New York Journal of Mathematics

New York J. Math. **31** (2025) 668–689.

A converse to Littlewood's theorem on random analytic functions

Yin Cai, Guozheng Cheng, Xiang Fang* and Chao Liu

Abstract. We reformulate the Littlewood theorem on random analytic functions in the Hardy spaces as a problem of determining the random symbol spaces, and we show that, under a general randomization scheme, the symbol space is always a subspace of $H^2(\mathbb{D})$ (Corollary 2.6). We then characterize completely when the symbol space is precisely $H^2(\mathbb{D})$ (Theorem 1.1). This result extends Littlewood's theorem and can also be considered a converse of the theorem, since previous literature has focused solely on the sufficiency part of the results. We establish an analog of the Fernique theorem by determining the optimal integrability exponent within the L^p-scale (Theorem 1.2), and we propose a conjecture concerning general Young functions. The issue of determining which vector spaces can emerge as symbol spaces is exemplified through examples.

Contents

1. Introduction

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D})$ be an analytic function over the unit disk in the complex plane, and let X_0 be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this note, by a random analytic function (RAF) we mean the series

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} a_n X_n z^n,$$

Received March 27, 2025.

2020 *Mathematics Subject Classification.* 30B20; 30H10.

Key words and phrases. Hardy spaces; Littlewood's theorem; random analytic function; integrability; Fernique's theorem.

*Corresponding author.

ISSN 1076-9803/2025

where $\{X_n\}_{n\geq 0}$ is a sequence of i.i.d. random variables.

The convergence radius of $\mathcal{R}f$ is almost surely a constant, and to ensure that $\mathcal{R}f$ is an analytic function over $\mathbb{D}$ for any $f \in H(\mathbb{D})$, it is necessary and sufficient to have $\limsup_{n\to\infty} |X_n|^{\frac{1}{n}} \leq 1$ a.s. This amounts to requiring

$$\sum_{n=0}^{\infty} \mathbb{P}(|X_0| > c^n) < \infty$$

for some, hence for all, $c \in (1, \infty)$. It is satisfied, in particular, if $X_0 \in L^p(\Omega, \mathcal{F}, \mathbb{P})$ for some $p > 0$.

The study of such series goes back to Émile Borel as early as 1896 ([2], [9, p. 37]). Many facets of $\mathcal{R}f$ have been thoroughly investigated over its long and rich history, placing significant emphasis on comprehending the distribution of its zero sets. Moreover, a substantial portion of the literature revolves around three canonical randomization methods; that is, when X_0 is Gaussian, Rademacher, or Steinhaus, as manifested in the preface of Kahane's monograph [9], where they are shorthanded as (G), (R), and (S), respectively.

A question occuring repeatedly to us, from colleagues and students alike, is that what happens to other randomization methods? This question, extensively examined in the context of random polynomials when the goal is to understand their zero sets, remains comparatively less explored for random analytic functions. In this vein, a significant contribution is the investigation of the so-called "pits property," a study carried out by Offord in a series of papers [17, 18] for rather general X_0, following the celebrated joint work of Littlewood and Offord on the Rademacher case [16]. This topic has been continued until recent years with more general conditions in the form of a stationary Gaussian process, introduced and studied in [3].

Extensions beyond the three canonical randomization methods in the literature often take one of the following three forms (or a combination of them):

- Symmetric distributions: That is, X_0 and $-X_0$ have the same distribution, which allows Kahane to introduce a trick, now called the reduction principle [9, p. 9], to reduce the study of $\mathcal{R}f$ to the Rademacher case in many scenarios. More information can be found in [9, Chapter 5]. Another benefit in this instance is the presence of the Lévy inequality [12, Theorem III.2, p. 129], which extends the Kolmogorov inequality.
- Subgaussian and subexponential distributions: A random variable X is called *subgaussian* if there exists $\tau > 0$ such that for every $t \in \mathbb{R}$ we have $\mathbb{E}\exp(tX) \leq e^{\frac{1}{2}\tau^2 t^2}$ [9, p. 82]. A random variable Y is *subexponential* if there exists $\lambda > 0$ such that $\mathbb{E}\exp(\lambda|Y|) < \infty$ [24, p. 31]. In these

cases, the rapid decay of probability tails and various strong integrability results, such as the Fernique theorem, often yield robust estimates.

