j \leq M - 1\).

Note that the condition \(\xi^{2M} = 1\) forces \(\xi^M = \pm 1\). In particular, for a fixed value of \(\zeta = \pm 1\), there are exactly \(M\) choices for \(\xi\). Thus we obtain:

**Lemma 2.4.** If the pointed fusion category Vec\(^{\xi}_{Z_M}\) admits a braiding then \(\zeta = \pm 1\). In addition we have:

1. If \(M\) is odd, Vec\(^{\xi}_{Z_M}\) does not admit any braiding unless \(\zeta = 1\), and in this case, it admits exactly \(M\) braidings up to equivalence.
2. If \(M\) is even, then each of the categories Vec\(^{\xi}_{Z_M}\) and Vec\(^{-1}_{Z_M}\) admits exactly \(M\) braidings, up to equivalence.

**Example 2.5.** Let \(N \geq 1\) and let \(\xi \in k^\times\) be a \(2N+1\)th root of 1. It follows from formulas (2.5) that the braided fusion category associated to \(\xi\) is non-degenerate if and only if \(\xi\) is primitive. If this is the case, then the underlying fusion category is Vec\(^{-1}_{Z_{2N}}\).

Let \(\xi \in k^\times\) be a primitive 8th root of 1. Let \(C = C(\mathbb{Z}_4, \xi)\) be the corresponding (non-degenerate) braided fusion category. We get from formulas (2.5) that \(q(2) = \xi^4 = -1\). Hence in this case the subcategory \(\langle 2 \rangle \subseteq C\) is equivalent to sVec.

### 2.4. Ising categories.

An Ising category is a fusion category of Frobenius-Perron dimension 4 which is not pointed. Let \(I\) be an Ising fusion category. Then, up to isomorphism, \(I\) has a unique nontrivial invertible object \(\delta\) and a unique non-invertible simple object \(Z\). Thus \(\text{FPdim } Z = \sqrt{2}\) and the fusion rules of \(I\) are determined by the relation

\[
Z \otimes Z \cong 1 \oplus \delta.
\]

(2.6)

In view of the results of [20], there exist exactly 2 non-equivalent Ising fusion categories. The universal grading group of \(I\) is isomorphic to \(\mathbb{Z}_2\). The explicit formulas for the associators of Ising categories in [20] imply that if \(I^+\) and \(I^-\) are two non-equivalent Ising categories then, up to an equivalence of fusion categories, any of them is obtained from the other by twisting the associator by the 3-cocycle \(\omega_{-1}\) on \(\mathbb{Z}_2\) determined by the relation \(\omega_{-1}(1, 1, 1) = -1\).
Every Ising fusion category admits a braiding and all possible braidings are classified by the main result of [19] (see also [4]); in particular all such braidings are non-degenerate. The category $\mathcal{I}_{pt}$ is equivalent to the category $\text{sVec}$ of super-vector spaces as a braided fusion category.

2.5. Equivariantizations and de-equivariantizations. Let $\mathcal{C}$ be a fusion category with an action by tensor autoequivalences $\rho : G \to \text{Aut}_\otimes(\mathcal{C})$ of a finite group $G$. The equivariantization $\mathcal{C}^G$ of $\mathcal{C}$ under the action of $G$ is defined as the category of $G$-equivariant objects and $G$-equivariant morphisms of $\mathcal{C}$. Thus, an object of $\mathcal{C}^G$ is a pair $(X, (u_g)_{g \in G})$, where $X$ is an object of $\mathcal{C}$, $u_g : \rho^g(X) \to X$, $g \in G$, is an isomorphism such that

$$u_{gh} \circ \rho^2_{g,h} = u_g \circ \rho^g(u_h),$$

for all $g, h \in G$, where $\rho^2_{g,h} : \rho^g(\rho^h(X)) \to \rho^{gh}(X)$ is the monoidal structure of the action $\rho$. The tensor product of equivariant objects is defined by means of the monoidal structure of the action.

Let $\mathcal{C}$ be a fusion category and let $\mathcal{E} = \text{Rep}(G) \subseteq Z(\mathcal{C})$ be a Tannakian subcategory of the Drinfeld center $Z(\mathcal{C})$ of $\mathcal{C}$ that embeds into $\mathcal{C}$ via the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$. Then the algebra $A = k^G$ of $k$-valued functions on $G$ is a commutative algebra in $Z(\mathcal{C})$. The de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ by $\mathcal{E}$ is the fusion category defined as the category of left $A$-modules in $\mathcal{C}$. See [4] for details on equivariantizations and de-equivariantizations.

The operations of equivariantization and de-equivariantization are inverse to each other: $(\mathcal{C}^G)_G \cong \mathcal{C} \cong (\mathcal{C}_G)^G$. As for their Frobenius-Perron dimensions, we have

$$\text{FPdim}(\mathcal{C}) = |G| \text{FPdim}(\mathcal{C}_G), \quad \text{FPdim}(\mathcal{C}^G) = |G| \text{FPdim}(\mathcal{C}).$$

Given a Tannakian subcategory $\text{Rep}(G)$ of a braided fusion category $\mathcal{C}$, we have an exact sequence of fusion categories (see [2, Section 1]):

$$\text{Rep}(G) \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}_G,$$

where $\mathcal{C}_G$ is the de-equivariantization of $\mathcal{C}$ by $\text{Rep}(G)$ and $F$ is the forgetful functor. Hence $\text{Rep}(G)$ is the kernel of $F$, that is, the subcategory of $\mathcal{C}$ whose objects have trivial image under $F$.

3. Extensions of a rank 2 pointed fusion category

3.1. General Results. Recall that a generalized Tambara-Yamagami fusion category is a fusion category $\mathcal{C}$ which is not pointed and such that the tensor product of two non-invertible simple objects of $\mathcal{C}$ is a sum of invertible objects. See [13].

**Theorem 3.1.** Let $\mathcal{C}$ be a $G$-extension of a pointed fusion category $\text{Vec}_{\mathbb{Z}_2}^\omega$. Then the following hold:

1. If $\omega = -1$, then $\mathcal{C}$ is pointed.
(2) If $\omega = 1$, then $C$ is either pointed or a generalized Tambara-Yamagami fusion category. If the last possibility holds, then:

(i) Up to isomorphism, $C$ has $2n$ invertible objects and $n$ simple objects of Frobenius-Perron dimension $\sqrt{2}$, for some $n \geq 1$.

(ii) $C_{ad} \cong \text{Vec}_{\mathbb{Z}_2}$ as fusion categories, and $U(C) = G$ is of order $2n$.

**Proof.** Let $C = \oplus_{g \in G} C_g$ be a faithful grading such that $C_e = \text{Vec}_{\mathbb{Z}_2}^\omega$. Since this grading is faithful, every component $C_g$ has Frobenius-Perron dimension $2$. Since $C$ is weakly integral, the Frobenius-Perron dimension of every simple object is a square root of some integer [7, Proposition 8.27]. This implies that every component $C_g$ either contains 2 non-isomorphic invertible objects, or it contains a unique $\sqrt{2}$-dimensional simple object. If $C$ is not pointed, then the trivial component $C_e$ is pointed and there exists a component $C_g$ containing a unique $\sqrt{2}$-dimensional simple object. It follows from [11, Lemma 2.6] that $\omega$ is trivial. Then (1) holds.

