The main supergraph of finite groups

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Abstract. Let $G$ be a finite group. The main supergraph $S(G)$ is a graph with vertex set $G$ in which two vertices $x$ and $y$ are adjacent if and only if $o(x) | o(y)$ or $o(y) | o(x)$. In this paper, we will show that if $S(G) \cong S(S)$, where $S$ belongs to a large class of finite non-solvable groups, then $G \cong S$. This work is an important step in solving Thompson's problem.

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1. Introduction

Let $G$ be a finite group and $x \in G$. The order of $x$ is denoted by $o(x)$. The set of all element orders of $G$ is denoted by $\pi_e(G)$ and the set of all prime factors of $|G|$ is denoted by $\pi(G)$. We set $M_t(G) = \{|g \in G| \text{ the order of } g \text{ is } t|\}$. The other notations and terminologies in this paper are standard, and the reader is referred to [14] if necessary.

Define the graph $S(G)$ with the vertex set $G$ such that two vertices $x$ and $y$ are adjacent if and only if $o(x) | o(y)$ or $o(y) | o(x)$. This graph is called the main supergraph of $G$ and was introduced in [20]. The proper main supergraph $S^\prime(G)$ is the graph constructed from $S(G)$ by removing the identity element of $G$. We write $x \sim y$ when two vertices $x$ and $y$ are adjacent.

Definition 1.1. Let $G$ be a finite group. We say that $G$ is recognizable by its main supergraph if for every group $H$ we have $S(G) \cong S(H)$, then $G \cong H$.

Note that not all groups are recognizable by the main supergraph. For example, we have $S(\mathbb{Z}_4) \cong S(\mathbb{Z}_2 \times \mathbb{Z}_2)$, but $\mathbb{Z}_4$ is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Definition 1.2. Two finite groups $G_1$ and $G_2$ are called of the same order type if and only if $M_t(G_1) = M_t(G_2)$ for all $t$.
In 1987, J. G. Thompson [37, Problem 12.37] posed the following problem:

**Thompson’s Problem.** Suppose that $G_1$ and $G_2$ are two finite groups of the same order type. If $G_1$ is solvable, is it true that $G_2$ is also necessarily solvable?

Another form of this problem can be stated as follows.

**Thompson’s Problem.** Suppose that $G_1$ and $G_2$ are two finite groups of the same order type. If $G_1$ is non-solvable, is it true that $G_2$ is also necessarily non-solvable?

By definition of the main supergraph, it is clear that if $G_1$ and $G_2$ are groups with the same order type, then $\mathcal{S}(G_1) \cong \mathcal{S}(G_2)$. So, if a finite group $G$ is recognizable by the main supergraph, then for $G$ Thompson’s problem is true.

In [3], the authors of this paper proved that alternating groups of degree $p$, $p + 1$, $p + 2$ and symmetric groups of degree $p$ are recognizable by their main supergraph. Also, in [6], [4], [2] and [5], it is proved that the groups $L_2(p)$, $PGL_2(p)$, where $p$ is prime, all sporadic simple groups, $L_2(q)$, small Ree groups $^2G_2(3^{2n+1})$, where $n$ is a natural number and Suzuki Ree groups are recognizable by their main supergraph.

The prime graph of $G$, is denoted by $\Gamma(G)$ and is defined as follows. The vertex set of $\Gamma(G)$ is $\pi(G)$ and two distinct vertices $p$ and $q$ are adjacent if and only if $G$ contains an element of order $pq$. Let $t(G)$ be the number of connected components of $\Gamma(G)$ and $\pi_i = \pi_i(G)$, $1 \leq i \leq t(G)$ be the connected components of $\Gamma(G)$. For a group of even order we let $2 \in \pi_1(G)$. Then the order of $G$ can be expressed as the product of $m_1, m_2, \ldots, m_{t(G)}$, where $m_i$ ($1 \leq i \leq t(G)$) are positive integers with $\pi(m_i) = \pi_i$. These $m_i$ ($1 \leq i \leq t(G)$) are called the order components of $G$. We write $OC(G) = \{m_1, m_2, \ldots, m_{t(G)}\}$ and call it the set of order components of $G$ (see [9]).