- Under (technical) moment-type conditions such as $\sup_{n\geq 0} \frac{\mathbb{E}X_n^4}{(\mathbb{E}X_n^2)^2} < \infty$ in [9, Theorem 1, p. 54], the hypotheses I and II in [17], (1.14) in [18], or (2.2.1) in [3].

It is perhaps fair to say that all these three approaches are designed to offer technical convenience, and the nature of the impact of the distribution of X_0 on the behaviors of $\mathcal{R}f$ has received relatively limited attention in the past. In other words, what conditions are necessary instead of merely being sufficient? As far as our knowledge extends, there have been essentially no known results in this particular direction.

The purpose of this note is to explore the effect of the distribution of X_0 on the properties of random analytic functions, subject to as little constraint on X_0 as possible. We choose to study this problem in the setting of Littlewood's theorem, which is one of the first major results on random analytic functions, and is reformulated as follows. In 1930, Littlewood [15] proved that if $f \in H^2(\mathbb{D})$, then

$$(\mathcal{R}f)(z) = \sum_{n=0}^{\infty} \pm a_n z^n \in H^p(\mathbb{D})$$

for any $p \in (0, \infty)$ a.s., and if $f \notin H^2(\mathbb{D})$, then almost surely, $\mathcal{R}f$ has a radial limit almost nowhere. In particular, Littlewood's theorem implies that $\mathcal{R}f$ is a.s. in $H^p(\mathbb{D})$ if and only if f is in $H^2(\mathbb{D})$. A convenient way to reformulate this fact is to introduce the random symbol spaces. Let $\mathcal{X}$ be a Banach or p-Banach space, with $0 < p < 1$, of analytic functions over $\mathbb{D}$. Define

$$(\mathcal{X})_\star = \{f \in H(\mathbb{D}) : \mathbb{P}(\mathcal{R}f \in \mathcal{X}) = 1\}.$$

Then we reformulate Littlewood's theorem as

$$(H^p(\mathbb{D}))_\star = H^2(\mathbb{D}) \tag{1}$$

if X_0 is Rademacher. The same conclusion holds if X_0 is Steinhaus [14, 21] or Gaussian [9, p. 54] (or, see [19]).

In this note, our goal is to study the following:

Problem: Let $p \in (0, \infty)$. How to characterize those X_0's such that (1) holds?

There are three reasons why we choose to work with this problem.

(i) Since there is no research done in the literature which is of the nature of a converse problem, it stands to reason to start with one of the earliest and most fundamental results in this area.

(ii) The Littlewood theorem reformulated as (1) captures somehow the essence of randomization, at least in the framework of mixed norm spaces $H^{p,q,\alpha}(\mathbb{D})$ ($0 < p, q, \alpha < \infty$), in view of [4, Theorem 6], which states that

$$(H^{p,q,\alpha}(\mathbb{D}))_\star = H^{2,q,\alpha}(\mathbb{D}) \tag{2}$$

under the three classical randomization methods. Here, a function $f \in H(\mathbb{D})$ belongs to *the mixed norm space* $H^{p,q,\alpha}(\mathbb{D})$ if

$$||f||_{H^{p,q,\alpha}(\mathbb{D})} = \left(\int_0^1 (1-r)^{q\alpha-1} M_p(f,r)^q dr \right)^{\frac{1}{q}} < \infty,$$

where $M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p}$. In other words, (2) suggests that the effect of randomization on an analytic function is to orthogonalize the circular p-norm to a 2-norm, but keep intact the two radial parameters q and α in the measurement of $\mathcal{R}f$. In this sense, Littlewood's theorem pinpoints a characteristic feature of randomization, at least in the setting of mixed norm spaces. This feature is also distinctly illustrated for entire functions in [6].