Suppose that $C$ is not pointed. By [10, Theorem 3.10], $C$ is endowed with a faithful $\mathbb{Z}_2$-grading $C = \oplus_{h \in \mathbb{Z}_2} C^h$, where the trivial component $C^0$ is $C_{ad}$ and $C^1$ contains all $\sqrt{2}$-dimensional simple objects. Let $X, Y$ be non-invertible simple objects of $C$. Then $X, Y \in C^1$ and hence $X \otimes Y \in C^0$, which implies that $X \otimes Y$ is a direct sum of invertible objects. Hence $C$ is a generalized Tambara-Yamagami fusion category and (2) holds.

Assume that the number of non-isomorphic $\sqrt{2}$-dimensional simple objects is $n \geq 1$. Then $2n = \text{FPdim}(C^1) = \text{FPdim}(C^0)$. Hence $|G| = 2n$ and we get part (i).

Since $C_{ad} \subseteq C_e \cong \text{Vec}_{\mathbb{Z}_2}$, we know $C_{ad} = \text{Vec}$ or $\text{Vec}_{\mathbb{Z}_2}$. Since $C$ is not pointed, then $C_{ad}$ cannot be $\text{Vec}$. Therefore $C_{ad} = C_e$ and $G = U(C)$. In particular the order of $U(C)$ is $2n$. This proves part (ii). □

For a fusion category $C$, let $\text{cd}(C)$ denote the set of Frobenius-Perron dimensions of simple objects of $C$.

**Corollary 3.2.** Let $C$ be a non-pointed fusion category. Then $C$ is an extension of a rank 2 pointed fusion category if and only if $\text{cd}(C) = \{1, \sqrt{2}\}$.

**Proof.** In view of Theorem 3.1, it will be enough to show that the condition $\text{cd}(C) = \{1, \sqrt{2}\}$ implies that $C$ is an extension of a rank 2 pointed fusion category. So assume that $\text{cd}(C) = \{1, \sqrt{2}\}$.

As in the proof of Theorem 3.1 we get that $C$ is a generalized Tambara-Yamagami fusion category. Then, by [17, Proposition 5.2], the adjoint subcategory $C_{ad}$ coincides with the fusion subcategory generated by $G[X]$, for any $\sqrt{2}$-dimensional simple object $X$. Hence $\text{FPdim}(C_{ad}) = 2$ and $C$ is an extension of a rank 2 pointed fusion category. □

**Corollary 3.3.** Let $C$ be a $G$-extension of $\text{Vec}_{\mathbb{Z}_2}$. Assume that $C$ is not pointed. Then the following hold:
(1) The action of the group $G(C)$ by left (or right) tensor multiplication on the set of non-invertible simple objects of $C$ is transitive.
(2) The group $Z_2$ is a normal subgroup of $G(C)$.

Proof. Since $C$ is not pointed, Theorem 3.1 implies that $C$ is a generalized Tambara-Yamagami fusion category. The corollary then follows from [17, Lemma 5.1].

3.2. Braided extensions of $Vec_{Z_2}$. Throughout this subsection $C$ will be an extension of $Vec_{Z_2}$. In addition, we assume that $C$ is braided and not pointed.

Lemma 3.4. The adjoint subcategory $C_{ad}$ is equivalent to $sVec$ as braided fusion categories.

Proof. By Theorem 3.1, we know that $C_{ad} \cong Vec_{Z_2}$. By [6, Lemma 2.5], $C_{ad} = C_{ad} \cap C_{pt}$ is symmetric. Suppose on the contrary that $C_{ad}$ is Tannakian. Then $C_{ad} \cong \text{Rep}(Z_2)$ as braided fusion categories and $C$ is a $Z_2$-equivariantization of a fusion category $C_{Z_2}$.

The forgetful functor $F : C \rightarrow C_{Z_2}$ is a tensor functor and the image of every object in $C_{ad}$ under $F$ is a trivial object of $C_{Z_2}$. Let $\delta$ be the unique nontrivial simple object of $C_{ad}$. If $X$ is a non-invertible simple object of $C$ then $X \otimes X^* \cong 1 \oplus \delta$. Hence $F(X \otimes X^*) \cong F(X) \otimes F(X)^* \cong 1 \oplus 1$, which implies that $F(X)$ is not simple. Then the decomposition of $F(X) \otimes F(X)^*$ must contain at least four simple direct summands. This contradiction shows that $C_{ad}$ cannot be Tannakian, and therefore $C_{ad} \cong sVec$, as claimed.

Recall that if $D$ is a fusion category with commutative Grothendieck ring and $A$ is a fusion subcategory of $D$, the commutator of $A$ in $D$, denoted by $A^{com}$, is the fusion subcategory of $D$ generated by all simple objects $X$ of $D$ such that $X \otimes X^*$ is contained in $A$ [10].

Lemma 3.5. The following relations hold:

1. $(C_{ad})' = C_{pt}$ and $Z_2(C) \subseteq C_{pt}$.
2. $Z_2(C_{pt}) = C_{ad} \vee Z_2(C)$.

Proof. (1) By [4, Proposition 3.25], a simple object $X \in C$ belongs to $(C_{ad})'$ if and only if it belongs to $Z_2(C)^{com}$; that is, if and only if $X \otimes X^* \in Z_2(C)$. If $X$ is not invertible then $X \otimes X^* \cong 1 \oplus \delta$ and hence $\delta \otimes X \cong X$, where $\delta$ is unique nontrivial simple object of $C_{ad}$. Hence $sVec \subseteq Z_2(C)$. But by Lemma 3.4, $C_{ad} \cong sVec$. This is impossible by [14, Lemma 5.4] which says that if $sVec \subseteq Z_2(C)$ then $\delta \otimes Y \not\cong Y$ for any $Y \in C$. Therefore, $(C_{ad})' \subseteq C_{pt}$ is pointed. By Proposition 2.1, $(C_{ad})' \supseteq (C_{pt})^\ast = C_{pt} \vee Z_2(C)$. Hence we have

$$C_{pt} \supseteq (C_{ad})' \supseteq C_{pt} \vee Z_2(C) \supseteq C_{pt},$$

which shows that $(C_{ad})' = C_{pt}$ and $Z_2(C) \subseteq C_{pt}$.

(2) By part (1), we have

$$Z_2(C_{pt}) = C_{pt} \cap (C_{pt})' = C_{pt} \cap (C_{ad})'' = C_{pt} \cap (C_{ad} \vee Z_2(C)) = C_{ad} \vee Z_2(C),$$
the third equality by Lemma 2.3. This proves part (2).

□

4. \textbf{N-Ising categories}

In what follows we shall denote by $\mathcal{I}$ the semisimplification of the representation category of $U_{-q}(\mathfrak{sl}_2)$, where $q = \exp(i\pi/4)$. Then $\mathcal{I}$ is an Ising fusion category; see Subsection 2.4.