In this paper, first we prove if $G$ and $S$ are two arbitrary groups such that $\mathcal{S}(G) \cong \mathcal{S}(S)$, then their order components are equal. Then, we conclude that if $S$ is one of the nonabelian simple groups

- $A_p$, where $p$ and $p - 2$ are primes,
- $A_n$, where $n = p$, $p + 1$, $p + 2$,
- $L_n(q)$, where $n = 2, 3, 5$,
- $U_p(q), L_p+1(q)$,
- $L_p+1(2)$,
- $U_{p+1}(q)$,
- $C_n(q)$, where $q$ is an even prime power,
- $C_2(q)$, where $q > 5$,
- $E_6(q)$,
- $F_4(q)$, where $q = 2^n > 2$,
- $G_2(q) (3 \mid q)$,
- $G_2(q) (2 < q \equiv 1 \pmod{3})$,
- $^2G_2(q)$,
- $^3D_4(q)$,
- $^2D_n(3)$, where $9 \leq n = 2^m + 1$ not a prime,
- $^2D_{p+1}(q)$, where $5 < p \neq 2^m - 1$,
\[ 2D_p(3), \ \text{where} \ p \geq 5 \text{ is a prime number not of the form } 2^m + 1, \]
\[ 2D_n(2), \ \text{where} \ n = 2^m + 1 \geq 5, \]
\[ D_{p+1}(2), \]
\[ D_{p+1}(3), \]
\[ 2D_n(2), \ \text{where} \ n = 2^m, \]
Suzuki Ree groups,
Sporadic simple groups,
\[ 2E_6(q), \]
\[ L_{p+1}(q), \]
\[ U_n(q), \ \text{where} \ n = 3, 5, 11, \]
almost sporadic simple groups, except Aut(McL) and Aut(J_2),
\[ S_n, \text{for} \ n = p, p + 1, \ \text{where} \ p \geq 3 \text{ is a prime number}, \]
then \( S \) is recognizable by the main supergraph.

**Main Theorem.** Let \( G \) and \( S \) be two arbitrary groups such that \( S(G) \cong S(S) \). Then \( OC(G) = OC(S) \).

To get the main result of this paper, we need to the following lemma.

**Lemma 1.3.** Let \( OC(G) = OC(S) \), where \( S \) is one of the nonabelian simple groups listed before the main theorem, then \( G \cong S \).

**Proof.** See \([1, 7, 8, 10, 11, 12, 13, 17, 15, 18, 16, 21, 24, 25, 26, 27, 23, 22, 28, 29, 31, 32, 36, 34, 35, 33, 30, 38, 39]\). \(\Box\)

**Corollary 1.4.** Let \( G \) be a finite group listed in the above lemma. Then \( G \) is recognizable by the main supergraph.

**Corollary 1.5.** Let \( G \) be a finite group listed in the above lemma. Then Thompson's problem is true for \( G \).

According to research conducted on non-solvable groups of order less than 2000 by using GAP and Corollary 1.4, we pose the following two conjectures:

**Conjecture 1.6.** Let \( S \) be a finite simple group and \( G \) be an arbitrary finite group such that \( S(G) \cong S(S) \). Then \( G \cong S \).

**Conjecture 1.7.** Let \( S \) be a finite non-solvable group and \( G \) be an arbitrary finite group such that \( S(G) \cong S(S) \). Then \( G \) is a non-solvable group.

It is clear that if Conjecture 1.7 is true, then Thompson's problem is also true. Note that there exist non-solvable groups \( S \) and \( G \) such that \( S(G) \cong S(S) \), but \( G \) and \( S \) are not isomorphic. For example, if \( G = \mathbb{Z}_4 \times A_5 \) and \( S = \langle a, b \rangle \), where

\[ a = (1, 20, 17, 5, 12)(2, 3, 9, 19, 10)(4, 14, 22, 11, 6)(7, 8, 15, 13, 16) \]

(in fact \( S \) has structure description \( SL(2, 5) : \mathbb{Z}_2 \)), then \( S(G) \cong S(S) \), but \( G \) and \( S \) are not isomorphic.

The next lemma is used in the proof of the main theorem.
Lemma 1.8. [40, Theorem 3] Let $G$ be a finite group. Then the number of elements whose order is a multiple of $n$ is either zero, or a multiple of the greatest divisor of $|G|$ that is prime to $n$.

2. Proof of main theorem

By definition of the main supergraph and our assumption, we have $|G| = |S|$ and $S^*(S) \cong S^*(G)$. First, let $S^*(G)$ be a connected graph. We show that $\Gamma(G)$ and $\Gamma(S)$ are connected. If $\pi(G) = \{p\}$, then $\Gamma(G) = \Gamma(S)$ has one vertex. So, $\Gamma(G)$ and $\Gamma(S)$ are connected. Let $|\pi(G)| \geq 2$. If $\Gamma(G)$ or $\Gamma(S)$ are disconnected, then $\Gamma(G)$ or $\Gamma(S)$ have two or more connected components. By definition of the main supergraph $S^*(G)$ is a disconnected graph, which is a contradiction.

