Variational inequalities for the differences of averages over lacunary sequences

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Abstract. Let $f$ be a locally integrable function defined on $\mathbb{R}$, and let $(n_k)$ be a lacunary sequence. Define the operator $A_{n_k}$ by

$$A_{n_k}f(x) = \frac{1}{n_k} \int_0^{n_k} f(x-t) \, dt.$$ 

We prove various types of new inequalities for the variation operator

$$V_s f(x) = \left( \sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^s \right)^{1/s}$$

when $2 \leq s < \infty$.

An increasing sequence $(n_k)$ of real numbers is called lacunary if there exists a constant $\beta > 1$ such that $n_{k+1}/n_k \geq \beta$ for all $k = 0, 1, 2, \ldots$.

Let $f$ be a locally integrable function defined on $\mathbb{R}$. Let $(n_k)$ be a lacunary sequence and define the operator $A_{n_k}$ by

$$A_{n_k}f(x) = \frac{1}{n_k} \int_0^{n_k} f(x-t) \, dt.$$ 

It is clear that

$$A_{n_k}f(x) = \frac{1}{n_k} \chi_{(0,n_k)} \ast f(x)$$

where $\ast$ stands for convolution. Consider the variation operator

$$V_s f(x) = \left( \sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^s \right)^{1/s}$$

for $2 \leq s < \infty$. The boundedness of the variation operator $V_s f$ provides an estimate on the speed (or rate) of convergence of the sequence $\{A_{n_k}f\}$.

Various types of inequalities for the two-sided variation operator

$$V_s^+ f(x) = \left( \sum_{k=-\infty}^{\infty} \left| \int_x^{x+2^n} f(t) \, dt - \left( \frac{1}{2^{n-1}} \int_x^{x+2^{n-1}} f(t) \, dt \right) \right|^s \right)^{1/s}$$

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when $2 \leq s < \infty$ have been proven by the author in Demir, S. [1]. In this research we prove that same types of inequalities are also true for any lacunary sequence $(n_k)$ for the one-sided variation operator $V_s f(x)$ for $2 \leq s < \infty$.

**Lemma 1.** Let $(n_k)$ be a lacunary sequence with the lacunarity constant $\beta$, i.e., $n_{k+1}/n_k \geq \beta > 1$ for all $k = 0, 1, 2, \ldots$. If $1 \leq s < \infty$, then there exists a sequence $(m_j)$ such that

$$\beta^2 \geq \frac{m_{j+1}}{m_j} \geq \beta > 1$$

for all $j$ and

$$\left(\sum_{k=1}^{\infty} |A_{n_k} f(x) - A_{n_{k-1}} f(x)|^s\right)^{1/s} \leq \left(\sum_{j=1}^{\infty} |A_{m_j} f(x) - A_{m_{j-1}} f(x)|^s\right)^{1/s}.$$

**Proof.** Let us start our construction by first choosing $m_0 = n_0$. If

$$\beta^2 \geq \frac{n_1}{n_0} \geq \beta,$$

define $m_1 = n_1$. If $n_1/n_0 > \beta^2$, let $m_1 = \beta n_0$. Then we have

$$\beta^2 \geq \frac{m_1}{m_0} = \frac{\beta n_0}{n_0} = \beta \geq \beta.$$

Also,

$$\frac{n_1}{m_1} \geq \frac{\beta^2 n_0}{\beta n_0} = \beta.$$

Again, if $n_1/m_1 \leq \beta^2$, then choose $m_2 = n_1$. If this is not the case, choose $m_2 = \beta^2 n_0 \leq n_1$. By the same calculation as before, $m_0, m_1, m_2$ are part of a lacunary sequence satisfying

$$\beta^2 \geq \frac{m_{k+1}}{m_k} \geq \beta > 1.$$

To continue the sequence, either $m_3 = n_1$ if $n_1/m_2 \leq \beta^2$ or $m_3 = \beta^2 n_0$ if $n_1/m_2 > \beta^2$.

