A class of prime fusion categories of dimension $2^N$

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Abstract. We study a class of strictly weakly integral fusion categories $\mathcal{C}_{N,\zeta}$, where $N \geq 1$ is a natural number and $\zeta$ is a $2^N$th root of unity, that we call $N$-Ising fusion categories. An $N$-Ising fusion category has Frobenius-Perron dimension $2^{N+1}$ and is a graded extension of a pointed fusion category of rank 2 by the cyclic group of order $\mathbb{Z}_2N$. We show that every braided $N$-Ising fusion category is prime and also that there exists a slightly degenerate $N$-Ising braided fusion category for all $N > 2$. We also prove a structure result for braided extensions of a rank 2 pointed fusion category in terms of braided $N$-Ising fusion categories.

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1. Introduction

Among the most basic examples of fusion categories, the pointed fusion categories are those whose simple objects are invertible. A pointed fusion category is determined by its group of invertible objects $G$ and the cohomology class of a 3-cocycle $\omega$ on $G$, who is responsible for the associativity constraint. We denote by $\text{Vec}_G^\omega$ the pointed fusion category associated to the pair $(G, \omega)$. 
Let $G$ be a finite group. A fusion category $\mathcal{C}$ is called a $G$-extension of a fusion category $\mathcal{D}$ if it admits a faithful grading by the group $G$,
\[
\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g,
\]
such that $\mathcal{C}_g \otimes \mathcal{C}_h \subseteq \mathcal{C}_{gh}$, for all $g, h \in G$, and the trivial homogeneous component is equivalent to $\mathcal{D}$ [10]. Thus, a fusion category $\mathcal{C}$ is pointed if and only if $\mathcal{C}$ is a $G$-extension of the fusion category $\text{Vec}$ of finite dimensional vector spaces, for some finite group $G$.

An Ising category is a fusion category of Frobenius-Perron dimension 4 which is not pointed. Ising categories appear in Conformal Field Theory related to 2-dimensional Ising models.

Every Ising fusion category is a $\mathbb{Z}_2$-extension of the rank 2 pointed fusion category $\text{Vec}_{\mathbb{Z}_2}$ and it belongs to the class of fusion categories classified by Tambara and Yamagami in [20]; in particular there exist exactly 2 Ising fusion categories up to equivalence, and they are a 3-cocycle twist of each other.

By the main result of [19], every Ising fusion category admits exactly 4 non-equivalent braidings. In particular all such braidings are non-degenerate. Several properties of Ising fusion categories are studied in [4, Appendix B]. See Subsection 2.4.

In this paper we study a family of examples of fusion categories that are obtained from Ising fusion categories and share some features with them. We call them $N$-Ising fusion categories. They are special instances of the cyclic extensions of adjoint categories of ADE type classified in [5] and are defined as follows: Let $\mathcal{J}$ be the semisimplification of the representation category of $U_q(sl_2)$, with $q = \exp(i\pi/4)$. Then $\mathcal{J}$ is an Ising fusion category. Let $Z$ be the non-invertible simple object of $\mathcal{J}$. Then an $N$-Ising category is defined as a 3-cocycle twist of the fusion subcategory of $\mathcal{J} \boxtimes \text{Vec}_{\mathbb{Z}_2} \otimes 1$; c.f. Section 4. (The definition of a 3-cocycle twist of a group-graded fusion category is recalled in Subsection 2.2.)

A 1-Ising fusion category is thus an Ising fusion category. For every $N \geq 1$, an $N$-Ising fusion category has Frobenius-Perron dimension $2^{N+1}$ and is a graded extension of a pointed fusion category of rank 2 by the cyclic group of order $\mathbb{Z}_{2^N}$. In addition every $N$-Ising fusion category is strictly weakly integral. Its group of invertible objects is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_{2^{N-1}}$ and it has $2^{N-1}$ simple objects of Frobenius-Perron dimension $\sqrt{2}$, none of which is self-dual except in the case $N = 1$.

As graded extensions of $\text{Vec}_{\mathbb{Z}_2}$, $N$-Ising fusion categories are parameterized by the integer $N$ and a $2^{N+1}$th root of unity $\zeta$. The corresponding category is denoted $\mathcal{J}_{N,\zeta}$. We use the notation $\mathcal{J}_N$ to indicate the category $\mathcal{J}_{N,1}$.
Every $N$-Ising fusion category $\mathcal{I}_{N,\pm 1}$ admits the structure of a braided fusion category. We show that a braided $N$-Ising fusion category is always prime (Corollary 4.8), that is, it does not contain any nontrivial non-degenerate fusion subcategory. We also show that with respect to any possible braiding, an $N$-Ising fusion category is non-degenerate if and only if $N = 1$. In addition, we prove that a slightly degenerate braided $N$-Ising category exists if $N > 2$. See Subsection 4.1. We point out that the classification of slightly degenerate fusion categories of Frobenius-Perron dimension 8 in [21, Proposition 4.6] implies that a 2-Ising fusion category cannot be slightly degenerate.

Observe that, as shown in [5], when $N \geq 2$ there is another family of non-pointed $\mathbb{Z}_2^N$-extensions of $\text{Vec}_{\mathbb{Z}_2}$ which is not equivalent to any $N$-Ising fusion category. However, the fusion categories in this family do not admit any braiding (Theorem 5.3).

Our main result for braided extensions of a rank 2 pointed fusion category is the following theorem:

**Theorem 5.5.** Let $\mathcal{C}$ be a non-pointed braided fusion category and suppose that $\mathcal{C}$ is an extension of a rank 2 pointed fusion category. Then $\mathcal{C}$ is equivalent as a fusion category to $\mathcal{I}_N \boxtimes \mathcal{B}$, for some $N \geq 1$, where $\mathcal{I}_N$ is a braided $N$-Ising fusion category, and $\mathcal{B}$ is a pointed braided fusion category. Furthermore, the categories $\mathcal{I}_N$ and $\mathcal{B}$ projectively centralize each other in $\mathcal{C}$.

The notion of projective centralizer of a fusion subcategory, introduced in [4], is recalled in Subsection 2.2.

Theorem 5.5 is proved in Section 5. Its proof relies on the classification results of [5]. We point out that Theorem 5.5 applies in particular when $\mathcal{C}$ is a slightly degenerate braided fusion category with generalized Tambara-Yamagami fusion rules, that is, when $\mathcal{C}$ is slightly degenerate, not pointed, and the tensor product of two non-invertible simple objects decomposes as a sum of invertible objects.

The paper is organized as follows. In Section 2 we discuss some preliminary notions and results on fusion categories that will be relevant in the rest of the paper. Section 3 contains some basic results on the structure of a general group extension of a rank 2 pointed fusion category and on braided such extensions that will be needed in the sequel. In Section 4 we introduce $N$-Ising categories and study their main properties. In Section 5 we give a proof of our main result on braided extensions of a rank 2 pointed fusion category.

### 2. Preliminaries

We shall work over an algebraically closed field $k$ of characteristic zero. A fusion category over $k$ is a $k$-linear semisimple rigid tensor category with
finitely many isomorphism classes of simple objects, finite-dimensional vector spaces of morphisms and such that the unit object 1 is simple. We refer the reader to [7], [4] for the main notions on fusion categories and braided fusion categories used throughout.

An object of a fusion category \( C \) is called trivial if it is isomorphic to \( 1^\oplus n \) for some natural number \( n \).

Let \( C \) be a fusion category. The tensor product in \( C \) induces a ring structure in the Grothendieck ring \( K(C) \) of \( C \). By [7, Section 8], there is a unique ring homomorphism \( \text{FPdim} : K(C) \to \mathbb{R} \) such that \( \text{FPdim}(X) \geq 1 \) for all nonzero \( X \in C \). The number \( \text{FPdim}(X) \) is called the Frobenius-Perron dimension of \( X \). The Frobenius-Perron dimension of \( C \) is defined by

\[
\text{FPdim}(C) = \sum_{X \in \text{Irr}(C)} \text{FPdim}(X)^2,
\]

where \( \text{Irr}(C) \) is the set of isomorphism classes of simple objects in \( C \).

A simple object \( X \in C \) is called invertible if \( X \otimes X^* \cong 1 \), where \( X^* \) is the dual of \( X \). Thus \( X \) is invertible if and only if \( \text{FPdim}(X) = 1 \). A fusion category \( C \) is called pointed if every simple object of \( C \) is invertible. Pointed fusion categories whose group of invertible objects is isomorphic to \( G \) are classified by the orbits of the action of the group \( \text{Out}(G) \) in \( H^3(G, k^\times) \). The pointed fusion category corresponding to the class of a 3-cocycle \( \omega \) will be denoted by \( \text{Vec}_\omega G \).

The largest pointed subcategory of \( C \), denoted \( C_{pt} \), is the fusion subcategory generated by all invertible simple objects. The set \( G = G(C) \) of isomorphism classes of invertible objects of \( C \) is a finite group with multiplication given by tensor product. The inverse of \( X \in G \) is its dual \( X^* \). The group \( G \) acts on the set \( \text{Irr}(C) \) by left tensor product multiplication. Let \( G[X] \) be the stabilizer of \( X \in \text{Irr}(C) \) under this action. Then we have a decomposition

\[
X \otimes X^* = \bigoplus_{g \in G[X]} \bigoplus_{Y \in \text{Irr}(C) - G[X]} \dim \text{Hom}(Y, X \otimes X^*) \ Y.
\]

### 2.1. Group extensions of fusion categories

Let \( G \) be a finite group. A fusion category \( C \) is graded by \( G \) if \( C \) has a direct sum decomposition into full abelian subcategories \( C = \bigoplus_{g \in G} C_g \) such that \( C_g \otimes C_h \subseteq C_{gh} \), for all \( g, h \in G \). If \( C_g \neq 0 \), for all \( g \in G \), then the grading is called faithful. When the grading is faithful, \( C \) is called a \( G \)-extension of the trivial component \( C_e \).