Recall that there exist exactly 2 non-equivalent such fusion categories, say $\mathcal{I}$ and $\mathcal{I}^-$. So that $\mathcal{I}^-$ is obtained from $\mathcal{I}$ by twisting the associator by the 3-cocycle $\alpha$ on $\mathbb{Z}_2$ such that $\alpha(1, 1, 1) = -1$.

We shall use the notation $\mathcal{I}$ to indicate either of the categories $\mathcal{I}$ or $\mathcal{I}^-$. As in Subsection 2.4 we shall denote by $\delta$ the unique nontrivial invertible object of $\mathcal{I}$ and $Z$ the unique non-invertible simple object.

Let $M \geq 2$ be an even natural number. Consider the fusion subcategory $\mathcal{C}_M$ of $\mathcal{I} \boxtimes \text{Vec}_{Z^M}$ generated by the object $Z \boxtimes 1$. The relation (2.6) implies that $\mathcal{C}_M$ has $M/2$ non-invertible simple objects:

$$Z_j = Z \boxtimes (2j + 1), \quad 0 \leq j \leq \frac{M}{2} - 1,$$

and $M$ invertible objects:

$$\delta^i \boxtimes (2j), \quad 0 \leq i \leq 1, \quad 0 \leq j \leq \frac{M}{2} - 1.$$

Thus $\text{FPdim} Z_j = \sqrt{2}$, for all $j = 0, \ldots, M/2 - 1$ and $\text{FPdim} \mathcal{C}_M = 2M$.

\textbf{Remark 4.1.} Every fusion category $\mathcal{C}_M$, $M \geq 2$, admits a braiding; to see this it suffices to consider any braiding in $\mathcal{I} \boxtimes \text{Vec}_{Z^M}$ and restrict it to $\mathcal{C}_M$.

The categories $\mathcal{C}_M$ have generalized Tambara-Yamagami fusion rules. Let us denote by $a = 1 \boxtimes 2 \in \mathcal{C}_M$. Explicitly, the fusion rules of $\mathcal{C}_M$ are determined as follows: the group of invertible objects is a direct product $\langle \delta \rangle \boxtimes \langle a \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{M/2}$ and

$$Z_j \otimes Z_\ell \cong a^{j+\ell+1} \oplus \delta a^{j+\ell+1}, \quad 0 \leq j, \ell \leq \frac{M}{2} - 1.$$

\textbf{Remark 4.2.} The categories $\mathcal{C}_M$ are particular cases of the construction in [5] of fusion categories which are cyclic extensions of fusion categories of adjoint ADE type. Note that the adjoint subcategory of $\mathcal{C}_M$ coincides with the subcategory generated by $\delta$. In particular, $\mathcal{C}_M$ is a $\mathbb{Z}_M$-extension of the fusion category of adjoint $A^{(1)}_3$ type $\mathcal{I}_{ad} \cong \text{Vec}_{\mathbb{Z}_2}$.

\textbf{Remark 4.3.} The construction of the categories $\mathcal{C}_M$ can be generalized replacing the cyclic group $\mathbb{Z}_M$ by any finite Abelian group $A$ as follows: We may suppose that $A = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}$, where $d_1, \ldots, d_r \geq 1$. Let $e_1, \ldots, e_r$ be the canonical generators of $A$. Then the fusion subcategory of $\mathcal{I} \boxtimes A$ generated by the simple objects $Z \boxtimes e_j$, $1 \leq j \leq r$, is an $A$-graded extension of $\text{Vec}_{\mathbb{Z}_2}$. Observe that all the fusion categories arising in this way admit a
braiding (c.f. Remark 4.1). In fact, the examples arising from this construction boil down to the ones obtained from cyclic groups, in view of Theorem 5.5 below.

Let $N \geq 1$. In what follows we shall use the notation $\mathcal{I}_N$ to indicate the fusion category $\mathcal{C}_{2N}$ defined above.

**Example 4.4.** As pointed out before, the category $\mathcal{I}_1 = \mathcal{I}$ is an Ising fusion category. In particular, it is non-degenerate. The category $\mathcal{I}_2$ has two non-isomorphic simple objects $Z_1$ and $Z_2$ of Frobenius-Perron dimension $\sqrt{2}$. The group of invertible objects is $\langle \delta \rangle \times \langle a \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and we have the fusion rules

$$Z_1^* \cong Z_2, \quad Z_1^{\otimes 2} \cong a \oplus \delta a \cong Z_2^{\otimes 2}.$$ 

In particular, $\mathcal{I}_2$ does not contain any Ising fusion subcategory.

More generally, the fusion rules (4.3) imply that $\mathcal{C}_M$ contains a non-invertible self-dual simple object if and only if $M/2$ is odd. If this is the case, such self-dual simple object must generate an Ising fusion subcategory. From the non-degeneracy of Ising fusion categories we obtain, for each $M$ such that $M/2$ is odd, an equivalence fusion categories $\mathcal{C}_M \cong \mathcal{I}_N \boxtimes \mathcal{B}$ or $\mathcal{C}_M \cong \mathcal{I}_N^{-} \boxtimes \mathcal{B}$, where $\mathcal{B}$ is a pointed fusion category. Furthermore, these are equivalences of braided fusion categories regardless of the choice of the braiding in the category $\mathcal{C}_M$. This feature is generalized in Theorem 4.5 below.

**Theorem 4.5.** Let $M \geq 2$ be an even natural number. Suppose that $M = 2^N m$, where $N \geq 1$ and $m \geq 1$ is odd. Then there is an equivalence of fusion categories $\mathcal{C}_M \cong \mathcal{I}_N \boxtimes \mathcal{B}$, where $\mathcal{B}$ is a pointed fusion category. Moreover, with respect to any braiding in $\mathcal{C}_M$, this is an equivalence of braided fusion categories for an appropriate braiding in $\mathcal{I}_N$.

**Proof.** It will be enough to show that $\mathcal{C}_M \cong \mathcal{I}_N \boxtimes \mathcal{B}$ as fusion categories. Indeed, if this is the case, then regardless of the braiding we consider in $\mathcal{C}_M$, the fusion subcategories $\mathcal{I}_N$ and $\mathcal{B}$ must centralize each other, since their Frobenius-Perron dimensions are coprime; see [4, Proposition 3.32].

By assumption, $\mathbb{Z}_M$ is the direct sum of the subgroup generated by $m$ and the subgroup $S \cong \mathbb{Z}_m$ generated by $2^N$. Let $\mathcal{D}_1 \cong \text{Vec}_{\mathbb{Z}_m}$ denote the fusion subcategory of $\mathcal{C}_M$ generated by $1 \boxtimes S$.

We have an equivalence of fusion categories $\text{Vec}_{\mathbb{Z}_2^N} \cong \langle m \rangle \subseteq \text{Vec}_{\mathbb{Z}_M}$, where $\langle m \rangle$ is the fusion subcategory generated by $m$ in $\text{Vec}_{\mathbb{Z}_M}$. Thus the non-invertible simple object $Z \boxtimes m$ of $\mathcal{C}_M$ generates a fusion subcategory $\mathcal{D}_2$ equivalent to $\mathcal{I}_N$.