Since $\beta > 1$, this process will end at some $k_0$ such that $m_{k_0} = n_1$. The remaining elements $m_k$ are constructed in the same manner as the original $n_k$, with necessary terms added between two consecutive $n_k$ to obtain the inequality

$$\beta^2 \geq \frac{m_{k+1}}{m_k} \geq \beta > 1.$$

Let now

$$J(k) = \{j : n_{k-1} < m_j \leq n_k\}.$$

Then we have

$$A_{n_k} f(x) - A_{n_{k-1}} f(x) = \sum_{j \in J(k)} (A_{m_j} f(x) - A_{m_{j-1}} f(x)).$$
and thus we get
\[ |A_{n_k}f(x) - A_{n_{k-1}}f(x)| = \left| \sum_{j \in I(k)} (A_{m_j}f(x) - A_{m_{j-1}}f(x)) \right| \]
\[ \leq \sum_{j \in I(k)} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|. \]

This implies that
\[ \sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)| \leq \sum_{k=1}^{\infty} \sum_{j \in I(k)} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|. \]
\[ = \sum_{j=1}^{\infty} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|. \]

Thus, we have
\[ \left( \sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^s \right)^{1/s} \leq \left( \sum_{j=1}^{\infty} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|^s \right)^{1/s}. \]

and this completes the proof. □

**Remark 2.** We know from Lemma 1 that
\[ \left( \sum_{k=1}^{\infty} |A_{n_k}f(x) - A_{n_{k-1}}f(x)|^s \right)^{1/s} \leq \left( \sum_{j=1}^{\infty} |A_{m_j}f(x) - A_{m_{j-1}}f(x)|^s \right)^{1/s}. \]

and the new sequence \((m_j)\) satisfies
\[ \beta^2 \geq \frac{m_{j+1}}{m_j} \geq \beta > 1 \]
for all \(j \in \mathbb{Z}^+.\) Therefore, we can assume without loss of generality that
\[ \beta^2 \geq \frac{n_{k+1}}{n_k} \geq \beta > 1 \]
for all \(k \in \mathbb{Z}^+\) when we are proving any result for \(V_s(x).\)

Since
\[ \frac{1}{n_k} = \frac{n_1}{n_2} \cdot \frac{n_2}{n_3} \cdot \frac{n_3}{n_4} \cdot \ldots \cdot \frac{n_{k-1}}{n_k}, \]
we can also assume that
\[ \frac{1}{n_k} \leq \frac{1}{\beta^{2(k-1)}} \]
for all \(k = 0, 1, 2, \ldots.\)

**Lemma 3.** Let \((n_k)\) be a lacunary sequence, and let \(\gamma\) denote the smallest positive integer satisfying
\[ \frac{1}{\beta} + \frac{1}{\beta^\gamma} \leq 1. \]
If $i \geq j + \gamma$, $0 < y \leq n_j$ and $n_j < x < n_{i+1}$, then

$$\chi_{(y,y+n_j)}(x) - \chi_{(0,n_j)}(x) = 0$$

unless $k = i$ in which case

$$\chi_{(y,y+n_i)}(x) - \chi_{(0,n_i)}(x) = \chi_{(n_i,y+n_i)}.$$  

**Proof.** Since $(n_k)$ is a lacunary sequence, there exists a constant $\beta > 1$ such that $n_{k+1}/n_k \geq \beta$ for all $k$. We can assume that

$$\beta^2 \geq \frac{n_{k+1}}{n_k} \geq \beta$$  \hfill (1)

for all $k$ by Remark 2. Since we have

$$\frac{n_l}{n_k} = \frac{n_l}{n_{l+1}} \cdot \frac{n_{l+1}}{n_{l+2}} \cdot \ldots \cdot \frac{n_{k-1}}{n_k}$$

