The main supergraph of finite groups

Alireza Khalili Asboei and Seyed Sadegh Salehi Amiri

Abstract. Let $G$ be a finite group. The main supergraph $S(G)$ is a graph with vertex set $G$ in which two vertices $x$ and $y$ are adjacent if and only if $o(x) | o(y)$ or $o(y) | o(x)$. In this paper, we will show that if $S(G) \cong S(S)$, where $S$ belongs to a large class of finite non-solvable groups, then $G \cong S$. This work is an important step in solving Thompson’s problem.

Contents

1. Introduction 1057
2. Proof of main theorem 1060
3. Acknowledgment 1061
References 1061

1. Introduction

Let $G$ be a finite group and $x \in G$. The order of $x$ is denoted by $o(x)$. The set of all element orders of $G$ is denoted by $\pi_e(G)$ and the set of all prime factors of $|G|$ is denoted by $\pi(G)$. We set $M_i(G) = \{|g \in G| \text{ the order of } g \text{ is } i\}$. The other notations and terminologies in this paper are standard, and the reader is referred to [14] if necessary.

Define the graph $S(G)$ with the vertex set $G$ such that two vertices $x$ and $y$ are adjacent if and only if $o(x) \mid o(y)$ or $o(y) \mid o(x)$. This graph is called the main supergraph of $\mathcal{P}(G)$ (power graph of $G$) and was introduced in [20]. The proper main supergraph $S^p(G)$ is the graph constructed from $S(G)$ by removing the identity element of $G$. We write $x \sim y$ when two vertices $x$ and $y$ are adjacent.

Definition 1.1. Let $G$ be a finite group. We say that $G$ is recognizable by its main supergraph if for every group $H$ we have $S(G) \cong S(H)$, then $G \cong H$.

Note that not all groups are recognizable by the main supergraph. For example, we have $S(\mathbb{Z}_4) \cong S(\mathbb{Z}_2 \times \mathbb{Z}_2)$, but $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Definition 1.2. Two finite groups $G_1$ and $G_2$ are called of the same order type if and only if $M_t(G_1) = M_t(G_2)$ for all $t$. 

Received August 16, 2021.
2010 Mathematics Subject Classification. 20D08, 05C25.
Key words and phrases. same order type groups, non-solvable, main supergraph, Thompson’s problem.
In 1987, J. G. Thompson [37, Problem 12.37] posed the following problem:

**Thompson’s Problem.** Suppose that $G_1$ and $G_2$ are two finite groups of the same order type. If $G_1$ is solvable, is it true that $G_2$ is also necessarily solvable?

Another form of this problem can be stated as follows.

**Thompson’s Problem.** Suppose that $G_1$ and $G_2$ are two finite groups of the same order type. If $G_1$ is non-solvable, is it true that $G_2$ is also necessarily non-solvable?

By definition of the main supergraph, it is clear that if $G_1$ and $G_2$ are groups with the same order type, then $\delta(G_1) \cong \delta(G_2)$. So, if a finite group $G$ is recognizable by the main supergraph, then for $G$ Thompson’s problem is true.

In [3], the authors of this paper proved that alternating groups of degree $p$, $p + 1$, $p + 2$ and symmetric groups of degree $p$ are recognizable by their main supergraph. Also, in [6], [4], [2] and [5], it is proved that the groups $L_2(p)$, $\text{PGL}_2(p)$, where $p$ is prime, all sporadic simple groups, $L_2(q)$, small Ree groups $2^{2G_2}(3^{2n+1})$, where $n$ is a natural number and Suzuki Ree groups are recognizable by their main supergraph.

The prime graph of $G$, is denoted by $\Gamma(G)$ and is defined as follows. The vertex set of $\Gamma(G)$ is $\pi(G)$ and two distinct vertices $p$ and $q$ are adjacent if and only if $G$ contains an element of order $pq$. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and $\pi_i = \pi_i(G)$, $1 \leq i \leq t(G)$ be the connected components of $\Gamma(G)$. For a group of even order we let $2 \in \pi_1(G)$. Then the order of $G$ can be expressed as the product of $m_1, m_2, \ldots, m_{t(G)}$, where $m_i (1 \leq i \leq t(G))$ are positive integers with $\pi(m_i) = \pi_i$. These $m_i (1 \leq i \leq t(G))$ are called the order components of $G$. We write $\text{OC}(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$ and call it the set of order components of $G$ (see [9]).