Consider the braiding on $\mathcal{C}_M$ induced by some braiding in $\mathcal{I}$ and the trivial half-braiding in $\text{Vec}_{\mathbb{Z}_M}$. With respect to such braiding, the fusion subcategories $\mathcal{D}_1$ and $\mathcal{D}_2$ centralize each other. In addition, since $\text{FPdim} \mathcal{D}_1 = m$ and $\text{FPdim} \mathcal{D}_2 = 2^{N+1}$ are coprime, then $\mathcal{D}_1 \cap \mathcal{D}_2 \cong \text{Vec}$. Therefore, $\mathcal{D}_1 \vee \mathcal{D}_2 \cong \mathcal{D}_1 \boxtimes \mathcal{D}_2$, by [15, Proposition 7.7]. Since $\text{FPdim} (\mathcal{D}_1 \boxtimes \mathcal{D}_2) =$
$2^{N+1}m = \text{FPdim} \mathcal{C}_M$, then $\mathcal{C}_M = \mathcal{D}_1 \vee \mathcal{D}_2 \cong \mathcal{D}_1 \boxtimes \mathcal{D}_2 \cong \mathcal{I}_N \boxtimes \text{Vec}_{\mathbb{Z}_m}$, as was to be shown.

Let $\omega$ be a 3-cocycle on $\mathbb{Z}_M$. Recall from Subsection 2.2 that $\mathcal{C}_M^\omega$ denotes the fusion category obtained from $\mathcal{C}_M$ by twisting the associator with $\omega$.

It follows from [5, Lemma 2.12] that, for every 3-cocycle $\omega$ on $\mathbb{Z}_M$, the fusion category $\mathcal{C}_M^\omega$ has a concrete realization as the fusion subcategory of $\mathcal{I} \otimes \text{Vec}_{\mathbb{Z}_M}^\omega$ generated by the simple object $Z \boxtimes 1$.

For every $M$th root of 1, $\zeta \in k^\times$, we shall denote by $\mathcal{C}_{M,\zeta}$ the fusion category obtained from $\mathcal{C}_M$ by twisting the associator with the 3-cocycle $\omega_\zeta$ defined by formula (2.3). Letting $M = 2^N$, we obtain $2^N$ fusion categories $\mathcal{I}_{N,\zeta}$ which are 3-cocycle twists of $\mathcal{I}_N = \mathcal{I}_{N,1}$. For $\zeta_1 \neq \zeta_2$, the fusion categories $\mathcal{I}_{N,\zeta_1}$ and $\mathcal{I}_{N,\zeta_2}$ are non-equivalent as $\mathbb{Z}_{2^N}$-extensions of $\text{Vec}_{\mathbb{Z}_2}$. We stress that, for fixed $N$, all the categories $\mathcal{I}_{N,\zeta}$ share the same fusion rules.

**Definition 4.6.** For $N \geq 1$, $\zeta \in G_{2^N}$, the category $\mathcal{I}_{N,\zeta}$ will be called an $N$-Ising fusion category.

Recall that a fusion category $\mathcal{C}$ has an exact factorization into a product of two fusion subcategories $\mathcal{D}_1$ and $\mathcal{D}_2$ if every simple object of $\mathcal{C}$ has a unique expression of the form $X \otimes Y$, where $X$ and $Y$ are simple objects of $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively. See [9].

It follows from Theorem 4.5 that every fusion category $\mathcal{C}_{M,\zeta}$ has an exact factorization into a product of a pointed fusion subcategory and an $N$-Ising fusion subcategory. The next theorem shows that this decomposition is sharp.

**Theorem 4.7.** Let $N \geq 1$ and let $\zeta \in k^\times$ be a $2^N$th root of 1. Then every proper fusion subcategory of $\mathcal{I}_{N,\zeta}$ is pointed. In particular, the category $\mathcal{I}_{N,\zeta}$ does not admit any proper exact factorization.

**Proof.** It is enough to show the first statement. Let $\mathcal{C} = \mathcal{I}_{N,\zeta}$. Let us identify the universal grading group of $\mathcal{C}$ with the cyclic group $\mathbb{Z}_{2^N}$ of order $2^N$. Let $X = Z \boxtimes 1 \in \mathcal{C}_1$, so that $X$ is a faithful simple object of $\mathcal{C}$. Then the rank of $\mathcal{C}_{2m-1}$ is 1 and the rank of $\mathcal{C}_{2m}$ is 2, for all $m \geq 1$. Since $2m - 1$ is also a generator of $U(\mathcal{C})$, we have that every non-invertible simple object of $\mathcal{C}$ is faithful. This implies that $\mathcal{C}$ contains no proper non-pointed fusion subcategories, as claimed.

Recall that a braided fusion category is called prime if it contains no nontrivial non-degenerate fusion subcategories.

As a consequence of Theorem 4.7 we obtain the primeness of the braided $N$-Ising categories:

**Corollary 4.8.** Let $N \geq 1$ and let $\mathcal{I}_N$ be an $N$-Ising fusion category. Assume that $\mathcal{I}_N$ admits a braiding. Then $\mathcal{I}_N$ is prime.
4.1. Braidings on $\mathcal{N}$-Ising categories. In this subsection we discuss braidings on $N$-Ising fusion categories. If $N = 1$, then $\mathcal{I}_{N,\pm 1}$ are Ising fusion categories and therefore they admit (necessarily non-degenerate) braidings.

**Remark 4.9.** Observe that if a non-degenerate braided fusion category is equivalent to a 3-cocycle twist of one the categories $\mathcal{C}_M$, then $M/2$ must be odd. In fact, by [17, Lemma 5.4 (ii)], every non-degenerate fusion category with generalized Tambara-Yamagami fusion rules has a non-invertible self-dual simple object. In particular, with respect to any possible braiding, an $N$-Ising fusion category is non-degenerate if and only if $N = 1$.

Let $M \geq 1$ be any even natural number. Consider the braiding in $\mathcal{C}_M$ induced by some fixed braiding in $\mathcal{I}$ and the trivial braiding in $\text{Vec}_{\mathbb{Z}^M}$. Then the M"uger center $\mathcal{Z}_2(\mathcal{C}_M)$ is $\mathcal{C}_M \cap \mathcal{C}_M'$, where $\mathcal{C}_M'$ is the M"uger centralizer of $\mathcal{C}_M$ in $\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}^M}$. Since $\mathcal{C}_M$ is generated by the simple object $\mathbb{1} \boxtimes \text{Vec}_{\mathbb{Z}^M}$, then $\mathcal{C}_M' = \mathbb{1} \boxtimes \text{Vec}_{\mathbb{Z}^M}$ and therefore $\mathcal{Z}_2(\mathcal{C}_M) \cong \text{Vec}_{\mathbb{Z}^M/2}$ is Tannakian. Hence for this particular braiding, the category $\mathcal{C}_M$ is not slightly degenerate neither.

Note that, by Lemma 2.4, each of the categories $\text{Vec}_{\mathbb{Z}^N}$ and $\text{Vec}_{\mathbb{Z}^{-1}^N}$ admits a braiding. Hence $\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}^N}$ and $\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}^{-1}^N}$ admit a braiding and therefore the same holds for their fusion subcategories $\mathcal{I}_{N,1}$ and $\mathcal{I}_{N,-1}$.