and

$$\frac{1}{\beta} \leq \frac{n_k}{n_{k+1}} \leq \frac{1}{\beta^{k-l}}$$

for all $k$, we see that

$$\frac{1}{\beta^{2(k-l)}} \leq \frac{n_l}{n_k} \leq \frac{1}{\beta^{k-l}}$$  \hfill (2)

for all $k > l$. Let $\gamma$ denote the smallest positive integer satisfying

$$\frac{1}{\beta} + \frac{1}{\beta^\gamma} \leq 1.$$  

We see from (2) that

$$n_j + n_k \leq n_{k+1}$$  \hfill (3)

for all $k \geq j + \gamma - 1$. It is easy to see that for $k > i$,

$$0 < y \leq n_j \leq n_i < x < n_{i+1} \leq n_k < y + n_k,$$

and this implies that

$$\left[\chi_{(y,y+n_k)}(x) - \chi_{(0,n_k)}(x)\right] \cdot \chi_{(n_i,n_{i+1})}(x) = 0.$$  

For $k \leq i - 1$, we see by (3) that

$$n_k < y + n_k \leq n_j + n_{i-1} \leq n_i.$$  

Then we have

$$\chi_{(y,y+n_k)}(x) \cdot \chi_{(n_i,n_{i+1})}(x) = \chi_{(0,n_k)}(x) \cdot \chi_{(n_i,n_{i+1})} = 0.$$  

Suppose now that $k = i$; by (3), we have

$$y < n_i < y + n_i \leq n_j + n_i \leq n_{i+1}$$

and this implies that

$$\chi_{(y,y+n_i)}(x) - \chi_{(0,n_i)}(x) = \chi_{(y,y+n_i)} \cdot \chi_{(n_i,n_{i+1})}(x) = \chi_{(n_i,y+n_i)}(x).$$  \hfill □
Let
\[ \phi_k(x) = \frac{1}{n_k} \chi_{(0,n_k]}(x) \]
and define the kernel operator \( K : \mathbb{R} \to \ell^s(\mathbb{Z}^+) \) as
\[ K(x) = \{ \phi_k(x) - \phi_{k-1}(x) \}_{k \in \mathbb{Z}^+}. \]
It is clear that
\[ V_s f(x) = \| K * f(x) \|_{\ell^s(\mathbb{Z}^+)} \]
\[ = \left( \sum_{k=1}^{\infty} |\phi_k * f(x) - \phi_{k-1} * f(x)|^s \right)^{1/s} \]
\[ = \left( \sum_{k=1}^{\infty} |A_{n_k} f(x) - A_{n_{k-1}} f(x)|^s \right)^{1/s} \]
where * denotes convolution, i.e.,
\[ K * f(x) = \int K(x - y) \cdot f(y) \, dy. \]

Let \( B \) be a Banach space. We say that the \( B \)-valued kernel \( K \) satisfies the \( D_r \) condition, for \( 1 \leq r < \infty \), and write \( K \in D_r \), if there exists a sequence \( \{ c_i \}_{i=1}^{\infty} \) of positive numbers such that \( \sum_i c_i < \infty \) and such that
\[ \left( \int_{S_l(|y|)} \| K(x - y) - K(x) \|_{B}^r \, dx \right)^{1/r} \leq c_l |S_l(|y|)|^{-1/r'}, \]
for all \( l \geq 1 \) and all \( y > 0 \), where \( S_l(|y|) \) denotes the spherical shell \( 2^l |y| < |x| < 2^{l+1} y \) and \( \frac{1}{r} + \frac{1}{r'} = 1. \)

When \( K \in D_1 \) we have the Hörmander condition:
\[ \int_{|x| > 2|y|} \| K(x - y) - K(x) \|_B \, dx \leq C \]
where \( C \) is a positive constant which does not depend on \( y > 0. \)