(iii) We shall introduce and study a so-called *"range problem"* for $(H^p(\mathbb{D}))_\star$ in Section 4; that is, given p, what vector space E can be realized as $(H^p(\mathbb{D}))_\star$ for some choice of X_0? Our preliminary investigation of this problem suggests that the case

$$E = H^2(\mathbb{D}) \tag{3}$$

is both generic and extremal. In particular, we shall show that if only X_0 is non-zero, then $(H^p(\mathbb{D}))_\star \subset H^2(\mathbb{D})$; see Theorem 1.5 or Corollary 2.6.

Our first main result solves completely the above problem, hence extending Littlewood's theorem in particular.

Theorem 1.1. *Let $\{X_n\}_{n\geq 0}$ be a sequence of non-constant i.i.d. random variables and $0 < p < \infty$.*

(i) *If $0 < p \leq 2$, then $(H^p(\mathbb{D}))_\star = H^2(\mathbb{D})$ if and only if $X_0 \in L^2(\Omega)$;*
(ii) *If $2 < p < \infty$, then $(H^p(\mathbb{D}))_\star = H^2(\mathbb{D})$ if and only if $X_0 \in L^2(\Omega)$ and $\mathbb{E}X_0 = 0$.*

When $\mathbb{E}X \neq 0$ in (ii) above, we shall see that $(H^p(\mathbb{D}))_\star = H^p(\mathbb{D})$.

Next, the investigation shifts towards examining the integrability of $||\mathcal{R}f||_{H^p(\mathbb{D})}$, which is now assumed to be a well-defined random variable. In particular, given $f \in (H^p(\mathbb{D}))_\star$, a fundamental issue is whether $||\mathcal{R}f||_{H^p(\mathbb{D})} < \infty$ a.s. implies that

$$\mathbb{E}||\mathcal{R}f||^t_{H^p(\mathbb{D})} < \infty, \tag{4}$$

which, in turn, allows one to introduce a (p-)Banach space structure on $(H^p(\mathbb{D}))_\star$. This is an important feature for the three canonical randomization methods, and one actually has the integrability of $||\mathcal{R}f||^t_{H^p(\mathbb{D})}$ for any $t > 0$, by Fernique's theorem [12, Theorem V.26, p. 255] for the Gaussian sequence, and by Kahane's inequality [12, Theorem V.2, p. 139] for the Rademacher and Steinhaus cases.

For a general X_0, the integrability of $\|\mathcal{R}f\|_{H^p(\mathbb{D})}$, under the assumption that $f \in (H^p(\mathbb{D}))_\star$, is no longer automatic. In particular, if $X_0 \notin L^1(\Omega)$, then (4) fails for $t = 1$ always. Our second main result is to identify the optimal integrability exponent for Littlewood-type phenomena. The following might be regarded as an L^p-version of the Fernique theorem which is for Gaussian vectors.

Theorem 1.2. *Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. random variables and $0 < p < \infty$.*

(i) *If $0 < q < \infty$, $\{a_n\}_{n\geq 0}$ is a non-zero sequence and*

$$\mathbb{E}\Big\| \sum_{n=0}^{\infty} a_n X_n z^n \Big\|_{H^p(\mathbb{D})}^q < \infty, \tag{5}$$

then $X_0 \in L^q(\Omega)$.

(ii) *If $2 \leq q < \infty$, $X_0 \in L^q(\Omega)$ and $\mathbb{E}X_0 = 0$, then, for all $\{a_n\}_{n\geq 0} \in \ell^2$, (5) holds.*

Moreover, $(H^p(\mathbb{D}))_\star$ is a (p-)Banach space under the norm $\big(\mathbb{E}\|\cdot\|_{H^p(\mathbb{D})}^t\big)^{1/t}$ if $1 \leq t \leq q$, $2 \leq q < \infty$ and $X_0 \in L^q(\Omega)$.

Recall that a functional $\|\cdot\| : E \to [0, \infty)$ is called a p-**norm** with $p \in (0, 1)$ if E is a complex vector space and $x, y \in E$, then

(i) $\|x\| > 0$ if $x \neq 0$, and $\|\lambda x\| = |\lambda|\|x\|$, $\lambda \in \mathbb{C}$; and
(ii) $\|x + y\|^p \leq \|x\|^p + \|y\|^p$.