**Remark 4.10.** Let $N \geq 1$ and let $\zeta \in \mathbb{G}_{2N}$. Suppose that $\mathcal{I}_{N,\zeta}$ admits a braiding. Then $\zeta = \pm 1$ or $\zeta = \pm \sqrt{-1}$.

Indeed, the pointed fusion subcategory $(\mathcal{I}_{N,\zeta})_{\text{pt}}$ is equivalent to $\langle \delta \rangle \boxtimes \langle 2 \rangle \cong \text{Vec}_{\mathbb{Z}^2} \boxtimes \text{Vec}_{\mathbb{Z}_{2N-1}^2}$, where $\bar{\omega}$ is the 3-cocycle on $\mathbb{Z}_{2N-1} \cong \langle 2 \rangle$ corresponding to the restriction of $\omega_{\zeta}$. Thus $\bar{\omega} = \omega_{\zeta^2}$. Since $\text{Vec}_{\mathbb{Z}_{2N-1}^2}$ admits a braiding, Lemma 2.4 implies that $\zeta^2 = \pm 1$. Therefore $\zeta = \pm 1$ or $\zeta = \pm \sqrt{-1}$, as claimed.

In addition, Lemma 3.4 implies that the adjoint subcategory $(\mathcal{I}_{N,\zeta})_{\text{ad}}$ is equivalent to $\text{sVec}$ as braided fusion categories.

**Lemma 4.11.** Let $\zeta \in \mathbb{G}_4$. Then a 2-Ising fusion category $\mathcal{I}_{2,\zeta}$ admits a braiding if and only if $\zeta = \pm 1$.

**Proof.** As observed in Remark 4.10, both $\mathcal{I}_{2,1}$ and $\mathcal{I}_{2,-1}$ admit a braiding.

Suppose conversely that $\mathcal{I}_{2,\zeta}$ admits a braiding. As pointed out in Remark 4.10, $\zeta = \pm 1$ or $\zeta = \pm \sqrt{-1}$. If $\zeta = \pm \sqrt{-1}$, then the pointed subcategory $\langle 2 \rangle$ must be equivalent as a fusion category to $\text{Vec}_{\mathbb{Z}_2^2}^{-1}$. In particular, $\langle 2 \rangle$ is non-degenerate, which contradicts the primeness of $\mathcal{I}_{2,\zeta}$ (see Corollary 4.8). Then we get that $\zeta = \pm 1$. □

**Lemma 4.12.** Suppose that $\mathcal{I}_N$, $N \geq 1$, is a braided $N$-Ising fusion category such that its M"uger center contains a fusion subcategory braided equivalent to the category $\text{sVec}$ of super-vector spaces. Then $\mathcal{I}_N$ is slightly degenerate.
Proof. Let $\mathcal{C} = \mathcal{I}_N$. Then the M"uger center $Z_2(\mathcal{C})$ is a pointed fusion category. Since the group of invertible objects of $\mathcal{C}$ coincides with $\langle \delta \rangle \boxtimes \langle 2 \rangle \cong Z_2 \times Z_{2N-1}$ and $Z_2(\mathcal{C}) \cap C_{ad} \cong \text{Vec}$, then the group of invertible objects of $Z_2(\mathcal{C})$ is cyclic. Combined with Lemma 5.1 below, the assumption implies that $Z_2(\mathcal{C}) \cong s\text{Vec}$ as braided fusion categories. Thus $\mathcal{C}$ is slightly degenerate. $\square$

It was shown in [21, Proposition 4.6] that every slightly degenerate fusion category of Frobenius-Perron dimension 8 is equivalent to a tensor product $s\text{Vec} \boxtimes D$, for some non-degenerate fusion category $D$ of dimension 4. In view of Theorem 4.7, this implies that a 2-Ising fusion category cannot be slightly degenerate.

The next example shows that, for all $N > 2$, the categories $\mathcal{I}_{N,-1}$ admit slightly degenerate braidings.

**Example 4.13.** Suppose that $N > 2$. Recall from Example 2.5 that the fusion category $\text{Vec}^Z_{Z_{2N}}$ admits a non-degenerate braiding if and only if $\zeta = -1$.

Consider the braiding in $\mathcal{I} \boxtimes \text{Vec}_{Z_{2N}}^{-1}$ induced by any fixed braiding in $\mathcal{I}$ and a non-degenerate braiding in $\text{Vec}_{Z_{2N}}^{-1}$. Then $\mathcal{I} \boxtimes \text{Vec}_{Z_{2N}}^{-1}$ is non-degenerate.

Regard $\mathcal{C} = \mathcal{I}_{N,-1}$ as a braided fusion category with the braiding induced from $\mathcal{I} \boxtimes \text{Vec}_{Z_{2N}}^{-1}$. Hence $Z_2(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}'$. Moreover, since $\text{FPdim} \mathcal{I}_{N,-1} = 2^{N+1}$ and $\mathcal{I} \boxtimes \text{Vec}_{Z_{2N}}^{-1}$ is non-degenerate, then $\text{FPdim} \mathcal{C}' = 2$. Since $\mathcal{C}$ is degenerate, then $\mathcal{C}' \subseteq \mathcal{C}$.

Since $\mathcal{I}$ is non-degenerate, then the nontrivial simple object of $\mathcal{C}'$ must be of the form $Y \boxtimes a$, where $a \in Z_{2N}$ is the unique element of order 2 and $Y = 1$ or $Y = \delta$. Suppose that $Y = 1$. Then $1 \boxtimes a$ centralizes $Z \boxtimes 1$ and therefore $a$ centralizes $1 \in Z_{2N}$. This implies that $a$ centralizes $\text{Vec}_{Z_{2N}}^{-1}$, which contradicts the non-degeneracy of $\text{Vec}_{Z_{2N}}^{-1}$. Threfore $Y = \delta$.

Let $q$ be the quadratic form on $\langle \delta \rangle \boxtimes Z_{2N-1}$ associated to the induced braiding in $\mathcal{C}_{pt}$. The observations in Example 2.5, imply that $q(a) = 1$. Since $\delta \boxtimes 0$ is the only nontrivial object of $\mathcal{C}_{ad} \cong s\text{Vec}$, then $q(\delta \boxtimes 0) = -1$. Using that $\delta \boxtimes 0$ centralizes $\mathcal{C}_{pt}$, we get that $q(\delta \boxtimes a) = q(\delta \boxtimes 0)q(1 \boxtimes a) = -1$. This implies that $Z_2(\mathcal{C}) \cong s\text{Vec}$. Then $\mathcal{C} = \mathcal{I}_{N,-1}$ is slightly degenerate.

If $N = 2$ then $a = 2$ and, as observed in Example 2.5, $\langle a \rangle \cong s\text{Vec}$. Hence $Z_2(\mathcal{I}_{2,-1}) = \langle \delta \boxtimes a \rangle \cong \text{Rep} Z_2$ is a Tannakian subcategory.