If \( C = \bigoplus_{g \in G} C_g \) is a faithful grading of \( C \), then [7, Proposition 8.20] shows that

\[
\text{FPdim}(C) = |G| \text{FPdim}(C_e), \quad \text{FPdim}(C_g) = \text{FPdim}(C_h), \quad \forall g, h \in G.
\]

It follows from the results of [10] that every fusion category \( C \) has a canonical faithful grading \( C = \bigoplus_{g \in U(C)} C_g \) with trivial component \( C_e = C_{ad} \), where \( C_{ad} \) is the adjoint subcategory of \( C \), that is, the fusion subcategory generated
by the simple constituents of $X \otimes X^*$, for all $X \in \text{Irr}(\mathcal{C})$. This grading is called the universal grading of $\mathcal{C}$, and $U(\mathcal{C})$ is called the universal grading group of $\mathcal{C}$. Any other faithful grading $\mathcal{C} = \oplus_{g \in G} \mathcal{C}_g$ of $\mathcal{C}$ is determined by a surjective group homomorphism $\pi : U(\mathcal{C}) \to G$. Hence the trivial component $\mathcal{C}_e$ contains $\mathcal{C}_{ad}$.

Let $G$ be a finite group and let $\mathcal{C}$ be a $G$-extension of a fusion category $\mathcal{D} \cong \mathcal{C}_e$. Let also $\omega \in Z^3(G, k^\times)$ be a 3-cocycle. We shall denote by $\mathcal{C}^\omega$ the fusion category obtained from $\mathcal{C}$ by twisting the associator with $\omega$. For $\omega_1, \omega_2 \in Z^3(G, k^\times)$, the categories $\mathcal{C}^\omega_1$ and $\mathcal{C}^\omega_2$ are equivalent as $G$-extensions of $\mathcal{D}$ if and only if the classes of $\omega_1$ and $\omega_2$ coincide in $H^3(G, k^\times)$. See [8].

### 2.2. Braided fusion categories

A braided fusion category $\mathcal{C}$ is a fusion category admitting a braiding $c$, that is, a family of natural isomorphisms: $c_{X,Y} : X \otimes Y \to Y \otimes X$, $X, Y \in \mathcal{C}$, obeying the hexagon axioms.

Let $\mathcal{C}$ be a braided fusion category. Two objects $X, Y \in \mathcal{C}$ are said to centralize each other if $c_{Y,X}c_{X,Y} = \text{id}_{X \otimes Y}$. The centralizer $\mathcal{D}'$ of a fusion subcategory $\mathcal{D} \subseteq \mathcal{C}$ is the full subcategory of objects which centralize every object of $\mathcal{D}$, that is

$$\mathcal{D}' = \{ X \in \mathcal{C} \mid c_{Y,X}c_{X,Y} = \text{id}_{X \otimes Y}, \forall Y \in \mathcal{D} \}.$$ 

The Müger center $Z_2(\mathcal{C})$ of a braided fusion category $\mathcal{C}$ is the centralizer $\mathcal{C}'$ of $\mathcal{C}$ itself. A braided fusion category $\mathcal{C}$ is called non-degenerate if $Z_2(\mathcal{C})$ is equivalent to the category $\text{Vec}$ of finite-dimensional vector spaces. A braided fusion category $\mathcal{C}$ is called slightly degenerate if $Z_2(\mathcal{C})$ is equivalent to the category $\text{sVec}$ of finite-dimensional super-vector spaces.

Two full subcategories $\mathcal{D}$ and $\tilde{\mathcal{D}}$ of $\mathcal{C}$ are said to projectively centralize each other if for all simple objects $X \in \mathcal{D}$ and $Y \in \tilde{\mathcal{D}}$, the squared braiding $c_{Y,X}c_{X,Y}$ is a scalar multiple of the identity $\text{id}_{X \otimes Y}$. See [4, Subsection 3.3].

Suppose that $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are fusion subcategories of $\mathcal{C}$ that projectively centralize each other. Then [4, Proposition 3.32] shows that there exist finite groups $G$ and $\tilde{G}$ endowed with a non-degenerate pairing $b : G \times \tilde{G} \to k^\times$ and faithful gradings $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$, $\tilde{\mathcal{D}} = \bigoplus_{g \in \tilde{G}} \tilde{\mathcal{D}}_g$, such that $\mathcal{D}_0 = \mathcal{D} \cap \tilde{\mathcal{D}}$, $\tilde{\mathcal{D}}_0 = \mathcal{D}' \cap \tilde{\mathcal{D}}$, and for all homogeneous simple objects $X \in \mathcal{D}_g$, $Y \in \tilde{\mathcal{D}}_h$, $g \in G$, $h \in \tilde{G}$, the squared braiding $c_{Y,X}c_{X,Y}$ is given by

$$c_{Y,X}c_{X,Y} = b(g, h) \text{id}_{X \otimes Y}.$$ 

A braided fusion category $\mathcal{C}$ is called symmetric if $Z_2(\mathcal{C}) = \mathcal{C}$. Hence the Müger center of a braided fusion category is a symmetric fusion category.

A symmetric fusion category $\mathcal{C}$ is called Tannakian if it is equivalent to the category $\text{Rep}(G)$ of finite-dimensional representations of a finite group $G$, as braided fusion categories.

Let $\mathcal{C}$ be a symmetric fusion category. Deligne proved that there exist a finite group $G$ and a central element $u$ of order 2, such that $\mathcal{C}$ is equivalent to
the category $\text{Rep}(G, u)$ of representations of $G$ on finite-dimensional super vector spaces, where $u$ acts as the parity operator [3].

The symmetric category $\mathcal{C}$ is either Tannakian or a $\mathbb{Z}_2$-extension of a Tannakian subcategory. Therefore, if $\text{FPdim}(\mathcal{C})$ is odd, then $\mathcal{C}$ is Tannakian. Moreover if $\text{FPdim}(\mathcal{C})$ is bigger than 2 then $\mathcal{C}$ necessarily contains a Tannakian subcategory. Also, a non-Tannakian symmetric fusion category of Frobenius-Perron dimension 2 is equivalent to the category $\text{sVec}$. See [4, Subsection 2.12].

The following proposition is a special case of Corollary 3.26 of [4].

**Proposition 2.1.** Let $\mathcal{C}$ be a braided fusion category. Then $\mathcal{C}_{ad} \subseteq (\mathcal{C}_{pt})'$. 

The following theorem is due to Drinfeld et al. In the case when $\mathcal{C}$ is modular, it is due to Müger [16, Theorem 4.2].

**Theorem 2.2.** [4, Theorem 3.13] Let $\mathcal{C}$ be a braided fusion category and let $\mathcal{D}$ be a non-degenerate subcategory of $\mathcal{C}$. Then $\mathcal{C}$ is braided equivalent to $\mathcal{D} \boxtimes \mathcal{D}'$, where $\mathcal{D}'$ is the centralizer of $\mathcal{D}$ in $\mathcal{C}$.

For a pair of fusion subcategories $\mathcal{A}, \mathcal{B}$ of $\mathcal{D}$, we use the notation $\mathcal{A} \vee \mathcal{B}$ to indicate the smallest fusion subcategory of $\mathcal{C}$ containing $\mathcal{A}$ and $\mathcal{B}$. The following result will be used frequently.

**Lemma 2.3.** [4, Corollary 3.11] Let $\mathcal{C}$ be a braided fusion category. If $\mathcal{D}$ is any fusion subcategory of $\mathcal{C}$ then $\mathcal{D}'' = \mathcal{D} \vee \mathcal{Z}_2(\mathcal{C})$.

### 2.3. Pointed braided fusion categories.

We recall in this subsection some facts related to the classification of pointed braided fusion categories. We refer the reader to [12], [18], [4] for a detailed exposition.

Let $G$ be a finite abelian group. An *abelian 3-cocycle* on $G$ with values in $k^\times$ is a pair $(\omega, \sigma)$, where $\omega : G \times G \times G \to k^\times$ is a normalized 3-cocycle and $\sigma : G \times G \to k^\times$ is a 2-cochain such that

$$\omega(a, b, c) \omega(b, c, a) \sigma(a, bc) = \omega(b, a, c) \sigma(a, b) \sigma(a, c),$$

for all $a, b, c \in G$. Abelian 3-cocycles form an abelian group $Z^3_{ab}(G, k^\times)$. Let $B^3_{ab}(G, k^\times) \subseteq Z^3_{ab}(G, k^\times)$ be the subgroup of abelian coboundaries, that is, abelian 3-cocycles of the form $(du, u(u_{21})^{-1})$ where $u : G \times G \to k^\times$ is a normalized 2-cochain, $du(a, b, c) = u(b, c) u(ab, c)^{-1} u(a, bc) u(a, b)^{-1}$, and $u_{21}$ is defined as $u_{21}(a, b) = u(b, a)$, for all $a, b, c \in G$.

The quotient $H^3_{ab}(G, k^\times) = Z^3_{ab}(G, k^\times) / B^3_{ab}(G, k^\times)$ is called the *abelian cohomology group* of $G$ with coefficients in $k^\times$. Every braiding of a pointed fusion category with group $G$ of invertible objects corresponds to an element of the group $H^3_{ab}(G, k^\times)$. In particular, given a normalized 3-cocycle $\omega$ and a 2-cochain $\sigma$ on $G$, we have that the rule

$$\sigma_{a,b} \text{id}_{ab} : a \otimes b \to b \otimes a, \quad a, b \in G,$$

defines a braiding in the fusion category $\text{Vec}_{G}^{\omega}$ if and only if $(\omega, \sigma) \in Z^3(G, k^\times)$. 
A quadratic form on $G$ with values in $k^\times$ is a map $q : G \to k^\times$ satisfying $q(g) = q(g^{-1})$, for all $g \in G$, and such that the map $\beta : G \times G \to k^\times$ defined by $\beta(a, b) = q(ab)q(a)^{-1}q(b)^{-1}$ is a symmetric bicharacter on $G$. If $q$ is a quadratic form on $G$, then the pair $(G, q)$ is called a pre-metric group.

To every abelian 3-cocycle $(\omega, \sigma)$ on $G$ one can associate a quadratic form on $G$ defined by

$$q(g) = \sigma(g, g), \quad g \in G. \quad(2.2)$$

A result of Eilenberg and Mac Lane states that this correspondence defines a group isomorphism between the abelian cohomology group $H^3_{ab}(G, k^\times)$ and the abelian group of quadratic forms on $G$.