If (E, d), with $d(x, y) = \|x - y\|^p$, is a complete metric space, then it is called a p-**Banach space**.

As complements to Theorem 1.2, we include two additional types of considerations:

- exponential integrability, and
- almost surely integrability.

For the former, i.e., when we take a closer look at the integrability problem associated with the Young functions $\varphi_q(x) = e^{x^q} - 1$ $(0 < q < \infty)$, we observe that an application of Talagrand's result [23, Theorem 4] (or see [11, Theorem 6.21, p. 172]) yields a neat generalization, for $H^p(\mathbb{D})$-valued random vectors, of the Fernique theorem, which corresponds to $q = 2$ below.

Theorem 1.3. *Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. random variables in $L^{\varphi_q}(\Omega)$, $\varphi_q(x) = e^{x^q} - 1$ $(1 < q \leq 2)$, $1 \leq p < \infty$, and $\mathbb{E}X_0 = 0$. Then for each $\{a_n\}_{n\geq 0} \in \ell^2$, there exists some $\lambda > 0$ such that*

$$\mathbb{E}\exp\Big(\lambda \Big\| \sum_{n=0}^{\infty} a_n X_n z^n \Big\|_{H^p(\mathbb{D})}^q\Big) < \infty. \tag{6}$$

The necessity of the Banach space $H^p(\mathbb{D})$ is clearly not essential, and to what extent it can be generalized is a worthy problem. On the other hand, even the case $q = 2$ is broader than Fernique's theorem and appears to be new. Recall that, for a Young function φ, $X \in L^\varphi(\Omega)$ [22] means that

$$\|X\|_{L^\varphi(\Omega)} = \inf\Big\{a > 0 : \ \mathbb{E}\Big(\varphi(\frac{|X|}{a})\Big) \le 1\Big\} < \infty.$$

A conjecture to extend the Fernique theorem to general Young functions is included after the proof of Theorem 1.3.

A dichotomy for almost sure behaviours. Lastly, if instead, we consider almost sure behaviors of $\mathcal{R}f$, then a strong dichotomy emerges, according to the coefficients are square-summable or not. This leads to our third type of integrability results.

Let $\{a_n\}_{n\ge0} \in \ell^2$ and $\{X_n\}_{n\ge0}$ be a sequence of i.i.d. random variables. If $X_0 \in L^2(\Omega)$ and symmetric, then, by [9, Theorem 1, p. 54],

$$\int_0^{2\pi} \exp\Big(\lambda\Big|\sum_{n=0}^{\infty} a_n X_n e^{int}\Big|^2\Big)dt < \infty \quad \text{a.s.} \tag{7}$$

for some $\lambda > 0$. This may be viewed as an extension of the classical Paley-Zygmund exponential estimates.

In contrast, if $\{a_n\}_{n\ge0} \notin \ell^2$, then (7) fails dramatically. The following theorems illustrate this phenomenon and they are inspired by [9, Proposition 2, p. 122].

Theorem 1.4. *Let $\{X_n\}_{n\ge0}$ be a sequence of i.i.d. non-zero symmetric random variables, and $\varphi(x)$ a non-negative function over $(0,\infty)$ with $\lim\limits_{x\to\infty}\varphi(x) = \infty$. If $\{a_n\}_{n\ge0} \notin \ell^2$ and $\sum\limits_{n=0}^{\infty} a_n X_n z^n \in H(\mathbb{D})$ a.s., then*

$$\limsup_{r\to1^-} \int_0^{2\pi} \varphi\Big(\Big|\sum_{n=0}^{\infty} a_n X_n r^n e^{in\theta}\Big|\Big)d\theta = \infty \quad a.s. \tag{8}$$

Theorem 1.5. *Let $\{X_n\}_{n\ge0}$ be a sequence of i.i.d. non-zero random variables, and φ an increasing function over $(0,\infty)$ such that $\lim\limits_{x\to\infty}\varphi(x) = \infty$ and $\varphi(x+y) \lesssim \varphi(x)+\varphi(y)$, $x, y \ge 0$. If $\{a_n\}_{n\ge0} \notin \ell^2$ and $\sum\limits_{n=0}^{\infty} a_n X_n z^n \in H(\mathbb{D})$ a.s., then (8) holds.*