In this paper, first we prove if $G$ and $S$ are two arbitrary groups such that $\delta(G) \cong \delta(S)$, then their order components are equal. Then, we conclude that if $S$ is one of the nonabelian simple groups

- $A_p$, where $p$ and $p - 2$ are primes,
- $A_n$, where $n = p, p + 1, p + 2$,
- $L_n(q)$, where $n = 2, 3, 5$,
- $U_p(q), L_{p+1}(q)$,
- $L_{p+1}(2)$,
- $U_{p+1}(q)$,
- $C_n(q)$, where $q$ is an even prime power,
- $C_2(q)$, where $q > 5$,
- $E_8(q)$,
- $F_4(q)$, where $q = 2^n > 2$,
- $G_2(q) (3 \mid q)$,
- $G_2(q) (2 < q \equiv 1 (\text{mod } 3))$,
- $2G_2(q)$,
- $3D_4(q)$,
- $2D_n(3)$, where $9 \leq n = 2m + 1$ not a prime,
- $2D_{p+1}(q)$, where $5 < p \neq 2m - 1$, 

\[ 2D(p, 3), \text{ where } p \geq 5 \text{ is a prime number not of the form } 2^m + 1, \]
\[ 2D_n(2), \text{ where } n = 2^m + 1 \geq 5, \]
\[ D_{p+1}(2), \]
\[ D_{p+1}(3), \]
\[ 2D_n(2), \text{ where } n = 2^m, \]
Suzuki Ree groups,
Sporadic simple groups,
\[ 2E_6(q), \]
\[ L_p(q), \]
\[ U_n(q), \text{ where } n = 3, 5, 11, \]
almost sporadic simple groups, except \( \text{Aut} \text{(McL) and Aut}(J_2), \)
\[ S_n, \text{ for } n = p, p + 1, \text{ where } p \geq 3 \text{ is a prime number}, \]
then \( S \) is recognizable by the main supergraph.

**Main Theorem.** Let \( G \) and \( S \) be two arbitrary groups such that \( S(G) \cong S(S) \). Then \( OC(G) = OC(S) \).

To get the main result of this paper, we need to the following lemma.

**Lemma 1.3.** Let \( OC(G) = OC(S) \), where \( S \) is one of the nonabelian simple groups listed before the main theorem, then \( G \cong S \).

**Proof.** See [1, 7, 8, 10, 11, 12, 13, 17, 15, 18, 16, 21, 24, 25, 26, 27, 23, 22, 28, 29, 31, 32, 36, 34, 35, 33, 30, 38, 39]. \( \square \)

**Corollary 1.4.** Let \( G \) be a finite group listed in the above lemma. Then \( G \) is recognizable by the main supergraph.

**Corollary 1.5.** Let \( G \) be a finite group listed in the above lemma. Then Thompson’s problem is true for \( G \).

According to research conducted on non-solvable groups of order less than 2000 by using GAP and Corollary 1.4, we pose the following two conjectures:

**Conjecture 1.6.** Let \( S \) be a finite simple group and \( G \) be an arbitrary finite group such that \( S(G) \cong S(S) \). Then \( G \cong S \).

**Conjecture 1.7.** Let \( S \) be a finite non-solvable group and \( G \) be an arbitrary finite group such that \( S(G) \cong S(S) \). Then \( G \) is a non-solvable group.

It is clear that if Conjecture 1.7 is true, then Thompson’s problem is also true. Note that there exist non-solvable groups \( S \) and \( G \) such that \( S(G) \cong S(S) \), but \( G \) and \( S \) are not isomorphic. For example, if \( G = Z_4 \times A_5 \) and \( S = < a, b > \), where
\[ a = (1, 20, 17, 5, 12)(2, 3, 9, 19, 10)(4, 14, 22, 11, 6)(7, 8, 15, 13, 16) \]
(in fact \( S \) has structure description \( SL(2, 5) : Z_2 \)), then \( S(G) \cong S(S) \), but \( G \) and \( S \) are not isomorphic.

The next lemma is used in the proof of the main theorem.
Lemma 1.8. [40, Theorem 3] Let $G$ be a finite group. Then the number of elements whose order is a multiple of $n$ is either zero, or a multiple of the greatest divisor of $|G|$ that is prime to $n$.

2. Proof of main theorem

By definition of the main supergraph and our assumption, we have $|G| = |S|$ and $S^*(S) \cong S^*(G)$. First, let $S^*(G)$ be a connected graph. We show that $\Gamma(G)$ and $\Gamma(S)$ are connected. If $\pi(G) = \{p\}$, then $\Gamma(G) = \Gamma(S)$ has one vertex. So, $\Gamma(G)$ and $\Gamma(S)$ are connected. Let $|\pi(G)| \geq 2$. If $\Gamma(G)$ or $\Gamma(S)$ are disconnected, then $\Gamma(G)$ or $\Gamma(S)$ have two or more connected components. By definition of the main supergraph $S^*(G)$ is a disconnected graph, which is a contradiction.

Now, let $S^*(G)$ be a disconnected graph. Then it has two or more connected components. Suppose that $K_1$ and $K_2$ are two connected components of $S^*(G)$. We show that if $x, y$ are two arbitrary vertices of $K_1$ and $K_2$, respectively such that $o(x) = r$ and $o(y) = s$, where $r$ and $s$ are primes, then $r$ and $s$ are not joined by an edge in the prime graph of $G$. Assume that $r$ and $s$ are joined by an edge in the prime graph of $G$. Then $rs \in \pi_e(G)$. So, there exists an element of order $rs$ in $G$. Assume $z \in G$ and $o(z) = rs$. By definition of the main supergraph $x \sim z$ and $y \sim z$. Thus $K_1$ and $K_2$ are connected, which is a contradiction.