Moreover, the functor that associates to every pointed fusion category $\mathcal{C}$ the pre-metric group $(G, q)$, where $G$ is the group of invertible objects of $\mathcal{C}$ and $q$ is the quadratic form (2.2), where $\sigma$ is the braiding of $\mathcal{C}$, defines an equivalence between the category of pointed fusion categories and braided functors up to braided isomorphism and the category of pre-metric groups.

Thus, two braided fusion categories $\mathcal{C}(G, q)$ and $\mathcal{C}(G, q')$ associated to the quadratic forms $q$ and $q'$ on $G$ are equivalent if and only if there exists an automorphism $\varphi$ of $G$ such that $q'(\varphi(g)) = q(g)$, for all $g \in G$.

The squared braiding of the braided fusion category $\mathcal{C}(G, q)$ associated to a quadratic form $q$ is given by the symmetric bilinear form $\beta : G \times G \to k^\times$ associated to $q$.

Let $M$ be a natural number and let $G = \mathbb{Z}_M$ be the cyclic group of order $M$. Let also $\zeta \in k^\times$ be an $M$th root of 1. Then $\zeta$ determines a 3-cocycle $\omega_\zeta$ on $\mathbb{Z}_M$ where, for all $0 \leq i, j, \ell \leq M - 1$,

$$\omega_\zeta(i, j, \ell) = \begin{cases} 1, & \text{if } j + \ell < M, \\ \zeta^i, & \text{if } j + \ell \geq M. \end{cases} \quad(2.3)$$

The assignment $\zeta \mapsto \omega_\zeta$ gives rise to a group isomorphism between the group $G_M$ of $M$th roots of 1 in $k^\times$ and the group $H^3(\mathbb{Z}_M, k^\times)$. In particular $H^3(\mathbb{Z}_M, k^\times) \cong \mathbb{Z}_M$.

We shall denote by $\text{Vec}_{\mathbb{Z}_M}^\zeta$ the pointed fusion category corresponding to the 3-cocycle $\omega_\zeta$. Thus $\text{Vec}_{\mathbb{Z}_M}^1 = \text{Vec}_{\mathbb{Z}_M}$ and, if $M$ is even, $\text{Vec}_{\mathbb{Z}_M}^{-1}$ is the pointed fusion category corresponding to the 3-cocycle $\omega_{-1}$ associated to $\zeta = -1 \in G_M$.

Let $\xi \in k^\times$ such that $\xi^{M^2} = 1 = \xi^{2M}$. Then the pair $(\omega_\xi, \sigma_\xi)$ is an abelian 3-cocycle on $G$ where, for all $0 \leq i, j, \ell \leq M - 1$,

$$\sigma_\xi(i, j) = \xi^{ij}. \quad(2.4)$$

Furthermore, this gives rise to a group isomorphism between $H^3_{ab}(\mathbb{Z}_M, k^\times)$ and the group $G_d$ of $d$th roots of 1 in $k^\times$, where $d = \gcd(M^2, 2M)$. See [12, pp. 49], [18, Subsection 2.5.2].
Thus $\text{Vec}_{\mathbb{Z}_M}^\xi$ is a braided fusion category whose squared braiding is given by $\beta_\xi(i,j) \, \text{id}_{i+j} : i + j \to i + j$, where $\beta_\xi : \mathbb{Z}_M \times \mathbb{Z}_M \to k^\times$ is the bilinear form defined as

$$\beta_\xi(i,j) = \xi^{2ij}, \quad 0 \leq i, j < M.$$ 

The quadratic form $q : \mathbb{Z}_M \to k^\times$ and the corresponding symmetric bilinear form on $\mathbb{Z}_M$ associated to the braiding (2.4) are given, respectively, by the formulas

$$q(j) = \xi^{j^2}, \quad \beta(i,j) = \xi^{2ij},$$

(2.5) for all $0 \leq i, j \leq M - 1$.

Note that the condition $\xi^{2M} = 1$ forces $\xi^M = \pm 1$. In particular, for a fixed value of $\zeta = \pm 1$, there are exactly $M$ choices for $\xi$. Thus we obtain:

**Lemma 2.4.** If the pointed fusion category $\text{Vec}_{\mathbb{Z}_M}^\zeta$ admits a braiding then $\zeta = \pm 1$. In addition we have:

1. If $M$ is odd, $\text{Vec}_{\mathbb{Z}_M}^\zeta$ does not admit any braiding unless $\zeta = 1$, and in this case, it admits exactly $M$ braidings up to equivalence.
2. If $M$ is even, then each of the categories $\text{Vec}_{\mathbb{Z}_M}$ and $\text{Vec}_{\mathbb{Z}_M}^{-1}$ admits exactly $M$ braidings, up to equivalence.

**Example 2.5.** Let $N \geq 1$ and let $\xi \in k^\times$ be a $2^{N+1}$th root of 1. It follows from formulas (2.5) that the braided fusion category associated to $\xi$ is non-degenerate if and only if $\xi$ is primitive. If this is the case, then the underlying fusion category is $\text{Vec}_{\mathbb{Z}_2N}^{-1}$.

Let $\xi \in k^\times$ be a primitive 8th root of 1. Let $C = C(\mathbb{Z}_4, \xi)$ be the corresponding (non-degenerate) braided fusion category. We get from formulas (2.5) that $q(2) = \xi^4 = -1$. Hence in this case the subcategory $\langle 2 \rangle \subseteq C$ is equivalent to $\text{sVec}$.

### 2.4. Ising categories

An Ising category is a fusion category of Frobenius-Perron dimension 4 which is not pointed. Let $\mathcal{I}$ be an Ising fusion category. Then, up to isomorphism, $\mathcal{I}$ has a unique nontrivial invertible object $\delta$ and a unique non-invertible simple object $Z$. Thus $\text{FPdim } Z = \sqrt{2}$ and the fusion rules of $\mathcal{I}$ are determined by the relation

$$Z^\otimes 2 \cong 1 \oplus \delta.$$ 

(2.6)

In view of the results of [20], there exist exactly 2 non-equivalent Ising fusion categories. The universal grading group of $\mathcal{I}$ is isomorphic to $\mathbb{Z}_2$. The explicit formulas for the associators of Ising categories in [20] imply that if $\mathcal{I}^+$ and $\mathcal{I}^-$ are two non-equivalent Ising categories then, up to an equivalence of fusion categories, any of them is obtained from the other by twisting the associator by the 3-cocycle $\omega_{-1}$ on $\mathbb{Z}_2$ determined by the relation $\omega_{-1}(1,1,1) = -1$. 

Every Ising fusion category admits a braiding and all possible braidings are classified by the main result of [19] (see also [4]); in particular all such braidings are non-degenerate. The category $\mathcal{I}_{pt}$ is equivalent to the category sVec of super-vector spaces as a braided fusion category.

2.5. Equivariantizations and de-equivariantizations. Let $\mathcal{C}$ be a fusion category with an action by tensor autoequivalences $\rho: G \to \text{Aut}_{\otimes}(\mathcal{C})$ of a finite group $G$. The equivariantization $\mathcal{C}^G$ of $\mathcal{C}$ under the action of $G$ is defined as the category of $G$-equivariant objects and $G$-equivariant morphisms of $\mathcal{C}$. Thus, an object of $\mathcal{C}^G$ is a pair $(X, (u_g)_{g \in G})$, where $X$ is an object of $\mathcal{C}$, $u_g: \rho^g(X) \to X$, $g \in G$, is an isomorphism such that

$$u_{gh} \circ \rho^2_{g,h} = u_g \circ \rho^g(u_h),$$

for all $g, h \in G$, where $\rho^2_{g,h}: \rho^g(\rho^h(X)) \to \rho^{gh}(X)$ is the monoidal structure of the action $\rho$. The tensor product of equivariant objects is defined by means of the monoidal structure of the action.

Let $\mathcal{C}$ be a fusion category and let $\mathcal{E} = \text{Rep}(G) \subseteq Z(\mathcal{C})$ be a Tannakian subcategory of the Drinfeld center $Z(\mathcal{C})$ of $\mathcal{C}$ that embeds into $\mathcal{C}$ via the forgetful functor $Z(\mathcal{C}) \to \mathcal{C}$. Then the algebra $A = k^G$ of $k$-valued functions on $G$ is a commutative algebra in $Z(\mathcal{C})$. The de-equivariantization $\mathcal{C}_G$ of $\mathcal{C}$ by $\mathcal{E}$ is the fusion category defined as the category of left $A$-modules in $\mathcal{C}$. See [4] for details on equivariantizations and de-equivariantizations.

The operations of equivariantization and de-equivariantization are inverse to each other: $(\mathcal{C}_G)^G \cong \mathcal{C} \cong (\mathcal{C}^G)_G$. As for their Frobenius-Perron dimensions, we have

$$\text{FPdim}(\mathcal{C}) = |G| \text{FPdim}(\mathcal{C}_G), \quad \text{FPdim}(\mathcal{C}^G) = |G| \text{FPdim}(\mathcal{C}).$$

Given a Tannakian subcategory $\text{Rep}(G)$ of a braided fusion category $\mathcal{C}$, we have an exact sequence of fusion categories (see [2, Section 1]):

$$\text{Rep}(G) \hookrightarrow \mathcal{C} \xrightarrow{F} \mathcal{C}_G,$$

where $\mathcal{C}_G$ is the de-equivariantization of $\mathcal{C}$ by $\text{Rep}(G)$ and $F$ is the forgetful functor. Hence $\text{Rep}(G)$ is the kernel of $F$, that is, the subcategory of $\mathcal{C}$ whose objects have trivial image under $F$.

3. Extensions of a rank 2 pointed fusion category

3.1. General Results. Recall that a generalized Tambara-Yamagami fusion category is a fusion category $\mathcal{C}$ which is not pointed and such that the tensor product of two non-invertible simple objects of $\mathcal{C}$ is a sum of invertible objects. See [13].