We shall prove only Theorem 1.4 in Section 3, since Theorem 1.5 follows from Theorem 1.4 and the device of symmetrization. In detail, let $\{\widetilde{X}_n\}_{n\ge0}$ be an independent copy of $\{X_n\}_{n\ge0}$, and let $Y_n = \widetilde{X}_n - X_n$. By the Kolmogorov zero-one law,

$$\mathbb{P}\Big(\limsup_{r\to 1^-}\int_0^{2\pi}\varphi\Big(\Big|\sum_{n=0}^{\infty}a_nX_nr^ne^{in\theta}\Big|\Big)d\theta<\infty\Big)\in\{0,1\}.$$

If one has $\mathbb{P}\Big(\limsup\limits_{r\to 1^-}\int_0^{2\pi}\varphi\Big(\Big|\sum_{n=0}^{\infty}a_nX_nr^ne^{in\theta}\Big|\Big)d\theta<\infty\Big)=1$, then, by the fact that

$$\varphi\Big(\Big|\sum_{n=0}^{\infty}a_nY_nr^ne^{in\theta}\Big|\Big)\lesssim\varphi\Big(\Big|\sum_{n=0}^{\infty}a_n\widetilde{X}_nr^ne^{in\theta}\Big|\Big)+\varphi\Big(\Big|\sum_{n=0}^{\infty}a_nX_nr^ne^{in\theta}\Big|\Big),$$

one has

$$\mathbb{P}\Big(\limsup_{r\to 1^-}\int_0^{2\pi}\varphi\Big(\Big|\sum_{n=0}^{\infty}a_nY_nr^ne^{in\theta}\Big|\Big)d\theta<\infty\Big)=1.$$

Together with Theorem 1.4 for $\{a_n\}_{n\geq 0}\notin\ell^2$, this leads to a contradiction.

The rest of this note is organized as follows. The proof of Theorem 1.1 is presented in Section 2. Then Section 3 treats three types of integrability results, including the proofs of Theorems 1.2, 1.3, and 1.4 in particular. In Section 4, various examples are presented to illustrate what vector spaces can arise as $(H^p(\mathbb{D}))_\star$ for some choice of X_0.

2. Proof of Theorem 1.1

We shall use the term *"a standard X sequence",* where

$$X\in\{\text{Rademacher, Steinhaus}, N(0,1)\}.$$

By this we mean a sequence of i.i.d. X random variables.

Proof of Theorem 1.1. We first consider the symmetric case. For convenience, we shall write H^p for $H^p(\mathbb{D})$ later on.

Claim A: *If $X_0\in L^2(\Omega)$ and is symmetric, then $(H^p)_\star=H^2$.*

To prove this claim, we shall use an exponential integrability result of Kahane [9, Theorem 1, p. 54] to ensure the needed L^p-integrability here. Kahane's arguments require a fouth-moment assumption, and a truncation trick allows one to get around this assumption. To achieve this, we prove the following first.

Lemma 2.1. *Let $\{a_n\}_{n\geq 0}$ be a sequence of complex numbers, and $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. non-zero random variables with $X_0\in L^2(\Omega)$. Then the following are equivalent:*

(i) $\sum_{n=0}^{\infty}|a_n|^2<\infty$;
(ii) $\sum_{n=0}^{\infty}\mathbb{E}\left(\min\{1,|a_nX_n|^2\}\right)<\infty$;
(iii) $\sum_{n=0}^{\infty}|a_nX_n|^2$ *converges a.s.*

Proof. The equivalence between (ii) and (iii) follows immediately from an application of [9, Theorem 6, p. 33]. We next show that (i) $\Longleftrightarrow$ (ii).