**Lemma 4.** Let \( \gamma \) denote the smallest positive integer satisfying
\[ \frac{1}{\beta} + \frac{1}{\beta^r} \leq 1, \]
and let \( 1 \leq r, s < \infty \), \( i \geq j + \gamma \), and \( 0 < y \leq n_j \). Then
\[ \left( \int_{n_j}^{n_{j+1}} \| K(x - y) - K(x) \|_{\ell^s(\mathbb{Z}^+)}^r \, dx \right)^{1/r} \leq C_i n_i^{1/r - 1}, \]
i.e., \( K \) satisfies the \( D_r \) condition for \( 1 \leq r < \infty \).
Proof. Let
\[ \Phi_k(x, y) = \phi_k(x - y) - \phi_k(x). \]
Then it is easy to check that
\[ K(x - y) - K(x) = \{ \Phi_k(x, y) - \Phi_{k-1}(x, y) \}_{k \in \mathbb{Z}^+}. \]
On the other hand, because of a property of the norm we have
\[
\| K(x - y) - K(x) \|_{\ell^s(\mathbb{Z}^+)} = \| \Phi_k(x, y) - \Phi_{k-1}(x, y) \|_{\ell^s(\mathbb{Z}^+)} \\
\leq \| \Phi_k(x, y) \|_{\ell^s(\mathbb{Z}^+)} + \| \Phi_{k-1}(x, y) \|_{\ell^s(\mathbb{Z}^+)} \\
\leq 2 \| \Phi_{k-1}(x, y) \|_{\ell^s(\mathbb{Z}^+)},
\]
where \( x \) and \( y \) are fixed and \( \| \Phi_{k-1}(x, y) \|_{\ell^s(\mathbb{Z}^+)} \) is the \( \ell^s(\mathbb{Z}^+) \)-norm of the sequence whose \( k^{\text{th}} \)-entry is \( \Phi_k(x, y) \).

We now have
\[
\left( \int_{n_{i+1}}^{n_i} \| K(x - y) - K(x) \|_{\ell^s(\mathbb{Z}^+)} \right)^{1/r} \\
\leq 2 \left( \int_{n_{i+1}}^{n_i} \| \Phi_{k-1}(x, y) \|_{\ell^s(\mathbb{Z}^+)} \right)^{1/r} \\
\leq 2 \left( \int_{n_{i+1}}^{n_i} \| \Phi_{k-1}(x, y) \|_{\ell^s(\mathbb{Z}^+)} \right)^{1/r} \\
= 2 \left( \int_{n_{i+1}}^{n_i} \left( \sum_{n_i < n_{k=1}} \frac{1}{n_{k-1}} \chi_{n_i, n_{k=1}}(x) \right)^r \right)^{1/r} \\
= 2 \left( \int_{n_{i+1}}^{n_i} \left( \sum_{n_i < n_{k=1}} \frac{1}{\beta^{2k-2}} \chi_{n_i, n_{k=1}}(x) \right)^r \right)^{1/r} \\
\leq 2 \left( \beta^2 + \frac{1}{1 - \beta^2} \right) \cdot \frac{1}{n_i} \cdot \left( \int_{n_{i+1}}^{n_i} \chi_{n_i, n_{k=1}}(x) \right)^{1/r} \\
= 2 \left( \beta^2 + \frac{1}{1 - \beta^2} \right) \cdot \frac{1}{n_i} \cdot y^{1/r} \\
\leq 2 \left( \beta^2 + \frac{1}{1 - \beta^2} \right) \frac{1}{\beta^{(i-j)/r}} n_i^{1/r-1}
\]
where in the last inequality we used
\[ y \leq n_{i} \leq \frac{n_i}{\beta^{i-j}} \]
by (2), and this completes our proof with
\[ C_i = 2 \left( \beta^2 + \frac{1}{1 - \beta^2} \right) \frac{1}{\beta^{(i-j)/r}}. \]
Lemma 5. Let \( \{n_k\} \) be a lacunary sequence. Then there exists a constant \( C > 0 \) such that
\[
\sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| < C
\]
for all \( x \in \mathbb{R} \), where \( \hat{\phi}_k(x) = \frac{1}{n_k} \chi_{(0,n_k)}(x) \), and \( \hat{\phi}_k \) is its Fourier transform.