Theorem 3.1. Let $\mathcal{C}$ be a $G$-extension of a pointed fusion category $\text{Vec}_{\mathbb{Z}_2}^\omega$. Then the following hold:

1. If $\omega = -1$, then $\mathcal{C}$ is pointed.
(2) If $\omega = 1$, then $\mathcal{C}$ is either pointed or a generalized Tambara-Yamagami fusion category. If the last possibility holds, then:

(i) Up to isomorphism, $\mathcal{C}$ has $2n$ invertible objects and $n$ simple objects of Frobenius-Perron dimension $\sqrt{2}$, for some $n \geq 1$.

(ii) $\mathcal{C}_{ad} \cong \text{Vec}_{\mathbb{Z}_2}$ as fusion categories, and $U(\mathcal{C}) = G$ is of order $2n$.

**Proof.** Let $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ be a faithful grading such that $\mathcal{C}_e = \text{Vec}_{\mathbb{Z}_2}$. Since this grading is faithful, every component $\mathcal{C}_g$ has Frobenius-Perron dimension 2. Since $\mathcal{C}$ is weakly integral, the Frobenius-Perron dimension of every simple object is a square root of some integer [7, Proposition 8.27]. This implies that every component $\mathcal{C}_g$ either contains 2 non-isomorphic invertible objects, or it contains a unique $\sqrt{2}$-dimensional simple object. If $\mathcal{C}$ is not pointed, then the trivial component $\mathcal{C}_e$ is pointed and there exists a component $\mathcal{C}_g$ containing a unique $\sqrt{2}$-dimensional simple object. It follows from [11, Lemma 2.6] that $\omega$ is trivial. Then (1) holds.

Suppose that $\mathcal{C}$ is not pointed. By [10, Theorem 3.10], $\mathcal{C}$ is endowed with a faithful $\mathbb{Z}_2$-grading $\mathcal{C} = \bigoplus_{h \in \mathbb{Z}_2} \mathcal{C}_h$, where the trivial component $\mathcal{C}^0$ is $\mathcal{C}_{ad}$ and $\mathcal{C}^1$ contains all $\sqrt{2}$-dimensional simple objects. Let $X, Y$ be non-invertible simple objects of $\mathcal{C}$. Then $X, Y \in \mathcal{C}^1$ and hence $X \otimes Y \in \mathcal{C}^0$, which implies that $X \otimes Y$ is a direct sum of invertible objects. Hence $\mathcal{C}$ is a generalized Tambara-Yamagami fusion category and (2) holds.

Assume that the number of non-isomorphic $\sqrt{2}$-dimensional simple objects is $n \geq 1$. Then $2n = \text{FPdim}(\mathcal{C}^1) = \text{FPdim}(\mathcal{C}^0)$. Hence $|G| = 2n$ and we get part (i).

Since $\mathcal{C}_{ad} \subseteq \mathcal{C}_e \cong \text{Vec}_{\mathbb{Z}_2}$, we know $\mathcal{C}_{ad} = \text{Vec}$ or $\text{Vec}_{\mathbb{Z}_2}$. Since $\mathcal{C}$ is not pointed, then $\mathcal{C}_{ad}$ cannot be $\text{Vec}$. Therefore $\mathcal{C}_{ad} = \mathcal{C}_e$ and $G = U(\mathcal{C})$. In particular the order of $U(\mathcal{C})$ is $2n$. This proves part (ii). \[\square\]

For a fusion category $\mathcal{C}$, let $\text{cd}(\mathcal{C})$ denote the set of Frobenius-Perron dimensions of simple objects of $\mathcal{C}$.

**Corollary 3.2.** Let $\mathcal{C}$ be a non-pointed fusion category. Then $\mathcal{C}$ is an extension of a rank 2 pointed fusion category if and only if $\text{cd}(\mathcal{C}) = \{1, \sqrt{2}\}$.

**Proof.** In view of Theorem 3.1, it will be enough to show that the condition $\text{cd}(\mathcal{C}) = \{1, \sqrt{2}\}$ implies that $\mathcal{C}$ is an extension of a rank 2 pointed fusion category. So assume that $\text{cd}(\mathcal{C}) = \{1, \sqrt{2}\}$.

As in the proof of Theorem 3.1 we get that $\mathcal{C}$ is a generalized Tambara-Yamagami fusion category. Then, by [17, Proposition 5.2], the adjoint subcategory $\mathcal{C}_{ad}$ coincides with the fusion subcategory generated by $G[X]$, for any $\sqrt{2}$-dimensional simple object $X$. Hence $\text{FPdim}(\mathcal{C}_{ad}) = 2$ and $\mathcal{C}$ is an extension of a rank 2 pointed fusion category. \[\square\]

**Corollary 3.3.** Let $\mathcal{C}$ be a $G$-extension of $\text{Vec}_{\mathbb{Z}_2}$. Assume that $\mathcal{C}$ is not pointed. Then the following hold:
(1) The action of the group $G(C)$ by left (or right) tensor multiplication on the set of non-invertible simple objects of $C$ is transitive.

(2) The group $\mathbb{Z}_2$ is a normal subgroup of $G(C)$.

Proof. Since $C$ is not pointed, Theorem 3.1 implies that $C$ is a generalized Tambara-Yamagami fusion category. The corollary then follows from [17, Lemma 5.1].

3.2. Braided extensions of $\text{Vec}_{\mathbb{Z}_2}$. Throughout this subsection $C$ will be an extension of $\text{Vec}_{\mathbb{Z}_2}$. In addition, we assume that $C$ is braided and not pointed.

Lemma 3.4. The adjoint subcategory $C_{ad}$ is equivalent to $s\text{Vec}$ as braided fusion categories.

Proof. By Theorem 3.1, we know that $C_{ad} \cong \text{Vec}_{\mathbb{Z}_2}$. By [6, Lemma 2.5], $C_{ad} = C_{ad} \cap C_{pt}$ is symmetric. Suppose on the contrary that $C_{ad}$ is Tannakian. Then $C_{ad} \cong \text{Rep}(\mathbb{Z}_2)$ as braided fusion categories and $C$ is a $\mathbb{Z}_2$-equivariantization of a fusion category $C_{\mathbb{Z}_2}$.

The forgetful functor $F: C \to C_{\mathbb{Z}_2}$ is a tensor functor and the image of every object in $C_{ad}$ under $F$ is a trivial object of $C_{\mathbb{Z}_2}$. Let $\delta$ be the unique nontrivial simple object of $C_{ad}$. If $X$ is a non-invertible simple object of $C$ then $X \otimes X^* \cong 1 \oplus \delta$. Hence $F(X \otimes X^*) \cong F(X) \otimes F(X)^* \cong 1 \oplus 1$, which implies that $F(X)$ is not simple. Then the decomposition of $F(X) \otimes F(X)^*$ must contain at least four simple direct summands. This contradiction shows that $C_{ad}$ cannot be Tannakian, and therefore $C_{ad} \cong s\text{Vec}$, as claimed.

Recall that if $D$ is a fusion category with commutative Grothendieck ring and $A$ is a fusion subcategory of $D$, the commutator of $A$ in $D$, denoted by $A^{com}$, is the fusion subcategory of $D$ generated by all simple objects $X$ of $D$ such that $X \otimes X^*$ is contained in $A$ [10].

Lemma 3.5. The following relations hold:

(1) $(C_{ad})' = C_{pt}$ and $\mathbb{Z}_2(C) \subseteq C_{pt}$.

(2) $\mathbb{Z}_2(C_{pt}) = C_{ad} \vee \mathbb{Z}_2(C)$.

Proof. (1) By [4, Proposition 3.25], a simple object $X \in C$ belongs to $(C_{ad})'$ if and only if it belongs to $\mathbb{Z}_2(C)^{com}$; that is, if and only if $X \otimes X^* \in \mathbb{Z}_2(C)$. If $X$ is not invertible then $X \otimes X^* \cong 1 \oplus \delta$ and hence $\delta \otimes X \cong X$, where $\delta$ is unique nontrivial simple object of $C_{ad}$. Hence $s\text{Vec} \subseteq \mathbb{Z}_2(C)$. By Lemma 3.4, $C_{ad} \cong s\text{Vec}$. This is impossible by [14, Lemma 5.4] which says that if $s\text{Vec} \subseteq \mathbb{Z}_2(C)$ then $\delta \otimes Y \not\cong Y$ for any $Y \in C$. Therefore, $(C_{ad})' \subseteq C_{pt}$ is pointed. By Proposition 2.1, $(C_{ad})' \supseteq (C_{pt})'' = C_{pt} \vee \mathbb{Z}_2(C)$. Hence we have

$$C_{pt} \supseteq (C_{ad})' \supseteq C_{pt} \vee \mathbb{Z}_2(C) \cong C_{pt},$$

which shows that $(C_{ad})' = C_{pt}$ and $\mathbb{Z}_2(C) \subseteq C_{pt}$.

(2) By part (1), we have

$$\mathbb{Z}_2(C_{pt}) = C_{pt} \cap (C_{pt})' = C_{pt} \cap (C_{ad})'' = C_{pt} \cap (C_{ad} \vee \mathbb{Z}_2(C)) = C_{ad} \vee \mathbb{Z}_2(C),$$
the third equality by Lemma 2.3. This proves part (2). □

4. \(N\)-Ising categories

In what follows we shall denote by \(\mathcal{I}\) the semisimplification of the representation category of \(U_{-q}(\mathfrak{sl}_2)\), where \(q = \exp(i\pi/4)\). Then \(\mathcal{I}\) is an Ising fusion category; see Subsection 2.4.

Recall that there exist exactly 2 non-equivalent such fusion categories, say \(\mathcal{I}\) and \(\mathcal{I}^\perp\). So that \(\mathcal{I}^\perp\) is obtained from \(\mathcal{I}\) by twisting the associator by the 3-cocycle \(\alpha\) on \(\mathbb{Z}_2\) such that \(\alpha(1,1,1) = -1\).

We shall use the notation \(\mathcal{I}\) to indicate either of the categories \(\mathcal{I}\) or \(\mathcal{I}^\perp\).

As in Subsection 2.4 we shall denote by \(\delta\) the unique nontrivial invertible object of \(\mathcal{I}\) and \(Z\) the unique non-invertible simple object.