(i) $\Longrightarrow$ (ii). If $\sum_{n=0}^{\infty} |a_n|^2 < \infty$, then

$$\begin{aligned}\sum_{n=0}^{\infty} \mathbb{E}\left(\min\{1, |a_n X_n|^2\}\right) &= \sum_{n=0}^{\infty} \mathbb{E}\mathbb{1}_{\left\{|X_n| \geq \frac{1}{|a_n|}\right\}} + \sum_{n=0}^{\infty} \mathbb{E}\left(|a_n X_n|^2 \mathbb{1}_{\left\{|X_n| < \frac{1}{|a_n|}\right\}}\right) \\ &\leq 2(\mathbb{E}|X_0|^2) \sum_{n=0}^{\infty} |a_n|^2.\end{aligned}$$

Hence (ii) holds.

(ii) $\Longrightarrow$ (i). We proceed by contradiction and assume that $\sum_{n=0}^{\infty} |a_n|^2 = \infty$. If $\lim_{n\to\infty} a_n = 0$, then there exists $\Omega_1 \subset \left\{|X_0| < \frac{1}{|a_n|}\right\}$ such that $\mathbb{P}(\Omega_1) > 0$ and $\mathbb{E}\left(|X_0|^2 \mathbb{1}_{\Omega_1}\right) > 0$ for all $n > N$ when N is large enough. Then

$$\sum_{n=0}^{\infty} \mathbb{E}\left(\min\{1, |a_n X_n|^2\}\right) \geq \sum_{n=N+1}^{\infty} |a_n|^2 \mathbb{E}\left(|X_0|^2 \mathbb{1}_{\Omega_1}\right) = \infty.$$

Otherwise, we may assume that $|a_{n_k}| \geq M$ for some $M > 0$ and a subsequence $\{n_k\}_{k\geq 1}$. Then

$$\sum_{n=0}^{\infty} \mathbb{E}\left(\min\{1, |a_n X_n|^2\}\right) \geq M^2 \sum_{k=1}^{\infty} \mathbb{E}\Big(\min\Big\{\frac{1}{M^2}, |X_0|^2\Big\}\Big) = \infty,$$

a contradiction. The proof of Lemma 2.1 is complete now. □

We continue with the proof of Claim A. If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in H^2$, then by Lemma 2.1 and [9, Theorem 1, p. 54], we deduce that $\sum_{n=0}^{\infty} a_n X_n e^{int}$ is exponentially integrable, hence $\sum_{n=0}^{\infty} a_n X_n e^{int} \in L^p(\mathbb{T})$ a.s. That is, $H^2 \subset (H^p)_\star$.

Conversely, if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in (H^p)_\star \setminus H^2$, then, by Lemma 2.1 again, $\sum_{n=0}^{\infty} |a_n X_n|^2$ diverges a.s. Let

$$\Delta \doteq \Big\{(\omega, \omega', \xi) \in \Omega \times \Omega' \times \mathbb{T} : \lim_{r\to 1^-} \Big(\sum_{n=0}^{\infty} a_n \epsilon_n(\omega') X_n(\omega) r^n \xi^n\Big) \text{ does not exist}\Big\},$$

where $\{\epsilon_n\}_{n\geq 0}$ is a standard Rademacher sequence on a probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ and independent to $\{X_n\}_{n\geq 0}$. According to [9, Theorem 4, p. 31], for almost surely $\omega \in \Omega$, and for all $\xi \in \mathbb{T}$, one has

$$\mathbb{P}'\Big(\lim_{r\to 1^-} \Big(\sum_{n=0}^{\infty} a_n \epsilon_n(\omega') X_n(\omega) r^n \xi^n\Big) \text{ does not exist}\Big) = 1.$$

Then, by the Fubinization principle [12, Proposition III.7, p. 26], for almost every $\xi \in \mathbb{T}$,

$$\mathbb{P}\Big(\lim_{r\to 1^-} \Big(\sum_{n=0}^{\infty} a_n X_n r^n \xi^n \Big) \text{ does not exist} \Big) = 1.$$

It implies that $\sum_{n=0}^{\infty} a_n X_n z^n \notin H^p$ a.s., a contradiction. This completes the proof of Claim A.