Observe that in these examples the pointed subcategory of $\mathcal{I}_{N,-1}$ is $\langle \delta \rangle \boxtimes \langle 2 \rangle \cong s\text{Vec} \boxtimes \text{Vec}_{Z_{2N-1}}$.

**Lemma 4.14.** Let $N > 2$. Consider a braiding in $\mathcal{I}_{N,\zeta}$ induced from a braiding in $\mathcal{I} \boxtimes \text{Vec}_{Z_{2N}}^{-1}$. Then $\mathcal{I}_{N,\zeta}$ is slightly degenerate if and only if the
induced braiding in $\text{Vec}_{Z_{2N}}^\zeta$ is non-degenerate. If this is the case, then $\zeta = -1$.

**Proof.** By Lemma 2.4, $\zeta = \pm 1$. In view of Example 2.5, it will be enough to prove the first statement. The 'if' direction was shown in Example 4.13. Suppose conversely that $I_{N,\zeta}$ is slightly degenerate. Note that with respect to any braiding in $\mathcal{I} \boxtimes \text{Vec}_{Z_{2N}}^\zeta$, the subcategory $1 \boxtimes \text{Vec}_{Z_{2N}}^\zeta$ must centralize $\mathcal{I} \boxtimes 0$ projectively. In view of [4, Proposition 3.32], this implies that if $a = 2^{N-1}$ is the unique element of order 2 of $Z_{2N}$, then $1 \boxtimes a$ centralizes $Z \boxtimes 1$. If $1 \boxtimes \text{Vec}_{Z_{2N}}^\zeta$ is degenerate, then its Müger center must contain $1 \boxtimes a$ and therefore $1 \boxtimes a$ centralizes $Z \boxtimes 1$. Since $1 \boxtimes a \in \mathcal{I}_{N,\zeta} = \langle Z \boxtimes 1 \rangle$, then $1 \boxtimes a \in \mathcal{Z}_2(\mathcal{I}_{N,\zeta})$. Hence $\mathcal{Z}_2(\mathcal{I}_{N,\zeta}) = \langle 1 \boxtimes a \rangle$. But, from Formula (2.5), $q(a) = 1$, where $q$ is the quadratic form in $Z_{2N}$ corresponding to the induced braiding in $1 \boxtimes \text{Vec}_{Z_{2N}}^\zeta$. Then $\mathcal{Z}_2(\mathcal{I}_{N,\zeta})$ is Tannakian against the assumption. This shows that $\text{Vec}_{Z_{2N}}^\zeta$ must be non-degenerate and finishes the proof of the lemma. □

**Remark 4.15.** Suppose $\mathcal{C}$ is a slightly degenerate $N$-Ising fusion category and $N > 2$. We have $\mathcal{C}_{pt} \cong \mathcal{C}_{ad} \boxtimes \mathcal{D}$, where $\mathcal{D} = \langle 1 \boxtimes 2 \rangle$ is a pointed fusion category whose group of invertible objects is cyclic of order $2^{N-1}$. This is in fact an equivalence of braided fusion categories since, by Lemma 3.5, $\mathcal{C}_{ad}$ centralizes $\mathcal{C}_{pt}$. Therefore

$$\mathcal{Z}_2(\mathcal{C}_{pt}) \cong \mathcal{C}_{ad} \boxtimes \mathcal{Z}_2(\mathcal{D}). \quad (4.4)$$

On the other hand, using again Lemma 3.5 and [15, Proposition 7.7], we find

$$\mathcal{Z}_2(\mathcal{C}_{pt}) = \mathcal{C}_{ad} \vee \mathcal{Z}_2(\mathcal{C}) \cong \mathcal{C}_{ad} \boxtimes \mathcal{Z}_2(\mathcal{C}). \quad (4.5)$$

From (4.4) and (4.5) we obtain that $\text{FPdim} \mathcal{Z}_2(\mathcal{D}) = 2$. Furthermore, if $\mathcal{Z}_2(\mathcal{D}) \cong \text{sVec}$, then Lemma 5.1 implies that $\text{sVec}$ is a direct factor of $\mathcal{D}$. This is possible only if $N = 2$.

Since $N > 2$, then $\mathcal{Z}_2(\langle 1 \boxtimes 2 \rangle)$ is Tannakian of dimension 2. Hence $\mathcal{Z}_2(\langle 1 \boxtimes 2 \rangle) \cong \langle 1 \boxtimes 2^{N-2} \rangle \cong \text{Rep} Z_2$ and the nontrivial object of $\mathcal{Z}_2(\mathcal{C})$ is $\delta \boxtimes 2^{N-2}$.

### 5. The structure of braided extensions of $\text{Vec}_{Z_2}$

Suppose that $\mathcal{B}$ is a pointed braided fusion category. Corollary A. 19 of [4] states that if the Müger center $\mathcal{Z}_2(\mathcal{B})$ of $\mathcal{B}$ coincides with the category $\text{sVec}$ of super-vector spaces, then the Müger center is a direct factor of $\mathcal{B}$, that is, $\mathcal{B} \cong \text{sVec} \boxtimes \mathcal{B}_0$, for some pointed (necessarily non-degenerate in this case) braided fusion category $\mathcal{B}_0$. However, the proof of [4, Corollary A. 19] only uses the fact that $\text{sVec} \subseteq \mathcal{Z}_2(\mathcal{B})$, in other words, it actually proves the following:
Lemma 5.1. Let $\mathcal{B}$ be a pointed braided fusion category. Suppose that the Müger center of $\mathcal{B}$ contains a fusion subcategory $\mathcal{D}$ braided equivalent to the category $\text{sVec}$ of super-vector spaces. Then $\mathcal{B} \cong \mathcal{D} \boxtimes \mathcal{B}_0$, for some pointed braided fusion category $\mathcal{B}_0$.

Let $\text{Vec}_{\mathbb{Z}_2}^\alpha$ be the pointed fusion category with associativity constraint given by the 3-cocycle $\alpha$, where

$$\alpha(a, b, c) = \begin{cases} 1, & b + c < 2M, \\ \exp\left(\frac{2\pi a}{M}\right), & b + c \geq 2M. \end{cases}$$

Consider the fusion category $\mathcal{D}_{2M}$ of $\mathcal{D} \boxtimes \text{Vec}_{\mathbb{Z}_2}^\alpha$, generated by the simple object $Z \boxtimes 1$. Let $(\mathcal{D}_{2M})_\mathcal{E}$ be the de-equivariantization of the fusion category $\mathcal{D}_{2M}$ by its (central) subcategory $\mathcal{E}$ generated by the invertible object $\delta \boxtimes M$.

The following result is a special instance of the classification of cyclic extensions of fusion categories of adjoint ADE type in [5].

Theorem 5.2. ([5, Lemma 3.10].) Up to twisting the associator by a 3-cocycle $\omega$ on $\mathbb{Z}_M$, every $\mathbb{Z}_M$-extension of $\text{Vec}_{\mathbb{Z}_2}$, $\otimes$-generated by a simple object of Frobenius-Perron dimension less than 2, is equivalent as a fusion category to some of the categories $C_M$ or, if 4 divides $M$, to some of the categories $(\mathcal{D}_{2M})_\mathcal{E}$.