Let \(M \geq 2\) be an even natural number. Consider the fusion subcategory \(C_M\) of \(\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}_M}\) generated by the object \(Z \boxtimes 1\). The relation (2.6) implies that \(C_M\) has \(M/2\) non-invertible simple objects:

\[Z_j = Z \boxtimes (2j + 1), \quad 0 \leq j \leq \frac{M}{2} - 1,\]  

(4.1)

and \(M\) invertible objects:

\[\delta^i \boxtimes (2j), \quad 0 \leq i \leq 1, \quad 0 \leq j \leq \frac{M}{2} - 1.\]  

(4.2)

Thus \(\text{FPdim} Z_j = \sqrt{2}\), for all \(j = 0, \ldots, M/2 - 1\) and \(\text{FPdim} C_M = 2M\).

Remark 4.1. Every fusion category \(C_M, M \geq 2\), admits a braiding; to see this it suffices to consider any braiding in \(\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}_M}\) and restrict it to \(C_M\).

The categories \(C_M\) have generalized Tambara-Yamagami fusion rules. Let us denote by \(a = 1 \boxtimes 2 \in C_M\). Explicitly, the fusion rules of \(C_M\) are determined as follows: the group of invertible objects is a direct product \(\langle \delta \rangle \boxtimes \langle a \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_{M/2}\) and

\[Z_j \otimes Z_\ell \cong a^{j+\ell+1} + \delta a^{j+\ell+1}, \quad 0 \leq j, \ell \leq \frac{M}{2} - 1.\]  

(4.3)

Remark 4.2. The categories \(C_M\) are particular cases of the construction in [5] of fusion categories which are cyclic extensions of fusion categories of adjoint ADE type. Note that the adjoint subcategory of \(C_M\) coincides with the subcategory generated by \(\delta\). In particular, \(C_M\) is a \(\mathbb{Z}_M\)-extension of the fusion category of adjoint \(A_3^{(1)}\) type \(\mathcal{I}_{ad} \cong \text{Vec}_{\mathbb{Z}_2}\).

Remark 4.3. The construction of the categories \(C_M\) can be generalized replacing the cyclic group \(\mathbb{Z}_M\) by any finite Abelian group \(A\) as follows: We may suppose that \(A = \mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_r}\), where \(d_1, \ldots, d_r \geq 1\). Let \(e_1, \ldots, e_r\) be the canonical generators of \(A\). Then the fusion subcategory of \(\mathcal{I} \boxtimes A\) generated by the simple objects \(Z \boxtimes e_j, 1 \leq j \leq r\), is an \(A\)-graded extension of \(\text{Vec}_{\mathbb{Z}_2}\). Observe that all the fusion categories arising in this way admit a
braiding (c.f. Remark 4.1). In fact, the examples arising from this construction boil down to the ones obtained from cyclic groups, in view of Theorem 5.5 below.

Let \( N \geq 1 \). In what follows we shall use the notation \( \mathcal{I}_N \) to indicate the fusion category \( \mathcal{C}_{2^N} \) defined above.

**Example 4.4.** As pointed out before, the category \( \mathcal{I}_1 = \mathcal{I} \) is an Ising fusion category. In particular, it is non-degenerate. The category \( \mathcal{I}_2 \) has two non-isomorphic simple objects \( Z_1 \) and \( Z_2 \) of Frobenius-Perron dimension \( \sqrt{2} \). The group of invertible objects is \( \langle \delta \rangle \times \langle a \rangle \cong Z_2 \times Z_2 \) and we have the fusion rules
\[
Z_1^z \cong Z_2, \quad Z_1^{z^2} \cong a \oplus \delta a \cong Z_2^{z^2}.
\]
In particular, \( \mathcal{I}_2 \) does not contain any Ising fusion subcategory.

More generally, the fusion rules (4.3) imply that \( \mathcal{C}_M \) contains a non-invertible self-dual simple object if and only if \( M/2 \) is odd. If this is the case, such self-dual simple object must generate an Ising fusion subcategory. From the non-degeneracy of Ising fusion categories we obtain, for each \( M \) such that \( M/2 \) is odd, an equivalence fusion categories \( \mathcal{C}_M \cong \mathcal{I} \boxtimes \mathcal{B} \) or \( \mathcal{C}_M \cong \mathcal{I}^- \boxtimes \mathcal{B} \), where \( \mathcal{B} \) is a pointed fusion category. Furthermore, these are equivalences of braided fusion categories regardless of the choice of the braiding in the category \( \mathcal{C}_M \). This feature is generalized in Theorem 4.5 below.

**Theorem 4.5.** Let \( M \geq 2 \) be an even natural number. Suppose that \( M = 2^N m \), where \( N \geq 1 \) and \( m \geq 1 \) is odd. Then there is an equivalence of fusion categories \( \mathcal{C}_M \cong \mathcal{I}_N \boxtimes \mathcal{B} \), where \( \mathcal{B} \) is a pointed fusion category. Moreover, with respect to any braiding in \( \mathcal{C}_M \), this is an equivalence of braided fusion categories for an appropriate braiding in \( \mathcal{I}_N \).

**Proof.** It will be enough to show that \( \mathcal{C}_M \cong \mathcal{I}_N \boxtimes \mathcal{B} \) as fusion categories. Indeed, if this is the case, then regardless of the braiding we consider in \( \mathcal{C}_M \), the fusion subcategories \( \mathcal{I}_N \) and \( \mathcal{B} \) must centralize each other, since their Frobenius-Perron dimensions are coprime; see [4, Proposition 3.32].

By assumption, \( Z_M \) is the direct sum of the subgroup generated by \( m \) and the subgroup \( S \cong Z_m \) generated by \( 2^N \). Let \( \mathcal{D}_1 \cong \text{Vec}_{Z_m} \) denote the fusion subcategory of \( \mathcal{C}_M \) generated by \( 1 \boxtimes S \).

We have an equivalence of fusion categories \( \text{Vec}_{Z_{2^N}} \cong \langle m \rangle \subseteq \text{Vec}_{Z_M} \), where \( \langle m \rangle \) is the fusion subcategory generated by \( m \) in \( \text{Vec}_{Z_M} \). Thus the non-invertible simple object \( Z \boxtimes m \) of \( \mathcal{C}_M \) generates a fusion subcategory \( \mathcal{D}_2 \) equivalent to \( \mathcal{I}_N \).

Consider the braiding on \( \mathcal{C}_M \) induced by some braiding in \( \mathcal{I} \) and the trivial half-braiding in \( \text{Vec}_{Z_M} \). With respect to such braiding, the fusion subcategories \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) centralize each other. In addition, since \( \text{FPdim} \mathcal{D}_1 = m \) and \( \text{FPdim} \mathcal{D}_2 = 2^{N+1} \) are coprime, then \( \mathcal{D}_1 \cap \mathcal{D}_2 \cong \text{Vec} \). Therefore, \( \mathcal{D}_1 \vee \mathcal{D}_2 \cong \mathcal{D}_1 \boxtimes \mathcal{D}_2 \), by [15, Proposition 7.7]. Since \( \text{FPdim}(\mathcal{D}_1 \boxtimes \mathcal{D}_2) = \)
Let $\omega$ be a 3-cocycle on $\mathbb{Z}_M$. Recall from Subsection 2.2 that $C_\omega M$ denotes the fusion category obtained from $C_M$ by twisting the associator with $\omega$.

It follows from [5, Lemma 2.12] that, for every 3-cocycle $\omega$ on $\mathbb{Z}_M$, the fusion category $C_\omega M$ has a concrete realization as the fusion subcategory of $I \otimes \text{Vec}_{\mathbb{Z}_M}^\omega$ generated by the simple object $Z \boxtimes 1$.

For every $M$th root of 1, $\zeta \in k^\times$, we shall denote by $C_{M,\zeta}$ the fusion category obtained from $C_M$ by twisting the associator with the 3-cocycle $\omega_\zeta$ defined by formula (2.3). Letting $M = 2^N$, we obtain $2^N$ fusion categories $I_{N,\zeta}$ which are 3-cocycle twists of $I_N = I_{N,1}$. For $\zeta_1 \neq \zeta_2$, the fusion categories $I_{N,\zeta_1}$ and $I_{N,\zeta_2}$ are non-equivalent as $\mathbb{Z}_{2^N}$-extensions of $\text{Vec}_{\mathbb{Z}_2}$. We stress that, for fixed $N$, all the categories $I_{N,\zeta}$ share the same fusion rules.

**Definition 4.6.** For $N \geq 1$, $\zeta \in G_{2^N}$, the category $I_{N,\zeta}$ will be called an $N$-Ising fusion category.

Recall that a fusion category $C$ has an **exact factorization** into a product of two fusion subcategories $D_1$ and $D_2$ if every simple object of $C$ has a unique expression of the form $X \otimes Y$, where $X$ and $Y$ are simple objects of $D_1$ and $D_2$, respectively. See [9].

It follows from Theorem 4.5 that every fusion category $C_{M,\zeta}$ has an exact factorization into a product of a pointed fusion subcategory and an $N$-Ising fusion subcategory. The next theorem shows that this decomposition is sharp.

**Theorem 4.7.** Let $N \geq 1$ and let $\zeta \in k^\times$ be a $2^N$th root of 1. Then every proper fusion subcategory of $I_{N,\zeta}$ is pointed. In particular, the category $I_{N,\zeta}$ does not admit any proper exact factorization.

**Proof.** It is enough to show the first statement. Let $C = I_{N,\zeta}$. Let us identify the universal grading group of $C$ with the cyclic group $\mathbb{Z}_{2^N}$ of order $2^N$. Let $X = Z \boxtimes 1 \in C_1$, so that $X$ is a faithful simple object of $C$. Then the rank of $C_{2m-1}$ is 1 and the rank of $C_{2m}$ is 2, for all $m \geq 1$. Since $2m - 1$ is also a generator of $U(C)$, we have that every non-invertible simple object of $C$ is faithful. This implies that $C$ contains no proper non-pointed fusion subcategories, as claimed.

Recall that a braided fusion category is called **prime** if it contains no nontrivial non-degenerate fusion subcategories.