For the non-symmetric case, let $\{\widetilde{X}_n\}_{n\geq 0}$ be an independent copy of $\{X_n\}_{n\geq 0}$, and we proceed with two cases.

Case (i): $0 < p \leq 2$.

We first show the sufficiency.

Claim B: *Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. random variables. Then*

$$(H^2)_\star = H^2$$

if and only if $X_0 \in L^2(\Omega)$.

Claim B is a consequence of the following lemma, whose necessity follows from Lemma 2.1.

Lemma 2.2. *Let $\{X_n\}_{n\geq 0}$ be a sequence of positive i.i.d. random variables, and $\{a_n\}_{n\geq 0}$ be a sequence of positive numbers. Then $X_0 \in L^1(\Omega)$ if and only if the following two statements are equivalent:*

(i) $\sum\limits_{n=0}^{\infty} a_n < \infty$;

(ii) $\sum\limits_{n=0}^{\infty} a_n X_n$ *converges a.s.*

Proof. For the sufficiency, we argue by contradiction. Let $X_0 \notin L^1(\Omega)$, $\Omega_n = \{X_1 \leq n\}$ and $b_n = \mathbb{E}(X_1 \mathbb{1}_{\Omega_n})$. Then $\{b_n\}_{n\geq 0}$ is increasing and $\lim\limits_{n\to\infty} b_n = \infty$. Next, observe that we can choose a positive sequence $\{a_n\}_{n\geq 0}$ such that

$$\sum_{n=0}^{\infty} a_n < \infty, \ \sum_{n=0}^{\infty} a_n b_n = \infty, \ a_0 = 1, \text{ and } a_n \leq \frac{1}{n}, \quad n = 1, 2, \cdots.$$

Then

$$\sum_{n=0}^{\infty} \mathbb{E}\left(a_n X_n \mathbb{1}_{\{a_n X_n \leq 1\}}\right) \geq \sum_{n=0}^{\infty} \mathbb{E}\left(a_n X_n \mathbb{1}_{\Omega_n}\right) = \sum_{n=0}^{\infty} a_n b_n = \infty.$$

By the three series theorem [12, Theorem III.5, p. 25], $\sum_{n=0}^{\infty} a_n X_n$ diverges a.s., hence a contradiction. $\square$

By Claim B, if $X_0 \in L^2(\Omega)$, then $H^2 = (H^2)_\star \subset (H^p)_\star$. Conversely, if $f(z) = \sum_{n=0}^\infty a_n z^n \in (H^p)_\star$, then $\sum_{n=0}^\infty a_n(\widetilde{X}_n - X_n)z^n \in H^p$ a.s. Now Claim A implies $(H^p)_\star \subset H^2$.

For the necessity of Case (i) in Theorem 1.1, we assume $(H^p)_\star = H^2$. If $X_0 \notin L^2(\Omega)$, then there exists $f(z) = \sum\limits_{k=0}^{\infty} a_k z^k \in H^2 \setminus (H^2)_\star$ since $(H^2)_\star \subsetneq H^2$, inferred from the proof of Lemma 2.2, which implies that

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty \text{ and } \sum_{k=0}^{\infty} |a_k X_k|^2 = \infty \text{ a.s.} \tag{9}$$

Let $g(z) = \sum_{k=0}^\infty a_k z^{n_k}$, a lacunary series with $\inf_{k\geq 0} \frac{n_{k+1}}{n_k} > 1$. Then $g \in H^2 \subset (H^p)_\star$. So $(\mathcal{R}g)(z) = \sum_{k=0}^\infty a_k X_{n_k} z^{n_k} \in H^p$ a.s. By [8, Theorem 6.2.2, p. 114], $\sum_{k=0}^\infty |a_k X_{n_k}|^2 < \infty$ a.s., contradicting (9). Hence, $X_0 \in L^2(\Omega)$, as desired.

Case (ii): $2 < p < \infty$.