Proof. First, note that we have
\[
I(x) = \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| = \sum_{k=1}^{\infty} \left| \frac{1 - e^{-ixn_k}}{xn_k} - \frac{1 - e^{-ixn_{k-1}}}{xn_{k-1}} \right|
\]
Let
\[
I(x) = \sum_{\{k:|x|n_k \geq 1\}} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| + \sum_{\{k:|x|n_k < 1\}} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)| = I_1(x) + I_2(x).
\]
Let us now fix \( x \in \mathbb{R} \) and let \( k_0 \) be the first \( k \) such that \( |x|n_k \geq 1 \). Since \( \hat{\phi}_k(x) \) is an even function, we can assume without the loss of generality that \( x \geq 0 \).

We clearly have
\[
I_1(x) \leq \sum_{\{k:|x|n_k \geq 1\}} \frac{4}{|x|n_k}.
\]
Since the sequence \( \{n_k\} \) is lacunary, there exists a constant \( \beta > 1 \) such that \( n_{k+1}/n_k \geq \beta \) for all \( k \in \mathbb{N} \). Also note that in the sum, \( I_1 \), the term with index \( n_{k_0} \) is the term with smallest index, since it is the first term that satisfies condition \( |x|n_k \geq 1 \) and the sequence \( \{n_k\} \) is increasing. On the other hand, we have
\[
\frac{n_{k_0}}{n_k} = \frac{n_{k_0}}{n_{k_0+1}} \cdot \frac{n_{k_0+1}}{n_{k_0+2}} \cdot \frac{n_{k_0+2}}{n_{k_0+3}} \cdots \frac{n_{k-1}}{n_k} \leq \frac{1}{\beta^k}.
\]
We now have
\[
I_1(x) \leq \sum_{\{k:|x|n_k \geq 1\}} \frac{4}{|x|n_k} = \frac{4n_{k_0}}{|x|n_{k_0}n_k} = \frac{4}{|x|n_{k_0}} \sum_{\{k:|x|n_k \geq 1\}} \frac{n_{k_0}}{n_k} \leq 4 \sum_{\{k:|x|n_k \geq 1\}} \frac{1}{\beta^k}
\]
since \( \frac{1}{|x|n_{k_0}} \leq 1 \) and \( \frac{n_{k_0}}{n_k} = \frac{1}{\beta^k} \). Also, since
\[
\sum_{k=1}^{\infty} \frac{1}{\beta^k} = \frac{1}{1 - \frac{1}{\beta}}
\]
we clearly see that \( I_1(x) \leq C_1 \) for some constant \( C_1 > 0 \).

To control the summation \( I_2 \) let us first define the function \( F \) as

\[
F(r) = \frac{1 - e^{-ir}}{r}.
\]

Then we have \( \hat{\phi}_k(x) = F(xn_k) \). Now by the Mean Value Theorem, there exists a constant \( \xi \in (xn_k, xn_{k+1}) \) such that

\[
|F(xn_{k+1}) - F(xn_k)| = |F'(\xi)||xn_{k+1} - xn_k|.
\]

Also, it is easy to verify that

\[
|F'(x)| \leq \frac{x + 2}{x^2},
\]

for \( x > 0 \).