As a consequence of Theorem 4.7 we obtain the primeness of the braided $N$-Ising categories:

**Corollary 4.8.** Let $N \geq 1$ and let $\mathcal{I}_N$ be an $N$-Ising fusion category. Assume that $\mathcal{I}_N$ admits a braiding. Then $\mathcal{I}_N$ is prime.
4.1. Braidings on $\mathcal{N}$-Ising categories. In this subsection we discuss braidings on $N$-Ising fusion categories. If $N = 1$, then $\mathcal{I}_{N,\pm 1}$ are Ising fusion categories and therefore they admit (necessarily non-degenerate) braidings.

**Remark 4.9.** Observe that if a non-degenerate braided fusion category is equivalent to a 3-cocycle twist of one the categories $\mathcal{C}_M$, then $M/2$ must be odd. In fact, by [17, Lemma 5.4 (ii)], every non-degenerate fusion category with generalized Tambara-Yamagami fusion rules has a non-invertible self-dual simple object. In particular, with respect to any possible braiding, an $N$-Ising fusion category is non-degenerate if and only if $N = 1$.

Let $M \geq 1$ be any even natural number. Consider the braiding in $\mathcal{C}_M$ induced by some fixed braiding in $\mathcal{I}$ and the trivial braiding in $\text{Vec}_{\mathbb{Z}_M}$. Then the M"uger center $Z_2(\mathcal{C}_M)$ is $\mathcal{C}_M \cap \mathcal{C}_M'$, where $\mathcal{C}_M'$ is the M"uger centralizer of $\mathcal{C}_M$ in $\mathcal{J} \otimes \text{Vec}_{\mathbb{Z}_M}$. Since $\mathcal{C}_M$ is generated by the simple object $\mathbb{Z}_M \otimes 1$, then $\mathcal{C}_M' = 1 \otimes \text{Vec}_{\mathbb{Z}_M}$ and therefore $Z_2(\mathcal{C}_M) \cong \text{Vec}_{\mathbb{Z}_M/2}$ is Tannakian. Hence for this particular braiding, the category $\mathcal{C}_M$ is not slightly degenerate neither.

Note that, by Lemma 2.4, each of the categories $\text{Vec}_{\mathbb{Z}_M}$ and $\text{Vec}_{\mathbb{Z}_M}^{-1}$ admits a braiding. Hence $\mathcal{J} \otimes \text{Vec}_{\mathbb{Z}_M}$ and $\mathcal{J} \otimes \text{Vec}_{\mathbb{Z}_M}^{-1}$ admit a braiding and therefore the same holds for their fusion subcategories $\mathcal{I}_{N,1}$ and $\mathcal{I}_{N,-1}$.

**Remark 4.10.** Let $N \geq 1$ and let $\zeta \in \mathbb{G}_{2N}$. Suppose that $\mathcal{J}_{N,\zeta}$ admits a braiding. Then $\zeta = \pm 1$ or $\zeta = \pm \sqrt{-1}$.

Indeed, the pointed fusion subcategory $(\mathcal{J}_{N,\zeta})_{\text{pt}}$ is equivalent to $\langle \delta \rangle \otimes \langle 2 \rangle \cong \text{Vec}_{\mathbb{Z}_2} \otimes \text{Vec}_{\mathbb{Z}_{2N^{-1}}}^\mathbb{Z}_{2N^{-1}}$, where $\hat{\omega}$ is the 3-cocycle on $\mathbb{Z}_{2N^{-1}} \cong \langle 2 \rangle$ corresponding to the restriction of $\omega_{\zeta}$. Thus $\hat{\omega} = \omega_{\zeta^2}$. Since $\text{Vec}_{\mathbb{Z}_{2N^{-1}}}^\mathbb{Z}_{2N^{-1}}$ admits a braiding, Lemma 2.4 implies that $\zeta^2 = \pm 1$. Therefore $\zeta = \pm 1$ or $\zeta = \pm \sqrt{-1}$, as claimed.

In addition, Lemma 3.4 implies that the adjoint subcategory $(\mathcal{J}_{N,\zeta})_{\text{ad}}$ is equivalent to $s\text{Vec}$ as braided fusion categories.

**Lemma 4.11.** Let $\zeta \in \mathbb{G}_4$. Then a 2-Ising fusion category $\mathcal{J}_{2,\zeta}$ admits a braiding if and only if $\zeta = \pm 1$.

**Proof.** As observed in Remark 4.10, both $\mathcal{J}_{2,1}$ and $\mathcal{J}_{2,-1}$ admit a braiding.

Suppose conversely that $\mathcal{J}_{2,\zeta}$ admits a braiding. As pointed out in Remark 4.10, $\zeta = \pm 1$ or $\zeta = \pm \sqrt{-1}$. If $\zeta = \pm \sqrt{-1}$, then the pointed subcategory $\langle 2 \rangle$ must be equivalent as a fusion category to $\text{Vec}_{\mathbb{Z}_2}^{-1}$. In particular, $\langle 2 \rangle$ is non-degenerate, which contradicts the primeness of $\mathcal{J}_{2,\zeta}$ (see Corollary 4.8). Then we get that $\zeta = \pm 1$.

**Lemma 4.12.** Suppose that $\mathcal{I}_N$, $N \geq 1$, is a braided $N$-Ising fusion category such that its M"uger center contains a fusion subcategory braided equivalent to the category $s\text{Vec}$ of super-vector spaces. Then $\mathcal{I}_N$ is slightly degenerate.
Proof. Let $\mathcal{C} = \mathcal{I}_N$. Then the Müger center $\mathcal{Z}_2(\mathcal{C})$ is a pointed fusion category. Since the group of invertible objects of $\mathcal{C}$ coincides with $\langle \delta \rangle \boxtimes \langle 2 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \mathbb{N}$ and $\mathcal{Z}_2(\mathcal{C}) \cap C_{ad} \cong \text{Vec}$, then the group of invertible objects of $\mathcal{Z}_2(\mathcal{C})$ is cyclic. Combined with Lemma 5.1 below, the assumption implies that $\mathcal{Z}_2(\mathcal{C}) \cong s\text{Vec}$ as braided fusion categories. Thus $\mathcal{C}$ is slightly degenerate.

It was shown in [21, Proposition 4.6] that every slightly degenerate fusion category of Frobenius-Perron dimension 8 is equivalent to a tensor product $s\text{Vec} \boxtimes \mathcal{D}$, for some non-degenerate fusion category $\mathcal{D}$ of dimension 4. In view of Theorem 4.7, this implies that a 2-Ising fusion category cannot be slightly degenerate.

The next example shows that, for all $N > 2$, the categories $\mathcal{I}_{N,-1}$ admit slightly degenerate braidings.

**Example 4.13.** Suppose that $N > 2$. Recall from Example 2.5 that the fusion category $\text{Vec}_{\mathbb{Z}_2 N}^\zeta$ admits a non-degenerate braiding if and only if $\zeta = -1$.

Consider the braiding in $\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}_2 N}^{-1}$ induced by any fixed braiding in $\mathcal{I}$ and a non-degenerate braiding in $\text{Vec}_{\mathbb{Z}_2 N}^{-1}$. Then $\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}_2 N}^{-1}$ is non-degenerate.

Regard $\mathcal{C} = \mathcal{I}_{N,-1}$ as a braided fusion category with the braiding induced from $\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}_2 N}^{-1}$. Hence $\mathcal{Z}_2(\mathcal{C}) = \mathcal{C} \cap \mathcal{C}'$. Moreover, since $\text{FPdim} \mathcal{I}_{N,-1} = 2^{N+1}$ and $\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}_2 N}^{-1}$ is non-degenerate, then $\text{FPdim} \mathcal{C}' = 2$. Since $\mathcal{C}$ is degenerate, then $\mathcal{C}' \subseteq \mathcal{C}$.

Since $\mathcal{I}$ is non-degenerate, then the nontrivial simple object of $\mathcal{C}'$ must be of the form $Y \boxtimes a$, where $a \in \mathbb{Z}_2 N$ is the unique element of order 2 and $Y = 1$ or $Y = \delta$. Suppose that $Y = 1$. Then $1 \boxtimes a$ centralizes $\mathbb{Z}_2 \boxtimes 1$ and therefore $a$ centralizes $1 \in \mathbb{Z}_2 N$. This implies that $a$ centralizes $\text{Vec}_{\mathbb{Z}_2 N}^{-1}$, which contradicts the non-degeneracy of $\text{Vec}_{\mathbb{Z}_2 N}^{-1}$. Therefore $Y = \delta$.

Let $q$ be the quadratic form on $\langle \delta \rangle \boxtimes \mathbb{Z}_2 N^{-1}$ associated to the induced braiding in $\mathcal{C}_{pt}$. The observations in Example 2.5, imply that $q(a) = 1$. Since $\delta \boxtimes 0$ is the only nontrivial object of $\mathcal{C}_{pt} \cong s\text{Vec}$, then $q(\delta \boxtimes 0) = -1$. Using that $\delta \boxtimes 0$ centralizes $\mathcal{C}_{pt}$, we get that $q(\delta \boxtimes a) = q(\delta \boxtimes 0)q(1 \boxtimes a) = -1$. This implies that $\mathcal{Z}_2(\mathcal{C}) \cong \text{Rep} \mathbb{Z}_2$ is a Tannakian subcategory.

If $N = 2$ then $a = 2$ and, as observed in Example 2.5, $\langle a \rangle \cong s\text{Vec}$. Hence $\mathcal{Z}_2(\mathcal{I}_{2,-1}) = \langle \delta \boxtimes a \rangle \cong \text{Rep} \mathbb{Z}_2$ is a Tannakian subcategory.

Observe that in these examples the pointed subcategory of $\mathcal{I}_{N,-1}$ is $\langle \delta \rangle \boxtimes \langle 2 \rangle \cong s\text{Vec} \boxtimes \text{Vec}_{\mathbb{Z}_2 N^{-1}}$.

**Lemma 4.14.** Let $N > 2$. Consider a braiding in $\mathcal{I}_N^\zeta$ induced from a braiding in $\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}_2 N}^\zeta$. Then $\mathcal{I}_N^\zeta$ is slightly degenerate if and only if the
induced braiding in $\text{Vec}_{\mathbb{Z}_2^N}^\zeta$ is non-degenerate. If this is the case, then $\zeta = -1$.