As an application of Theorem 5.2, we obtain:

Theorem 5.3. Let $\mathcal{C}$ be a non-pointed braided fusion category and suppose that $\mathcal{C}$ is a $\mathbb{Z}_M$-extension of the fusion category $\text{Vec}_{\mathbb{Z}_2}$. Then $\mathcal{C}$ is equivalent as a fusion category to $C_M^\omega$, for some 3-cocycle $\omega$ on $\mathbb{Z}_M$.

Proof. By assumption the braided fusion category $\mathcal{C}$ is nilpotent. Since $\mathcal{C}$ is not pointed, then $C_{ad} \cong \text{Vec}_{\mathbb{Z}_2}$ and therefore $U(\mathcal{C}) \cong \mathbb{Z}_M$. Then [17, Theorem 4.7] implies that $\mathcal{C}$ has a faithful simple object $X$ and in addition $X$ is not invertible. Since the homogeneous components of the $\mathbb{Z}_M$-grading of $\mathcal{C}$ have dimension 2, then FPdim $X = \sqrt{2}$ (see Theorem 3.1). Hence $\mathcal{C}$ is $\otimes$-generated by a simple object of Frobenius-Perron dimension less than 2.

In view of Theorem 5.2 we may assume that $\mathcal{C}$ is equivalent to a 3-cocycle twist of one of the fusion categories $(\mathcal{D}_{2M})_\mathcal{E}$, where $M$ is divisible by 4.

Consider the canonical dominant tensor functor $F : \mathcal{D}_{2M} \to (\mathcal{D}_{2M})_\mathcal{E}$, that is, the functor $F$ is the ‘free $A$-module functor’, where $A$ is the regular algebra determined by the Tannakian category $\mathcal{E}$.

The functor $F$ takes a simple object of Frobenius-Perron dimension $\sqrt{2}$ of $\mathcal{D}_{2M}$ to a simple object (of the same Frobenius-Perron dimension) of $(\mathcal{D}_{2M})_\mathcal{E}$. Then $F$ induces a surjective group homomorphism $G(\mathcal{D}_{2M}) \to G((\mathcal{D}_{2M})_\mathcal{E})$ whose kernel is the subgroup $(\delta \boxtimes M)$ generated by $\delta \boxtimes M$. Hence we obtain a group isomorphism $G((\mathcal{D}_{2M})_\mathcal{E}) \cong G(\mathcal{D}_{2M})/(\delta \boxtimes M)$. But $G(\mathcal{D}_{2M}) = (\delta) \boxtimes (2)$, so that $G((\mathcal{D}_{2M})_\mathcal{E}) \cong \mathbb{Z}_M$ is cyclic of order $M$. 


Then the group of invertible objects of \( C \) is cyclic of order \( M \). Since \( C \) is not pointed, then \( C \) has generalized Tambara-Yamagami fusion rules. Then the group of invertible objects of \( C \), being cyclic, must contain a unique subgroup of order 2. This subgroup is necessarily the group of invertible objects of the adjoint subcategory \( C_{ad} \cong \text{Vec}_{\mathbb{Z}_2} \).

By Lemmas 3.4 and 3.5, \( C_{ad} \cong \text{sVec} \) as braided fusion categories and \( \mathbb{Z}_2(C_{pt}) = C_{ad} \vee \mathbb{Z}_2(C) \). Then, by Lemma 5.1, \( C_{pt} \cong C_{ad} \boxtimes B \), for some pointed fusion category \( B \). Since \( G(C) \) is cyclic, we obtain that \( B \) has odd dimension \( n \). This implies that \( M/2 = n \) is odd, against the assumption.

The proof of the theorem is now complete.

\( \square \)

**Remark 5.4.** The proof of Theorem 5.3 shows that (twistings of) the fusion categories \((\mathcal{D}_M)_{\mathcal{E}}\) are not braided unless \( M/2 \) is odd, in which case they are equivalent to a twisting of the fusion category \( \mathcal{C}_M \). When \( M = 4 \), \((\mathcal{D}_M)_{\mathcal{E}}\) has Fermionic Moore-Reed fusion rules. It is known that there are four fusion categories admitting these fusion rules and none of them is braided; see [1], [13].

The following is the main result of this section:

**Theorem 5.5.** Let \( C \) be a non-pointed braided fusion category and suppose that \( C \) is an extension of a rank 2 pointed fusion category. Then \( C \) is equivalent as a fusion category to \( \mathcal{I}_N \boxplus B \), for some \( N \geq 1 \), where \( \mathcal{I}_N \) is a braided \( N \)-Ising fusion category, and \( B \) is a pointed braided fusion category. Furthermore, the categories \( \mathcal{I}_N \) and \( B \) projectively centralize each other in \( C \).

**Proof.** Let \( U(C) \) be the universal grading group of \( C \), denoted additively. Then \( U(C) \) is an Abelian group and \( C = \bigoplus_{a \in U(C)} C_a \), with \( C_0 = C_{ad} \cong \text{Vec}_{\mathbb{Z}_2} \). Then \( C_{ad} \cong \text{sVec} \) as braided fusion categories. We shall denote by \( \delta \) the unique non-invertible simple object of \( C_{ad} \).

Let us identify \( U(C) \) with a direct sum of cyclic groups \( \mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r} \), where the integers \( 2 \leq d_1, \ldots, d_r \) are such that \( d_j \mid d_{j+1} \), for all \( j = 1, \ldots, r-1 \). Let \( e_i \in U(C), 1 \leq i \leq r \), be the canonical generators: \( e_i \) has 1 in the \( i \)th component and 0 in the remaining components.

For each \( 1 \leq i \leq r \), let \( C_{e_i} \) be the homogeneous component of degree \( e_i \) of \( C \). Write the set \( \{1, \ldots, r\} \) as a disjoint union \( \{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\} \), where \( p + q = r \) and the indices \( i_1, \ldots, i_p, j_1, \ldots, j_q \) are such that
\[
\begin{align*}
i_1 &\leq \cdots \leq i_p, \\
&j_1 \leq \cdots \leq j_q,
\end{align*}
\]
the homogeneous components \( C_{e_{i_\ell}}, 1 \leq \ell \leq p \), contain a non-invertible simple object \( Z_{i_\ell} \), and the components \( C_{e_{j_s}}, 1 \leq s \leq q \), contain two non-isomorphic invertible objects \( a_{j_s} \) and \( b_{j_s} \).

**Claim 5.6.** The \( p + 2q \) simple objects
\[
Z_{i_1}, \ldots, Z_{i_p}, a_{j_1}, b_{j_1}, \ldots, a_{j_q}, b_{j_q},
\]
generate the fusion category \( C \).
Proof of the claim. Let $X$ be a simple object of $C$ and suppose that $X \in C_a$, $a \in U(C)$. Since $e_1, \ldots, e_r$ generate $U(C)$, then $a = t_1e_1 + \cdots + t_re_r$, for some non-negative integers $t_1, \ldots, t_r$. Then the tensor product
\[ Z_{i_1}^{t_1} \otimes \cdots \otimes Z_{i_p}^{t_p} \otimes x_{j_1}^{l_{j_1}} \cdots x_{j_q}^{l_{j_q}} \] (5.3)
belongs to $C_a$, where, for all $1 \leq s \leq q$, $x_{js} = a_{js}$ or $b_{js}$.