Now we have

\[
|F(xn_{k+1}) - F(xn_k)| = |F'(\xi)||xn_{k+1} - xn_k|
\leq \frac{\xi + 2}{\xi^2} |x|(n_{k+1} - n_k)
\leq \frac{xn_{k+1} + 2}{x^2n_k^2} |x|(n_{k+1} - n_k)
\leq \frac{2n_{k+1}}{n_k^2} (n_{k+1} - n_k).
\]

Thus, we have

\[
I_2(x) = \sum_{\{k : |x|n_k < 1\}} |F(xn_{k+1}) - F(xn_k)|
\leq \sum_{\{k : |x|n_k < 1\}} \frac{2}{|x|n_k} \cdot \frac{2n_{k+1}}{n_k^2} (n_{k+1} - n_k)
\leq 4n_{k+1}^2 \sum_{\{k : |x|n_k < 1\}} \frac{1}{n_k^2} |x| \left( \frac{1}{n_k} - \frac{1}{n_{k+1}} \right)
\leq \frac{16}{|x|} \left( \frac{1}{n_1} - \frac{1}{n_{k+1}} \right)
\leq \frac{16}{|x|n_{k+1}}
\leq 16.
\]

We thus conclude that

\[
I(x) = I_1(x) + I_2(x) \leq C_1 + 16 := C
\]
Lemma 6. Let $s \geq 2$ and $(n_k)$ be a lacunary sequence. Then there exists a constant $C > 0$ such that

$$\| V_s f \|_{L^2(\mathbb{R})} \leq C \| f \|_{L^2(\mathbb{R})}$$

for all $f \in L^2(\mathbb{R})$.

Proof. Since

$$\sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|^2 \leq \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|,$$

it is clear from Lemma 5 that there exists a constant $C > 0$ such that

$$\sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|^2 < C$$

for all $x \in \mathbb{R}$.

We now obtain

$$\| V_s f \|_{L^2(\mathbb{R})} = \int_{\mathbb{R}} \left( \sum_{k=1}^{\infty} |\hat{\phi}_k * f(x) - \hat{\phi}_{k-1} * f(x)|^2 \right)^{\frac{1}{2}} dx$$

$$\leq \int_{\mathbb{R}} \sum_{k=1}^{\infty} |\hat{\phi}_k * f(x) - \hat{\phi}_{k-1} * f(x)|^2 dx$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |\hat{\phi}_k * f(x) - \hat{\phi}_{k-1} * f(x)|^2 dx$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |(\hat{\phi}_k - \hat{\phi}_{k-1}) * f(x)|^2 dx$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |\Delta_k * f(x)|^2 dx \quad (\Delta_k(x) = \phi_k(x) - \phi_{k-1}(x))$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |\hat{\Delta_k} * f(x)|^2 dx \quad \text{(by Plancherel's theorem)}$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}} |\hat{\phi}_k(x)|^2 \cdot |\hat{f}(x)|^2 dx$$

$$= \int_{\mathbb{R}} \sum_{k=1}^{\infty} |\hat{\phi}_k(x)|^2 \cdot |\hat{f}(x)|^2 dx$$

$$= \int_{\mathbb{R}} \sum_{k=1}^{\infty} |\hat{\phi}_k(x) - \hat{\phi}_{k-1}(x)|^2 \cdot |\hat{f}(x)|^2 dx$$

$$\leq C \int_{\mathbb{R}} |\hat{f}(x)|^2 dx$$
where
\[ \|K * f(x)\|_{L^p(\mathbb{R})} = \mathcal{V}_s f(x). \]

Also, we have proved in Lemma 6 that \( T f = \{\phi_k - \phi_{k-1}\} \ast f \}_{k \in \mathbb{Z}^+} \) is a bounded operator from \( L^2(\mathbb{R}) \) to \( L^2(\ell^2(\mathbb{Z}^+)) \) since
\[ \|K * f(x)\|_{L^p(\mathbb{R})} = \mathcal{V}_s f(x). \]

Therefore, \( T f = \{\phi_k - \phi_{k-1}\} \ast f \}_{k \in \mathbb{Z}^+} \) is an \( \ell^2 \)-valued singular operator of convolution type for \( s \geq 2 \).