**Proof.** By Lemma 2.4, $\zeta = \pm 1$. In view of Example 2.5, it will be enough to prove the first statement. The 'if' direction was shown in Example 4.13. Suppose conversely that $\mathcal{I}_{\mathbb{N},\zeta}$ is slightly degenerate. Note that with respect to any braiding in $\mathcal{I} \sotimes \text{Vec}_{\mathbb{Z}_2^N}^\zeta$, the subcategory $1 \sotimes \text{Vec}_{\mathbb{Z}_2^N}^\zeta$ must centralize $\mathcal{I} \sotimes 0$ projectively. In view of [4, Proposition 3.32], this implies that if $a = 2^{N-1}$ is the unique element of order 2 of $\mathbb{Z}_2^N$, then $1 \sotimes a$ centralizes $\mathcal{I} \sotimes 1$. If $1 \sotimes \text{Vec}_{\mathbb{Z}_2^N}^\zeta$ is degenerate, then its M"uger center must contain $1 \sotimes a$ and therefore $1 \sotimes a$ centralizes $\mathcal{I} \sotimes 0$. Since $1 \sotimes a \in \mathcal{I}_N,\zeta = \langle \mathcal{I} \sotimes 1 \rangle$, then $1 \sotimes a \in \mathcal{Z}_2(\mathcal{I}_N,\zeta)$. Hence $\mathcal{Z}_2(\mathcal{I}_N,\zeta) = \langle 1 \sotimes a \rangle$. But, from Formula (2.5), $q(a) = 1$, where $q$ is the quadratic form in $\mathbb{Z}_2^N$ corresponding to the induced braiding in $1 \sotimes \text{Vec}_{\mathbb{Z}_2^N}^\zeta$. Then $\mathcal{Z}_2(\mathcal{I}_N,\zeta)$ is Tannakian against the assumption.

This shows that $\text{Vec}_{\mathbb{Z}_2^N}^\zeta$ must be non-degenerate and finishes the proof of the lemma. □

**Remark 4.15.** Suppose $\mathcal{C}$ is a slightly degenerate $\mathbb{N}$-Ising fusion category and $N > 2$. We have $\mathcal{C}_{pt} \cong \mathcal{C}_{ad} \sotimes \mathcal{D}$, where $\mathcal{D} = \langle 1 \sotimes 2 \rangle$ is a pointed fusion category whose group of invertible objects is cyclic of order $2^{N-1}$. This is in fact an equivalence of braided fusion categories since, by Lemma 3.5, $\mathcal{C}_{ad}$ centralizes $\mathcal{C}_{pt}$. Therefore

$$\mathcal{Z}_2(\mathcal{C}_{pt}) \cong \mathcal{C}_{ad} \sotimes \mathcal{Z}_2(\mathcal{D}). \tag{4.4}$$

On the other hand, using again Lemma 3.5 and [15, Proposition 7.7], we find

$$\mathcal{Z}_2(\mathcal{C}_{pt}) = \mathcal{C}_{ad} \vee \mathcal{Z}_2(\mathcal{C}) \cong \mathcal{C}_{ad} \sotimes \mathcal{Z}_2(\mathcal{C}). \tag{4.5}$$

From (4.4) and (4.5) we obtain that $\text{FPdim} \mathcal{Z}_2(\mathcal{D}) = 2$. Furthermore, if $\mathcal{Z}_2(\mathcal{D}) \cong \text{sVec}$, then Lemma 5.1 implies that $\text{sVec}$ is a direct factor of $\mathcal{D}$. This is possible only if $N = 2$.

Since $N > 2$, then $\mathcal{Z}_2(\langle 1 \sotimes 2 \rangle)$ is Tannakian of dimension 2. Hence $\mathcal{Z}_2(\langle 1 \sotimes 2 \rangle) \cong \langle 1 \sotimes 2^{N-2} \rangle \cong \text{Rep}_{\mathbb{Z}_2}$ and the nontrivial object of $\mathcal{Z}_2(\mathcal{C})$ is $\delta \sotimes 2^{N-2}$.

### 5. The structure of braided extensions of $\text{Vec}_{\mathbb{Z}_2}$

Suppose that $\mathcal{B}$ is a pointed braided fusion category. Corollary A. 19 of [4] states that if the Müger center $\mathcal{Z}_2(\mathcal{B})$ of $\mathcal{B}$ coincides with the category $\text{sVec}$ of super-vector spaces, then the Müger center is a direct factor of $\mathcal{B}$, that is, $\mathcal{B} \cong \text{sVec} \sotimes \mathcal{B}_0$, for some pointed (necessarily non-degenerate in this case) braided fusion category $\mathcal{B}_0$. However, the proof of [4, Corollary A. 19] only uses the fact that $\text{sVec} \subseteq \mathcal{Z}_2(\mathcal{B})$, in other words, it actually proves the following:
Lemma 5.1. Let $\mathcal{B}$ be a pointed braided fusion category. Suppose that the Müger center of $\mathcal{B}$ contains a fusion subcategory $\mathcal{D}$ braided equivalent to the category $\text{sVec}$ of super-vector spaces. Then $\mathcal{B} \cong \mathcal{D} \boxtimes \mathcal{B}_0$, for some pointed braided fusion category $\mathcal{B}_0$.

Let $\text{Vec}_{\mathbb{Z}_2}^\alpha$ be the pointed fusion category with associativity constraint given by the 3-cocycle $\alpha$, where

$$\alpha(a, b, c) = \begin{cases} 1, & b + c < 2M, \\ \exp\left(\frac{2\pi a}{M}\right), & b + c \geq 2M. \end{cases}$$

Consider the fusion category $\mathcal{D}_{2M}$ of $\mathcal{I} \boxtimes \text{Vec}_{\mathbb{Z}_2}^\alpha$ generated by the simple object $Z \boxtimes 1$. Let $((\mathcal{D}_{2M})_{\mathcal{E}})$ be the de-equivariantization of the fusion category $\mathcal{D}_{2M}$ by its (central) subcategory $\mathcal{E}$ generated by the invertible object $\delta \boxtimes M$.

The following result is a special instance of the classification of cyclic extensions of fusion categories of adjoint ADE type in [5].

Theorem 5.2. ([5, Lemma 3.10].) Up to twisting the associator by a 3-cocycle $\omega$ on $\mathbb{Z}_M$, every $\mathbb{Z}_M$-extension of $\text{Vec}_{\mathbb{Z}_2} \otimes$-generated by a simple object of Frobenius-Perron dimension less than 2, is equivalent as a fusion category to some of the categories $\mathcal{C}_M$ or, if 4 divides $M$, to some of the categories $((\mathcal{D}_{2M})_{\mathcal{E}})$.

As an application of Theorem 5.2, we obtain:

Theorem 5.3. Let $\mathcal{C}$ be a non-pointed braided fusion category and suppose that $\mathcal{C}$ is a $\mathbb{Z}_M$-extension of the fusion category $\text{Vec}_{\mathbb{Z}_2}$. Then $\mathcal{C}$ is equivalent as a fusion category to $\mathcal{C}_M^\alpha$, for some 3-cocycle $\omega$ on $\mathbb{Z}_M$.

Proof. By assumption the braided fusion category $\mathcal{C}$ is nilpotent. Since $\mathcal{C}$ is not pointed, then $\mathcal{C}_{ad} \cong \text{Vec}_{\mathbb{Z}_2}$ and therefore $U(\mathcal{C}) \cong \mathbb{Z}_M$. Then [17, Theorem 4.7] implies that $\mathcal{C}$ has a faithful simple object $X$ and in addition $X$ is not invertible. Since the homogeneous components of the $\mathbb{Z}_M$-grading of $\mathcal{C}$ have dimension 2, then $\text{FPdim} X = \sqrt{2}$ (see Theorem 3.1). Hence $\mathcal{C}$ is $\otimes$-generated by a simple object of Frobenius-Perron dimension less than 2.

In view of Theorem 5.2 we may assume that $\mathcal{C}$ is equivalent to a 3-cocycle twist of one of the fusion categories $(\mathcal{D}_{2M})_{\mathcal{E}}$, where $M$ is divisible by 4.

Consider the canonical dominant tensor functor $F : \mathcal{D}_{2M} \to ((\mathcal{D}_{2M})_{\mathcal{E}}$, that is, the functor $F$ is the ‘free $A$-module functor’, where $A$ is the regular algebra determined by the Tannakian category $\mathcal{E}$.

The functor $F$ takes a simple object of Frobenius-Perron dimension $\sqrt{2}$ of $\mathcal{D}_{2M}$ to a simple object (of the same Frobenius-Perron dimension) of $((\mathcal{D}_{2M})_{\mathcal{E}}$. Then $F$ induces a surjective group homomorphism $G(\mathcal{D}_{2M}) \to G((\mathcal{D}_{2M})_{\mathcal{E}})$ whose kernel is the subgroup $(\delta \boxtimes M)$ generated by $\delta \boxtimes M$. Hence we obtain a group isomorphism $G((\mathcal{D}_{2M})_{\mathcal{E}}) \cong G(\mathcal{D}_{2M})/(\delta \boxtimes M)$. But $G(\mathcal{D}_{2M}) = (\delta) \boxtimes (2)$, so that $G((\mathcal{D}_{2M})_{\mathcal{E}}) \cong \mathbb{Z}_M$ is cyclic of order $M$. 

Then the group of invertible objects of $C$ is cyclic of order $M$. Since $C$ is not pointed, then $C$ has generalized Tambara-Yamagami fusion rules. Then the group of invertible objects of $C$, being cyclic, must contain a unique subgroup of order 2. This subgroup is necessarily the group of invertible objects of the adjoint subcategory $C_{ad} \cong \text{Vec}_{\mathbb{Z}_2}$.

By Lemmas 3.4 and 3.5, $C_{ad} \cong s\text{Vec}$ as braided fusion categories and $Z_2(C_{pt}) = C_{ad} \vee Z_2(C)$. Then, by Lemma 5.1, $C_{pt} \cong C_{ad} \boxtimes B$, for some pointed fusion category $B$. Since $G(C)$ is cyclic, we obtain that $B$ has odd dimension $n$. This implies that $M/2 = n$ is odd, against the assumption. The proof of the theorem is now complete. □

Remark 5.4. The proof of Theorem 5.3 shows that (twistings of) the fusion categories $(D_2M)_E$ are not braided unless $M/2$ is odd, in which case they are equivalent to a twisting of the fusion category $C_M$. When $M = 4$, $(D_2M)_E$ has Fermionic Moore-Reed fusion rules. It is known that there are four fusion categories admitting these fusion rules and none of them is braided; see [1], [13].

