Random nilpotent groups of maximal step

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Abstract. Let $G$ be a random torsion-free nilpotent group generated by two random words of length $\ell$ in $U_n(\mathbb{Z})$. Letting $\ell$ grow as a function of $n$, we analyze the step of $G$, which is bounded by the step of $U_n(\mathbb{Z})$. We prove a conjecture of Delp, Dymarz, and Schaffer-Cohen, that the threshold function for full step is $\ell = n^2$.

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1. Introduction

A group $G$ is nilpotent if its lower central series,

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_r = \{1\}$$

defined by $G_{i+1} = [G, G_i]$, eventually terminates. The first index $r$ for which $G_r = \{1\}$ is called the step of $G$. One may ask what a generic nilpotent group looks like, including its step. Questions about generic properties of groups can be answered with random groups, first introduced by Gromov [5]. Since Gromov’s original few relators and density models are nilpotent with probability 0, they cannot tell us about generic properties of nilpotent groups. Thus, there is a need for new random group models that are nilpotent by construction.

Delp, et al. (2019) [3] introduced a model for random nilpotent groups, motivated by the observation that any finitely generated torsion-free nilpotent group can be embedded in the group $U_n(\mathbb{Z})$ of $n \times n$ upper triangular integer matrices with ones on the diagonal [4]. Note that, since any finitely generated nilpotent group contains a torsion-free subgroup of finite index, we are not losing much by restricting our attention to torsion-free groups. (Another model is considered in [2]).

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We construct a random subgroup of $U_n(Z)$ as follows. Let $E_{i,j}$ be the elementary matrix with 1’s on the diagonal, a 1 at position $(i, j)$ and 0’s elsewhere. Then $S = \{ E_{i,j}^n : 1 \leq i < n \}$ forms the standard generating set for $U_n(Z)$. We call the entries at positions $(i, i+1)$ the superdiagonal entries. Define a random walk of length $\ell$ to be a product

$$V = V_1 V_2 \ldots V_\ell$$

where each $V_i$ is chosen independently and uniformly from $S$. Let $V$ and $W$ be two independent random walks of length $\ell$. Then $G = \langle V, W \rangle$ is a random subgroup of $U_n(Z)$. We have step($G$) $\leq$ step($U_n(Z)$), and it is not hard to check that step($U_n(Z)$) $= n - 1$. If step($G$) $= n - 1$, we say $G$ has full step.

Now let $n \to \infty$ and $\ell = \ell(n)$ grow as a function of $n$. We say a proposition $P$ holds asymptotically almost surely (a.a.s.) if $\mathbb{P}[P] \to 1$ as $n \to \infty$. Delp et al. (2019) gave results on the step of $G$, depending on the growth rate of $\ell$ with respect to $n$. Recall that $f = o(g(n))$ means $f(n)/g(n) \to 0$ as $n \to \infty$ and $f = \omega(g(n))$ means $f(n)/g(n) \to \infty$ as $n \to \infty$.

**Theorem 1.1** (Delp-Dymarz-Schaffer-Cohen). Let $n, \ell(n) \to \infty$ and $G = \langle V, W \rangle$ where $V, W$ are independent random walks of length $\ell$. Then:

1. If $\ell(n) = o(\sqrt{n})$ then a.a.s. step($G$) $= 1$, i.e. $G$ is abelian.
2. If $\ell(n) = o(n^2)$ then a.a.s. step($G$) $< n - 1$.
3. If $\ell(n) = \omega(n^2)$ then a.a.s. step($G$) $= n - 1$, i.e. $G$ has full step.

In this paper we close the gap between cases 2 and 3.

**Theorem 1.2.** Let $n, \ell(n) \to \infty$ and $G = \langle V, W \rangle$. If $\ell(n) = \omega(n^2)$ then a.a.s. $G$ has full step.

To prove this requires a careful analysis of the nested commutators that generate $G_{n-1}$. In Section 1, we give a combinatorial criterion for a nested commutator of $V$’s and $W$’s to be nontrivial. In Section 2, we show this criterion is satisfied asymptotically almost surely when $V, W$ are random walks.

### 2. Nested commutators

Let $G = G_0 \geq G_1 \geq \ldots$ be the lower central series of $G$. We have

$$G_i = [G, G_{i-1}] = [G, [G, \ldots, [G, G] \ldots]]$$

In particular, $G_i$ includes all $i + 1$-fold nested commutators of elements of $G$. We restrict our attention to commutators where each factor is $V$ or $W$.