Now, let $S^*(G)$ be a disconnected graph. Then it has two or more connected components. Suppose that $K_1$ and $K_2$ are two connected components of $S^*(G)$. We show that $\pi(K_i)$ for every $1 \leq i \leq n$ is the set of vertices of one of the connected components of $\Gamma(G)$. Assume that $\Gamma_i$ is a component in the prime graph $G$ such that the vertices of $\Gamma_i$ are subset of $\pi(K_i)$. Thus, $rs \in \pi_e(G)$ for every $r \in V(\Gamma_i)$ and $s \in \pi(K_i) \setminus V(\Gamma_i)$. By definition of the main supergraph, we can conclude that $K_i$ is not connected, a contradiction. Therefore, there exists a one-to-one correspondence between connected components of $S^*(G)$ and $\Gamma(G)$. It follows that $\pi(K_i)$ for every $1 \leq i \leq n$ is the set of vertices of one connected component of $\Gamma(G)$.

Let $K$ be one of the connected components of $S^*(G)$. The vertices of $K$ are elements of $G$ and their orders are divided by some prime numbers. We will show how to find these prime numbers.

Suppose that $K_1$, $K_2$ are two arbitrary connected components of $S^*(G)$. Since $K_1$ and $K_2$ are isolated, we have $rs \not\in \pi_e(G)$, where $r \in \pi(K_1)$ and $s \in \pi(K_2)$.

Let $p \in \pi(K_1)$ be arbitrary. If $\pi(K_2) = \{p_1\}$, then considering $n = p_1$ in Lemma 1.8, $|P| | \sum_{t} M_t = (M_{p_1} + M_{p_1^2} + \cdots + M_{p_1^k}) = |K_2| (p_1^k \in \pi_e(G))$, where $P$ is a Sylow $p$-subgroup of $G$. Assume that $\pi(K_2) = \{p_1, p_2\}$. Considering $n = p_1p_2$ in Lemma 1.8, we have $|P| | \sum_{t} M_t = (\sum_{i} M_{p_1^i} M_{p_2^i} + \cdots + M_{p_1^k} M_{p_2^k})$. Similarly, $|P| | (M_{p_1^2} + M_{p_2^2} + \cdots + M_{p_2^k}) (p_2^k \in \pi_e(G))$. Therefore, $|P| | (M_{p_1} + M_{p_1^2} + \cdots + M_{p_1^k}) + (M_{p_2} + M_{p_2^2} + \cdots + M_{p_2^k}) + (\sum_{t} M_{p_1^i} M_{p_2^i} + \cdots + M_{p_1^k} M_{p_2^k}) = |K_2|$. If $\pi(K_2) = \{p_1, p_2, p_3\}$, then considering $n = p_1p_2p_3$, $p_1p_2$, $p_1p_3$ and $p_1$
in Lemma 1.8, $|P| \mid \sum t$ is a multiple of $p_1p_2p_3M_t$, $|P| \mid \sum t$ is a multiple of $p_1p_2M_t$, $|P| \mid \sum t$ is a multiple of $p_1p_2p_3M_t$ and $|P| \mid \sum t$ is a multiple of $p_1M_t$. Thus, $|P| \mid (M_{p_1} + M_{p_2} + \cdots + M_{p_1})$. Similarly, $|P| \mid (M_{p_1} + M_{p_2} + \cdots + M_{p_1})$ and $|P| \mid (M_{p_1} + M_{p_2} + \cdots + M_{p_1})$. Therefore, $|P| \mid |K_2|$. 

Arguing as above, if $\pi(K_2) = \{p_1, p_2, \ldots, p_t\}$, then $|P| \mid |K_i|$. Also, $|P| \mid |K_i|$ for every connected component $K_i (i \neq 1)$ of $S^*(G)$. Since $|P| \mid |G| = (1 + |K_1| + |K_2| + \cdots + |K_n|)$, we have $|P| \mid |K_1|$. Now, let $K$ be one of the connected components of $S^*(G)$. If $p \in \pi(G)$ is such that $|P| \mid |K|$ then $p \in \pi(K)$. 

Since there exists a one-to-one correspondence between connected components of $S^*(G)$ and $\Gamma(G)$ and $|G| = |S|$, we have $OC(G) = OC(S)$. This completes the proof of the main theorem.

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References


[15] DARAFSHEH, MOHAMMAD R. Characterizability of the group 2\textit{D}_m(3) by its order components, where \( p \geq 5 \) is a prime number not of the form \( 2^m + 1 \). \textit{Acta Math. Sin., (Engl. Ser.)} 24 (2008), no. 7, 1117–1126. MR2420882, Zbl 1011.4-007-6143-7. 1059


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