**Lemma 8.** Let \( A \) and \( B \) be Banach spaces. A singular integral operator \( T \) mapping \( A \)-valued functions into \( B \)-valued functions can be extended to an operator defined in all \( L^p_\alpha \), \( 1 \leq p < \infty \), and satisfying

1. \( \|Tf\|_{L^p_\alpha} < \|f\|_{L^p_\alpha}, \quad 1 \leq p < \infty \),
2. \( \|Tf\|_{WL^1_\alpha} < \|f\|_{L^1_\alpha} \),
3. \( \|Tf\|_{L^1_\alpha} < \|f\|_{H^1_\alpha} \),
4. \( \|Tf\|_{BM\alpha_0(B)} < \|f\|_{L^\infty_\alpha(A)}, \quad f \in L^\infty_\alpha(A), \)

where \( C_1, C_2, C_3 > 0 \), and \( L^\infty_\alpha(A) \) is the space of bounded functions with compact support.

**Proof.** This is Theorem 1.3 of Part II in Rubio de Francia, J. L. et al [5].

The following theorem is our first result:

**Theorem 9.** Let \( 2 \leq s < \infty \), and let \((n_k)\) be a lacunary sequence. Then there exists a constant \( C > 0 \) such that
\[ \|\mathcal{V}_s f\|_{L^1(\mathbb{R})} \leq C\|f\|_{H^1(\mathbb{R})} \]

for all \( f \in H^1(\mathbb{R}) \).

**Proof.** This follows from Remark 7 and Lemma 8 (iii) since \( \|K * f(x)\|_{L^\infty} = \mathcal{V}_s f(x) \).

**Remark 10.** We have proved that \( T f = \{\phi_k - \phi_{k-1}\} \ast f \}_{k \in \mathbb{Z}^+} \) is an \( \ell^s \)-valued singular operator of convolution type for \( s \geq 2 \). By applying Lemma 8 to this observation we also provide a different proof for the following known facts for \( s \geq 2 \) (see [4]) since
\[ \|K * f(x)\|_{L^\infty} = \mathcal{V}_s f(x). \]

1. \( \|\mathcal{V}_s f\|_{L^p(\mathbb{R})} < \|f\|_{L^p(\mathbb{R})}, \quad 1 \leq p < \infty \),
2. \( \|\mathcal{V}_s f\|_{WL^1(\mathbb{R})} < \|f\|_{L^1(\mathbb{R})} \),
3. \( \|\mathcal{V}_s f\|_{BM(\mathbb{R})} < \|f\|_{L^\infty(\mathbb{R})}, \quad f \in L^\infty(\mathbb{R}) \),

where \( C_1, C_2 > 0 \).
Let $w \in L^1_{\text{loc}}(\mathbb{R})$ be a positive function. We say that $w$ is an $A_p$ weight for some $1 < p < \infty$ if the following condition is satisfied:

$$\sup_I \left( \frac{1}{|I|} \int_I w(x) \, dx \right) \left( \frac{1}{|I|} \int_I w(x)^{-\frac{1}{p-1}} \, dx \right)^{p-1} < \infty,$$

where the supremum is taken over all intervals $I$ in $\mathbb{R}$. We say that the function $w$ is an $A_\infty$ weight if there exist $\delta > 0$ and $\varepsilon > 0$ such that given an interval $I$ in $\mathbb{R}$, for any measurable $E \subset I$,

$$|E| < \delta \cdot |I| \implies w(E) < (1 - \varepsilon) \cdot w(I).$$

Here

$$w(E) = \int_E w.$$

It is well known and easy to see that $w \in A_p \implies w \in A_\infty$ if $1 < p < \infty$. We say that $w \in A_1$ if given an interval $I$ in $\mathbb{R}$ there is a positive constant $C$ such that

$$\frac{1}{|I|} \int_I w(y) \, dy \leq C w(x)$$

for a.e. $x \in I$.