Suppose that $K_1, K_2, \ldots, K_n$ are all connected components of $S^*(G)$. Let $\pi(K)$ be all prime numbers that divide the order of vertices of $K$, where $K$ is one of the connected components of $S^*(G)$. We claim that $\pi(K_i)$ for every $1 \leq i \leq n$ is the set of vertices of one of the connected components of $\Gamma(G)$. Assume that $T_i$ is a component in the prime graph $G$ such that the vertices of $T_i$ are subset of $\pi(K_i)$. Thus, $rs \notin \pi_e(G)$ for every $r \in V(T_i)$ and $s \in \pi(K_i) \setminus V(T_i)$. By definition of the main supergraph, we can conclude that $K_i$ is not connected, a contradiction. Therefore, there exists a one-to-one correspondence between connected components of $S^*(G)$ and $\Gamma(G)$. It follows that $\pi(K_i)$ for every $1 \leq i \leq n$ is the set of vertices of one connected component of $\Gamma(G)$.

Let $K$ be one of the connected components of $S^*(G)$. The vertices of $K$ are elements of $G$ and their orders are divided by some prime numbers. We will show how to find these prime numbers.

Suppose that $K_1, K_2$ are two arbitrary connected components of $S^*(G)$. Since $K_1$ and $K_2$ are isolated, we have $rs \notin \pi_e(G)$, where $r \in \pi(K_1)$ and $s \in \pi(K_2)$.

Let $p \in \pi(K_1)$ be arbitrary. If $\pi(K_2) = \{p_1\}$, then considering $n = p_1$ in Lemma 1.8, $\sum_{\ell} M_{t} = (M_{p_1} + M_{p_2} + \cdots + M_{p_k}) = |K_2| (p_1^k \in \pi_e(G))$, where $P$ is a Sylow $p$-subgroup of $G$. Assume that $\pi(K_2) = \{p_1, p_2\}$. Considering $n = p_1p_2$ in Lemma 1.8, we have $\sum_{\ell} M_{t}$ is a multiple of $p_1, p_2$. On the other hand, considering $n = p_1$ in Lemma 1.8, $\sum_{\ell} M_{t}$ is a multiple of $p_1, M_{t} = (\sum_{i} M_{t}) + (M_{p_1} + M_{p_2} + \cdots + M_{p_k})$. It follows that $\sum_{\ell} M_{t} = (M_{p_1} + M_{p_2} + \cdots + M_{p_k})$. Similarly, $|P| \mid (M_{p_1} + M_{p_2} + \cdots + M_{p_k}) (p_2 \in \pi_e(G))$. Therefore, $|P| \mid (M_{p_1} + M_{p_2} + \cdots + M_{p_k}) + (M_{p_1} + M_{p_2} + \cdots + M_{p_k}) + (\sum_{i} M_{t}) = |K_2|$. If $\pi(K_2) = \{p_1, p_2, p_3\}$, then considering $n = p_1p_2p_3, p_1p_2, p_1p_3$ and $p_1$
in Lemma 1.8, \(|P|\) \(\mid\sum_{t} t\) is a multiple of \(p_1 p_2 p_3\), \(M_t\), \(|P|\) \(\mid\sum_{t} t\) is a multiple of \(p_1 p_2\), \(M_t\), \(|P|\) \(\mid\sum_{t} t\) is a multiple of \(p_1\), \(M_t\). Thus, \(|P|\) \(\mid (M_{p_1}+M_{p_2}+\cdots+M_{p_3})\). Similarly, \(|P|\) \(\mid (M_{p_3}+M_{p_2}+\cdots+M_{p_1})\). Therefore, \(|P|\) \(\mid |K_2|\).

Arguing as above, if \(\pi(K_2) = \{p_1, p_2, \ldots, p_t\}\), then \(|P|\) \(\mid |K_2|\). Also, \(|P|\) \(\mid |K_i|\) for every connected component \(K_i\) (\(i \neq 1\)) of \(S^*(G)\). Since \(|P|\) \(\mid |G| = (1 + |K_1| + |K_2| + \cdots + |K_n|)\), we have \(|P|\) \(\mid |K|\). Now, let \(K\) be one of the connected components of \(S^*(G)\). If \(p \in \pi(G)\) is such that \(|P|\) \(\nmid |K|\), then \(p \in \pi(K)\).

Since there exists a one-to-one correspondence between connected components of \(S^*(G)\) and \(\Gamma(G)\) and \(|G| = |S|\), we have \(OC(G) = OC(S)\). This completes the proof of the main theorem.

3. Acknowledgment

We thank the referee for some extremely helpful remarks, which allowed us to improve the paper.

References


This paper is available via http://nyjm.albany.edu/j/2022/28-43.html.