The following is the main result of this section:

Theorem 5.5. Let $C$ be a non-pointed braided fusion category and suppose that $C$ is an extension of a rank 2 pointed fusion category. Then $C$ is equivalent as a fusion category to $I_N \boxtimes B$, for some $N \geq 1$, where $I_N$ is a braided $N$-Ising fusion category, and $B$ is a pointed braided fusion category. Furthermore, the categories $I_N$ and $B$ projectively centralize each other in $C$.

Proof. Let $U(C)$ be the universal grading group of $C$, denoted additively. Then $U(C)$ is an Abelian group and $C = \bigoplus_{a \in U(C)} C_a$, with $C_0 = C_{ad} \cong \text{Vec}_{\mathbb{Z}_2}$. Then $C_{ad} \cong s\text{Vec}$ as braided fusion categories. We shall denote by $\delta$ the unique non-invertible simple object of $C_{ad}$.

Let us identify $U(C)$ with a direct sum of cyclic groups $\mathbb{Z}_{d_1} \oplus \cdots \oplus \mathbb{Z}_{d_r}$, where the integers $2 \leq d_1, \ldots, d_r$ are such that $d_j | d_{j+1}$, for all $j = 1, \ldots, r-1$. Let $e_i \in U(C)$, $1 \leq i \leq r$, be the canonical generators: $e_i$ has 1 in the $i$th component and 0 in the remaining components.

For each $1 \leq i \leq r$, let $C_{e_i}$ be the homogeneous component of degree $e_i$ of $C$. Write the set $\{1, \ldots, r\}$ as a disjoint union $\{i_1, \ldots, i_p\} \cup \{j_1, \ldots, j_q\}$, where $p + q = r$ and the indices $i_1, \ldots, i_p, j_1, \ldots, j_q$ are such that 

$$i_1 \leq \cdots \leq i_p \quad j_1 \leq \cdots \leq j_q, \quad (5.1)$$

the homogeneous components $C_{e_{i_\ell}}, 1 \leq \ell \leq p$, contain a non-invertible simple object $Z_{i_\ell}$, and the components $C_{e_{j_s}}, 1 \leq s \leq q$, contain two non-isomorphic invertible objects $a_{j_s}$ and $b_{j_s}$.

Claim 5.6. The $p + 2q$ simple objects 

$$Z_{i_1}, \ldots, Z_{i_p}, a_{j_1}, b_{j_1}, \ldots, a_{j_q}, b_{j_q}, \quad (5.2)$$

generate the fusion category $C$. 

Proof of the claim. Let $X$ be a simple object of $\mathcal{C}$ and suppose that $X \in \mathcal{C}_a$, $a \in U(\mathcal{C})$. Since $e_1, \ldots, e_r$ generate $U(\mathcal{C})$, then $a = t_1e_1 + \cdots + t_re_r$, for some non-negative integers $t_1, \ldots, t_r$. Then the tensor product
\begin{equation}
Z \otimes t_1 \otimes \cdots \otimes Z \otimes t_r \otimes x_j^1 \cdots x_j^q
\end{equation}
belongs to $\mathcal{C}_a$, where, for all $1 \leq s \leq q$, $x_j^s = a_j$ or $b_j$.

If $X$ is the unique simple object of $\mathcal{C}_a$ up to isomorphism, then the tensor product (5.3) must be isomorphic to a direct sum of copies of $X$. In particular $X$ is a simple constituent of (5.3) and therefore it belongs to the fusion subcategory generated by (5.2).

Note in addition that such a non-invertible simple object $X$ of $\mathcal{C}$ must exist, because $\mathcal{C}$ is not pointed. Thus if $t_1, \ldots, t_r$ are chosen as above, then (5.3) does not contain any invertible constituent. Hence some of the simple objects in (5.2) must be non-invertible, that is, $p \geq 1$. Since $Z \otimes Z^* \cong 1 \oplus \delta$, then we find that $\delta$ belongs to the fusion subcategory generated by (5.2).

Suppose next that the simple object $X \in \mathcal{C}_a$ is invertible. Then the only simple objects of $\mathcal{C}_a$ are, up to isomorphism, $X$ and $\delta X$. Also in this case, at least one of the objects $X$ or $\delta X$ is a simple constituent of (5.3) and therefore it belongs to the fusion subcategory generated by (5.2). Then so does the other, because $\delta$ belongs to this subcategory. This proves the claim. □

By Corollary 3.3 (1), the action of the group of invertible objects of $\mathcal{C}$ on the isomorphism classes of non-invertible simple objects is transitive. Then, for all $1 \leq \ell \leq p$,
\begin{equation}
Z_\ell \otimes Z^*_\ell \cong g_\ell \oplus \delta g_\ell,
\end{equation}
for some invertible object $1 \neq g_\ell$ such that
\begin{equation}
g_\ell \otimes Z_\ell \cong Z_\ell.
\end{equation}
In particular $g_1 = \delta$. Then $g_\ell$ and $\delta g_\ell$ are, up to isomorphism, the unique simple objects of $\mathcal{C}_{e_i_1 - e_i_{i'}}$.

Let $\mathcal{B}$ be the pointed fusion subcategory of $\mathcal{C}$ generated by the invertible objects
\begin{equation}
a_{j_1}, b_{j_2}, \ldots, a_{j_q}, b_{j_q}, g_1, g_2, \ldots, g_p.
\end{equation}

Since $\delta = g_1$ generates $\mathcal{C}_{ad}$, then $s\text{Vec} \cong \mathcal{C}_{ad} \subseteq \mathcal{B}$. But by Lemma 3.5, $\mathcal{C}_{ad}$ centralizes $\mathcal{B}$. Lemma 5.1 implies that $\mathcal{B} \cong \mathcal{C}_{ad} \boxtimes \mathcal{B}_0$ for some pointed fusion category $\mathcal{B}_0$. Note that the degree of homogeneity $b$ of a simple object of $\mathcal{B}_0$ is of the form
\[b = s_2(e_{i_1} - e_{i_2}) + \cdots + s_p(e_{i_1} - e_{i_p}) + n_1e_{j_1} + \cdots + n_qe_{j_q}
\]
\[= he_{i_1} - s_2e_{i_2} - \cdots - s_pe_{i_p} + n_1e_{j_1} + \cdots + n_qe_{j_q},
\]
for some non-negative integers $s_2, \ldots, s_p, n_1, \ldots, n_q$, where $h = s_2 + \cdots + s_p$.  


Let $Z = Z_{i_1}$. Relation (5.4) and Claim 5.6 imply that the fusion subcategory $\langle Z \rangle$ generated by $Z$ and $B_0$ generate $C$. By commutativity of the fusion rules of $C$, we obtain that every simple object $Y$ of $C$ decomposes in the form

$$Y \cong X \otimes g,$$

for some simple object $X$ of $\langle Z \rangle$ and some invertible object $g \in B_0$.

Suppose that $X, X' \in \langle Z \rangle$ and $g, g' \in B_0$ are simple objects such that

$$X \otimes g \cong X' \otimes g'.$$

Then $X \otimes g(g')^{-1} \in \langle Z \rangle$ and thus $g(g')^{-1}$ is a simple constituent of $Z^\otimes m$, for some $m \geq 0$. In particular, $g(g')^{-1}$ is homogeneous of degree $me_{i_1}$.

On the other hand, $g(g')^{-1} \in B_0$. Then

$$me_{i_1} = he_{i_1} - s_2 e_{i_2} - \cdots - s_p e_{i_p} + n_1 e_{j_1} + \cdots + n_q e_{i_q},$$

for some non-negative integers $s_2, \ldots, s_p, n_1, \ldots, n_q$, and $h = s_2 + \cdots + s_p$. Therefore $d_{i_1} | s_2, \ldots, d_{i_p} | s_p$ and $d_{j_1} | n_1, \ldots, d_{j_q} | n_q$. From condition (5.1), we have that $d_{i_1} | d_{i_2} | \cdots | d_{i_p}$. Hence $d_{i_1} | h$ and $g(g')^{-1} \in C_0 = C_{ad}$. Therefore $g(g')^{-1} \cong 1$, by the definition of $B_0$. Then $g \cong g'$ and also $X \cong X'$, by (5.7).

We have thus shown that the factorization (5.6) of a simple object of $C$ is unique up to isomorphism. By [9, Theorem 3.8], $C$ has an exact factorization into a product of its fusion subcategories $\langle Z \rangle$ and $B_0$. Since $C$ is braided, then $C \cong \langle Z \rangle \boxtimes B_0$ as fusion categories and the categories $\langle Z \rangle$ and $B_0$ projectively centralize each other, by [9, Corollary 3.9]. Since $\langle Z \rangle$ is a cyclic extension of $Vec_{Z_2}$, then Theorems 5.3 and 4.5 imply that $\langle Z \rangle \cong \mathcal{I}_{N, \zeta} \boxtimes D$, for some $N \geq 1$, where $\zeta$ is a $2^N$th root of 1, and $D$ is a pointed braided fusion category, such that $\mathcal{I}_{N, \zeta}$ and $D$ centralize each other. Letting $B = D \boxtimes B_0$, we obtain the theorem.

Keep the notation in Theorem 5.5. Observe that the equivalence stated in the theorem is in principle a tensor equivalence, rather than a braided equivalence. The following question was asked by the referee:

**Question 5.7.** Is there an explicit example where such an equivalence is actually not braided?

For instance, the answer to this question is negative if $FPdim C = 4m$, with $m$ odd. Indeed in this case we must have $N = 1$ and therefore the category $\mathcal{I}_N$ would be non-degenerate, forcing $C$ to be braided equivalent to a tensor product of $\mathcal{I}_N$ and the pointed braided fusion category $\mathcal{I}'_N$ (see Theorem 2.2).

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References


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