Let $\{0,1\}^d$ be the $d$-dimensional cube, or the set of all length $d$ binary vectors. For $x \in \{0,1\}^d$, $y \in \{0,1\}^e$ we define the norm $N(x) = \sum_{1 \leq i \leq d} x_i$ and the concatenation $xy \in \{0,1\}^{d+e}$. For example if $x = (1,0,0)$ and $y = (0,1)$ then $xy = (1,0,0,0,1) = 10^31$. 
We define a family of maps $C_d : \{0, 1\}^d \to G_d$ as follows.

$C_1(1) = V$
$C_1(0) = W$
$C_d(1x) = [V, C_{d-1}(x)]$
$C_d(0x) = [W, C_{d-1}(x)]$

Thus for example, $C_2(1101) = C_2(10001) = [V, [W, [W, [W, V]]]].$ We omit the subscript $d$ when it is obvious. To prove $G$ has full step, it suffices to find an $x \in \{0, 1\}^{n-1}$ such that $C(x)$ is nontrivial. We begin with Lemma 2.3 from [3], which gives a recursive formula for the entries of a nested commutator.

**Lemma 2.1.** Let $a \in \{0, 1\}, x \in \{0, 1\}^{d-1}.$ Then $C(ax) \in G_d$ and we have

$$C(ax)_{i,i+d} = C(a)_{i,j+1}C(x)_{i+1,j+1} - C(a)_{i,j-1}C(x)_{i+1,j}$$

and furthermore $C(ax)_{i,j} = 0$ for $j < i + d.$

In particular, for $d = n - 1$ only the upper rightmost entry $C(ax)_{1,n}$ can be nonzero.

From the formula, it is clear that $C(ax)_{i,i+d}$ is a degree-$d$ polynomial in the superdiagonal entries of $V$ and $W.$ Let us state this more precisely and analyze the coefficients of the polynomial.

**Lemma 2.2.** Let $d \geq 1.$ There exists a function $K_d : \{0, 1\}^d \times \{0, 1\}^d \to \mathbb{Z}$ such that for $1 \leq i \leq n - d$ we have

$$C(x)_{i,i+d} = \sum_{y \in \{0, 1\}^d} K_d(x, y) \prod_{i \leq j < i+d} V_{j,j+1}^{y_j} W_{j,j+1}^{1-y_j}$$

(1)

Furthermore, setting $K_d(x, y) = 0$ for $N(x) \neq N(y)$ we have a recursion

$$K_d(ax, byc) = K_1(a, b)K_{d-1}(x, yc) - K_1(a, c)K_{d-1}(x, by)$$

with base cases

$$K_1(0, 0) = K_1(1, 1) = 1$$
$$K_1(0, 1) = K_1(1, 0) = 0$$

Note that $K_d(x, y)$ does not depend on $i.$ We also drop the subscript $d$ since it can be inferred from $x$ and $y.$

**Proof.** Abbreviate

$$U(i, d, y) := \prod_{i \leq j < i+d} V_{j,j+1}^{y_j} W_{j,j+1}^{1-y_j}$$

We first prove inductively that there exist coefficients $K_d : \{0, 1\}^d \times \{0, 1\}^d \to \mathbb{Z}$ such that

$$C(x)_{i,i+d} = \sum_{y \in \{0, 1\}^d} K_d(x, y)U(i, d, y)$$
The case $d = 1$ is trivial. Assume it holds for $d - 1$. Let $a \in \{0, 1\}$ and $x \in \{0, 1\}^{d-1}$, then we have

$$C(ax)_{i,i+d} = C(a)_{i,i+1}C(x)_{i+1,i+d} - C(a)_{i+d-1,i+d}C(x)_{i,i+d-1}$$

Expanding $C(a)_{i,i+1}$ and $C(x)_{i+1,i+d}$, the first term is

$$= [K_1(a, 1)V_{i,i+1} + K_1(a, 0)W_{i,i+1}] \left[ \sum_{y \in \{0,1\}^{d-1}} K_{d-1}(x,y)U(i+1,d-1,y) \right]$$

$$= \sum_{y \in \{0,1\}^{d-1}} K_1(a,1)K_{d-1}(x,y)U(i,d,1y) + K_1(a,0)K_{d-1}(x,y)U(i,d,0y)$$

$$= \sum_{b,c \in \{0,1\}, y' \in \{0,1\}^{d-2}} K_1(a,b)K_{d-1}(x,y')U(i,d,by'c)$$