**Lemma 11.** Let $A$ and $B$ be Banach spaces, and $T$ be a singular integral operator mapping $A$-valued functions into $B$-valued functions with kernel $K \in D_r$, where $1 < r < \infty$. Then, for all $1 < \rho < \infty$, and for all $(f_j) \in L^p_A(w) \cap L^p_A(\mathbb{R}^n)$, the weighted inequalities

$$\left\| \left( \sum f_j^\rho \right)^{1/\rho} \right\|_{L^p(w)} \leq C_{\rho,p} \left( w \right) \left( \sum f_j^\rho \right)^{1/\rho} \left( \sum f_j^\rho \right)^{1/\rho} \left( \sum f_j^\rho \right)^{1/\rho}$$

hold if $w \in A_{\rho/r'}$ and $r' \leq p < \infty$, or if $w \in A_{r'}$, and $1 < p \leq r'$. Likewise, if $w(x)^{r'} \in A_1$, then the weak type inequality

$$w \left( \left\{ x : \left( \sum f_j(x)^\rho \right)^{1/\rho} > \lambda \right\} \right) \leq C_{\rho}(w) \frac{1}{\lambda} \int \left( \sum f_j(x)^\rho \right)^{1/\rho} w(x) \, dx$$

holds for all $(f_j) \in L^1_A(w) \cap L^1_A(\mathbb{R}^n)$.

**Proof.** This is Theorem 1.6 of Part II in Rubio de Francia, J. L. et al [5].

Our next result is the following:
Theorem 12. Let \( 2 \leq s < \infty \). Then, for all \( 1 < \rho < \infty \), and for all \( (f_j) \in L^p(w) \cap L^p(\mathbb{R}) \), the weighted inequalities

\[
\left\| \sum_j (V_s f_j)^\rho \right\|_{L^p(w)} \leq C_{p,\rho}(w) \left\| \sum_j |f_j|^\rho \right\|_{L^p(w)}
\]

hold if \( w \in A_{p/r'} \) and \( r' \leq p < \infty \), or if \( w \in A_p' \) and \( 1 < p \leq r' \). Likewise, if \( w(x)^{r'} \in A_1 \), then the weak type inequality

\[
w \left( \left\{ x : \left( \sum_j (V_s f_j(x))^{\rho} \right)^{1/\rho} > \lambda \right\} \right) \leq C_p(w) \frac{1}{\lambda} \int \left( \sum_j |f_j(x)|^{\rho} \right)^{1/\rho} w(x) \, dx
\]

holds for all \( (f_j) \in L^1(w) \cap L^1(\mathbb{R}) \).

Proof. We have proved for \( 2 \leq s < \infty \) that \( T f = \{(\phi_k - \phi_{k-1}) * f\}_{k \in \mathbb{Z}}^+ \) is an \( \ell^s \)-valued singular integral operator of convolution type and its kernel operator \( K(x) = \{(\phi_k(x) - \phi_{k-1}(x))\}_{k \in \mathbb{Z}}^+ \) satisfies \( D_s \) condition for \( 1 \leq r < \infty \). Thus, the result follows from Lemma 11 and the fact that \( \| K * f(x) \|_{\ell^s(\mathbb{Z})} = V_s f(x) \). \( \square \)

In particular we have the following corollary:

Corollary 13. Let \( 2 \leq s < \infty \). Then the weighted inequalities

\[
\|V_s f\|_{L^p(w)} \leq C_{p,\rho}(w) \|f\|_{L^p(w)}
\]

hold for all \( (f_j) \in L^p_A(w) \cap L^p_A(\mathbb{R}^n) \) if \( w \in A_{p/r'} \) and \( r' \leq p < \infty \), or if \( w \in A_p' \) and \( 1 < p \leq r' \). Likewise, if \( w(x)^{r'} \in A_1 \), then the weak type inequality

\[
w \left( \left\{ x : V_s f(x) > \lambda \right\} \right) \leq C_p(w) \frac{1}{\lambda} \int |f(x)| w(x) \, dx
\]

holds for all \( (f_j) \in L^1(w) \cap L^1(\mathbb{R}) \).

References


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