If $X$ is the unique simple object of $C_a$ up to isomorphism, then the tensor product (5.3) must be isomorphic to a direct sum of copies of $X$. In particular $X$ is a simple constituent of (5.3) and therefore it belongs to the fusion subcategory generated by (5.2).

Note in addition that such a non-invertible simple object $X$ of $C$ must exist, because $C$ is not pointed. Thus if $t_1, \ldots, t_r$ are chosen as above, then (5.3) does not contain any invertible constituent. Hence some of the simple objects in (5.2) must be non-invertible, that is, $p \geq 1$. Since $Z_{i_1} \otimes Z_{i_1}^* \cong 1 \oplus \delta$, then we find that $\delta$ belongs to the fusion subcategory generated by (5.2).

Suppose next that the simple object $X \in C_a$ is invertible. Then the only simple objects of $C_a$ are, up to isomorphism, $X$ and $\delta X$. Also in this case, at least one of the objects $X$ or $\delta X$ is a simple constituent of (5.3) and therefore it belongs to the fusion subcategory generated by (5.2). Then so does the other, because $\delta$ belongs to this subcategory. This proves the claim. $\Box$

By Corollary 3.3 (1), the action of the group of invertible objects of $C$ on the isomorphism classes of non-invertible simple objects is transitive. Then, for all $1 \leq \ell \leq p$,
\[ Z_{i_1} \otimes Z_{i_1}^* \cong g_\ell \oplus \delta g_\ell, \]
for some invertible object $1 \neq g_\ell$ such that
\[ g_\ell \otimes Z_{i_1} \cong Z_{i_\ell}. \] (5.4)
In particular $g_1 = \delta$. Then $g_\ell$ and $\delta g_\ell$ are, up to isomorphism, the unique simple objects of $C_{e_{i_1} - e_{i_\ell}}$.

Let $\mathcal{B}$ be the pointed fusion subcategory of $C$ generated by the invertible objects
\[ a_{j_1}, b_{j_1}, \ldots, a_{j_q}, b_{j_q}, g_1, g_2, \ldots, g_p. \] (5.5)

Since $\delta = g_1$ generates $C_{ad}$, then $sVec \cong C_{ad} \subseteq \mathcal{B}$. But by Lemma 3.5, $C_{ad}$ centralizes $\mathcal{B}$. Lemma 5.1 implies that $\mathcal{B} \cong C_{ad} \boxtimes \mathcal{B}_0$ for some pointed fusion category $\mathcal{B}_0$. Note that the degree of homogeneity $b$ of a simple object of $\mathcal{B}_0$ is of the form
\[ b = s_2(e_{i_1} - e_{i_2}) + \cdots + s_p(e_{i_1} - e_{i_p}) + n_1e_{j_1} + \cdots + n_qe_{j_q} \]
\[ = he_{i_1} - s_2e_{i_2} - \cdots - s_pe_{i_p} + n_1e_{j_1} + \cdots + n_qe_{j_q} \]
for some non-negative integers $s_2, \ldots, s_p, n_1, \ldots, n_q$, where $h = s_2 + \cdots + s_p$. 
Let \( Z = Z_{i_1} \). Relation (5.4) and Claim 5.6 imply that the fusion subcategory \( \langle Z \rangle \) generated by \( Z \) and \( B_0 \) generate \( C \). By commutativity of the fusion rules of \( C \), we obtain that every simple object \( Y \) of \( C \) decomposes in the form

\[
Y \cong X \otimes g, \quad (5.6)
\]

for some simple object \( X \) of \( \langle Z \rangle \) and some invertible object \( g \in B_0 \).

Suppose that \( X, X' \in \langle Z \rangle \) and \( g, g' \in B_0 \) are simple objects such that

\[
X \otimes g \cong X' \otimes g'. \quad (5.7)
\]

Then \( X \otimes g(g')^{-1} \in \langle Z \rangle \) and thus \( g(g')^{-1} \) is a simple constituent of \( Z^\otimes m \), for some \( m \geq 0 \). In particular, \( g(g')^{-1} \) is homogeneous of degree \( me_{i_1} \).

On the other hand, \( g(g')^{-1} \in B_0 \). Then

\[
me_{i_1} = he_{i_1} - s_2 e_{i_2} - \cdots - s_p e_{i_p} + n_1 e_{j_1} + \cdots + n_q e_{i_q},
\]

for some non-negative integers \( s_2, \ldots, s_p, n_1, \ldots, n_q \), and \( h = s_2 + \cdots + s_p \). Therefore \( d_{i_1} | s_2, \ldots, d_{i_p} | s_p \) and \( d_{j_1} | n_1, \ldots, d_{j_q} | n_q \). From condition (5.1), we have that \( d_{i_1} | d_{i_2} | \cdots | d_{i_p} \). Hence \( d_{i_1} | h \) and \( g(g')^{-1} \in C_0 = C_{ad} \). Therefore \( g(g')^{-1} \cong 1 \), by the definition of \( B_0 \). Then \( g \cong g' \) and also \( X \cong X' \), by (5.7).

We have thus shown that the factorization (5.6) of a simple object of \( C \) is unique up to isomorphism. By [9, Theorem 3.8], \( C \) has an exact factorization into a product of its fusion subcategories \( \langle Z \rangle \) and \( B_0 \). Since \( C \) is braided, then \( C \cong \langle Z \rangle \boxtimes B_0 \) as fusion categories and the categories \( \langle Z \rangle \) and \( B_0 \) projectively centralize each other, by [9, Corollary 3.9]. Since \( \langle Z \rangle \) is a cyclic extension of \( \text{Vec}_{Z_2} \), then Theorems 5.3 and 4.5 imply that \( \langle Z \rangle \cong \mathcal{I}_{N, \zeta} \boxtimes D \), for some \( N \geq 1 \), where \( \zeta \) is a \( 2^N \)th root of 1, and \( D \) is a pointed braided fusion category, such that \( \mathcal{I}_{N, \zeta} \) and \( D \) centralize each other. Letting \( B = D \boxtimes B_0 \), we obtain the theorem. \( \square \)

Keep the notation in Theorem 5.5. Observe that the equivalence stated in the theorem is in principle a tensor equivalence, rather than a braided equivalence. The following question was asked by the referee:

**Question 5.7.** Is there an explicit example where such an equivalence is actually not braided?

For instance, the answer to this question is negative if \( \text{FPdim} \mathcal{C} = 4m \), with \( m \) odd. Indeed in this case we must have \( N = 1 \) and therefore the category \( \mathcal{I}_N \) would be non-degenerate, forcing \( C \) to be braided equivalent to a tensor product of \( \mathcal{I}_N \) and the pointed braided fusion category \( \mathcal{I}'_N \) (see Theorem 2.2).

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References


A CLASS OF PRIME FUSION CATEGORIES OF DIMENSION $2^N$


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