Similarly, the second term is

$$= \sum_{b,c \in \{0,1\}, y' \in \{0,1\}^{d-2}} K_1(a,c)K_{d-1}(x,by')U(i,d,by'c)$$

Combining, we get

$$C(ax)_{i,i+d} = \sum_{b,c \in \{0,1\}, y \in \{0,1\}^{d-2}} [K_1(a,b)K_{d-1}(x,yc) - K_1(a,c)K_{d-1}(x,by)] U(i,d,byc)$$

and setting $K_d(ax, byc) = K_1(a,b)K_{d-1}(x,yc) - K_1(a,c)K_{d-1}(x,by)$, the lemma is proved for $d$. It is also easy to see inductively that $K_d(x, y) = 0$ for $N(x) \neq N(y)$, so we may add the condition $N(x) = N(y)$ under the sum.

We now have a strategy for choosing $x \in \{0, 1\}^{n-1}$ such that $C(x)$ is nontrivial. In the random model, it may happen that $V_{i,i+1} = 0$ for some $i$. Define the vector $v \in \{0, 1\}^{n-1}$ by $v_j = 1$ if $V_{i,i+1} \neq 0$ and $v_j = 0$ otherwise. For now assume $0 < N(v) < n-1$. If we choose $x$ such that $N(x) = N(v)$, then Equation 1 simplifies to

$$C_{n-1}(x)_{1,n} = K_d(x, v) \prod_{1 \leq i < n} V_{i,i+1}^{v_i} W_{i,i+1}^{1-v_i}.$$

If we assume there is no $i$ such that $V_{i,i+1} = W_{i,i+1} = 0$, the product of matrix entries is nonzero. So, we just need to choose $x$ such that $K_d(x, v) \neq 0$. We can do this with some additional assumptions on $v$.

**Lemma 2.3.** Let $v \in \{0, 1\}^{n-1}$ with $0 < N(v) < n-1$. Write $v = 1^{a_1}01^{a_2} \ldots 1^{a_k}01^{a_k}$. Assume that $a_i \geq 1$ for all $i$, i.e., there are no adjacent 0's, and that $a_1 \neq a_k$. Then there exists $x \in \{0, 1\}^{n-1}$ such that $K(x, v) \neq 0$.

We will prove in section 2 that all assumptions used hold asymptotically almost surely.
Proof. Using the recursion from Lemma 2.2, the following identities are easily verified by induction:

1. If \(a, b \geq 0\), then 
   \[K(1^{a+b}00^1, 1^a01^b) = \binom{a + b}{a}(-1)^b\]

2. If \(a, b \geq 1, c \geq 0\) with \(c < \min(a, b)\), then 
   \[K(1^c0x, 1^a0y1^b) = 0\]

3. Let \(a, b \geq 0\). If \(a < b\) then 
   \[K(1^a0x, 1^a0y1^b) = K(x, y1^b)\]
   If \(b < a\) then 
   \[K(1^b0x, 1^ax01^b) = K(x, 1^a0y)(-1)^{b+1}\]

4. If \(a, b \geq 0\) then 
   \[K(1^{a+b}0^2x, 1^a0y10^1b) = 2\binom{a + b}{a}(-1)^bK(x, 1y1)\]

Let \(v = 1^{a_1}a_2 \ldots 0 \ldots 1^{a_k}\). First assume \(k = 2\ell\) is even. We set 
\[x = 1^{a_1+a_2+2a_3+\ldots+a_{\ell-1}0^2} \ldots 1^{a_\ell+a_{\ell+1}0}\]
Then applying identity 4 repeatedly followed by identity 1, we obtain 
\[K(x, v) = 2\ell(-1)^{a_2+a_4+a_6+\ldots+a_{\ell+1}}\binom{a_1+a_2+2\ell+1}{a_1}\binom{a_2+a_4+2\ell}{a_2}\ldots\binom{a_\ell+a_{\ell+1}}{a_\ell}\]
If \(k\) is odd, we apply identity 3 once and proceed as before. \(\square\)

3. Asymptotics

In Section 1, we derived a combinatorial condition on the superdiagonal entries of \(V\) and \(W\) sufficient for \(G\) to have full step. Define 
\[\mathcal{V} = \{i : 1 \leq i < n, V_{i,j+1} = 0\}\]
\[\mathcal{W} = \{i : 1 \leq i < n, W_{i,j+1} = 0\}\]
Then, to apply Lemma 2.3, we need that

1. \(\mathcal{V}\) and \(\mathcal{W}\) are nonempty.
2. \(\mathcal{V} \cap \mathcal{W} = \emptyset\).
3. \(\mathcal{V}\) has no adjacent elements.
4. \(\min \mathcal{V} \neq n - \max \mathcal{V}\).

If condition (1) does not hold, then Theorem 1.2 follows by a modification of Lemma 5.4 in [3].

We now show that in the random model, if \(\ell = \omega(n^2)\), then the superdiagonal entries satisfy conditions (2)-(4) asymptotically almost surely. Recall that \(V\) and \(W\) are random walks 
\[V = V_1V_2 \ldots V_\ell\]
\[W = W_1W_2 \ldots W_\ell\]
where each \( V_i, W_i \) is chosen independently and uniformly from the generating set \( S = \{ E_{i,j+1}^{\pm 1} : 1 \leq i < n \} \).

Define

\[
\sigma_j(Z) = \begin{cases} 
1 & \text{if } Z = E_{j,j+1} \\
-1 & \text{if } Z = E_{j,j+1}^{-1} \\
0 & \text{otherwise}
\end{cases}
\]

Then we have

\[
V_{i,i+1} = \sum_{j=1}^\ell \sigma_\ell(V_j).
\]

When \( \ell \gg n \), the superdiagonal entries \( V_{i,i+1} \) behave roughly like independent random walks on \( \mathbb{Z} \). We restate Corollary 3.2 from [3].

**Lemma 3.1.** Suppose \( \ell = o(n) \). Then uniformly for \( 1 \leq k_1 < k_2 < \cdots < k_d < n \) we have

\[
P[k_i \in V \cap W \text{ for all } i] \sim \left( \frac{n}{2\pi \ell} \right)^d
\]

By the union bound, we have \( P[V \cap W \neq \emptyset] \ll n^2/\ell \to 0 \). Thus, condition (2) holds a.a.s. For conditions (3) and (4), we will need a bound on the size of \( V \).

**Lemma 3.2.** Fix \( \varepsilon > 0 \). Then \( P[|V| > \varepsilon\sqrt{n}] \to 0 \) as \( n \to \infty \).

**Proof.** Define random variables

\[
X_i = \begin{cases} 
1 & V(i,i+1) = 0 \\
0 & V(i,i+1) \neq 0
\end{cases}
\]

So \( |V| = \sum_i X_i \). From Lemma 3.1 we have \( E[X_i] \ll \sqrt{n/\ell} \) and \( E[X_i X_j] \ll n/\ell \) for \( 1 \leq i < j < n \). Hence \( E[|V|] \ll \sqrt{n^3/\ell} \) and \( \text{Var}[|V|] \ll n^3/\ell \). By Chebyshev's inequality,

\[
P[|V| \geq \varepsilon \sqrt{n}] \leq P \left[ |V| - \sqrt{n^3/\ell} \geq \sqrt{n}(\varepsilon - \sqrt{n^3/\ell}) \right]
\]

\[
\leq \frac{1}{(\varepsilon - \sqrt{n^3/\ell})^2(\ell/n^3)} \to 0
\]

Observe that the distribution of \( V \) is invariant under permutation. In other words, for a fixed set \( S \subset \{1, \ldots, n-1\} \) and a permutation \( \pi \) on \( \{1, \ldots, n-1\} \) we have

\[
P[V = S] = P[V = \pi S]
\]
and hence,\[ \mathbb{P}[V = S] = \frac{1}{\binom{n-1}{|S|}} \mathbb{P}[|V| = |S|] \]

Let \( A(k) \) be the number of sets \( S \subset \{1, \ldots, n-1\} \) of size \( k \) with at least one pair of adjacent elements. We have
\[ A(k) \leq (n-2) \binom{n-3}{k-2}. \]

Let \( B(k) \) be the number of sets \( S \) for which \( \min S = n - \max S \). Summing over the possible values of \( \min S \) we have
\[ B(k) \leq \sum_{1 \leq \alpha \leq n/2} \binom{n-1-2\alpha}{k-2}. \]

One easily checks
\[ \frac{A(k) + B(k)}{\binom{n-1}{k}} \leq \frac{2k^2}{n}. \]

For \( k \leq \varepsilon\sqrt{n} \), this is \( \leq 2\epsilon^2 \). On the other hand, \( \mathbb{P}[|V| > \varepsilon\sqrt{n}] \to 0 \), so we are done.

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