For the sufficiency, let $X_0 \in L^2(\Omega)$ and $\mathbb{E}X_0 = 0$. The inclusion $(H^p)_\star \subset H^2$ follows from arguments similar to those in the proof of Case (i). We now show the converse. Let $f(z) = \sum_{n=0}^\infty a_n z^n \in H^2$ and we need only to show that $H^2 \subset (H^p)_\star$ for $p \geq 2$. By Claim A, $\sum_{n=0}^\infty a_n(\widetilde{X}_n - X_n)z^n \in H^p$ a.s. Moreover, $\sum_{n=0}^N a_n(\widetilde{X}_n - X_n)z^n$ converges in H^p a.s. as $N \to \infty$. Then

$$\left(\mathbb{E}\left\| \sum_{n=0}^{\infty} a_n(\widetilde{X}_n - X_n)z^n \right\|_{H^p}^2\right)^{1/2} \leq \tau_{2,p} \liminf_{N\to\infty} \left(\sum_{n=1}^{N} \mathbb{E}\|a_n(\widetilde{X}_n - X_n)z^n\|_{H^p}^2\right)^{1/2} < \infty$$

for some constant $\tau_{2,p}$. This inequality follows from [7, Proposition 7.1.4, p. 57], since H^p has type 2 [12, Theorem IV.9, p. 188]. Then, by [12, Proposition II.13, p. 128] and the hypothesis $\mathbb{E}X_0 = 0$,

$$\lim_{r\to 1^-} \left(\mathbb{E}\left\| \sum_{n=0}^{\infty} a_n X_n r^n z^n \right\|_{H^p}^2\right)^{1/2} \leq \lim_{r\to 1^-} \left(\mathbb{E}\left\| \sum_{n=0}^{\infty} a_n(\widetilde{X}_n - X_n) r^n z^n \right\|_{H^p}^2\right)^{1/2} < \infty.$$

This implies that $\sum_{n=0}^\infty a_n X_n z^n \in H^p$ a.s., as desired.

For the necessity in Case (ii), the conclusion $X_0 \in L^2(\Omega)$ follows from arguments similar to those in Case (i). If, by contradiction, $\mathbb{E}X_0 \neq 0$, then we let $Y_n = X_n - \mathbb{E}X_0$. We just proved that $\sum_{n=0}^\infty a_n Y_n z^n \in H^p$ a.s. if $\sum_{n=0}^\infty a_n z^n \in H^2$. On the other hand, pick any $\sum_{n=0}^\infty b_n z^n \in H^2 \setminus H^p$ when $p > 2$, and observe that

$$\sum_{n=0}^{\infty} b_n X_n z^n = \sum_{n=0}^{\infty} b_n(Y_n + \mathbb{E}X_0)z^n \notin H^p \quad \text{a.s.}$$

This contradiction concludes the proof of Theorem 1.1.

$\square$

We end this section with a few corollaries.

Non-symmetric randomization. A motivating example, to us, is to understand the curious case when X_0 is the uniform distribution on $[0,1]$, and when

$$\mathbb{P}(X_0 = 0) = \mathbb{P}(X_0 = 1) = \frac{1}{2}.$$

Simple as they are, they are seldom considered in literature. If $\mathbb{E}X_0 \neq 0$, then using

$$\sum_{n=0}^{\infty} a_n X_n z^n = \sum_{n=0}^{\infty} a_n\big((X_n - \mathbb{E}X_0) + \mathbb{E}X_0\big)z^n,$$

we obtain

Corollary 2.3. *Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. random variables. If $X_0 \in L^2(\Omega)$ and $\mathbb{E}X_0 \neq 0$, then, for any $2 < p < \infty$, $(H^p)_\star = H^p$.*

Clearly, $H^p \subset H^2$ here. It is indeed a general fact that the non-symmetric symbol space is always smaller than or equal to the symmetrized symbol space. In detail, let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. non-zero random variables, not necessarily in $L^2(\Omega)$, $\{\widetilde{X}_n\}_{n\geq 0}$ an independent copy of $\{X_n\}_{n\geq 0}$, and $Y_n = \widetilde{X}_n - X_n$. It can be verified that

$$(H^p)_\star^X \subset (H^p)_\star^Y. \tag{10}$$

Here $(H^p)_\star^X$ and $(H^p)_\star^Y$ are the random symbol spaces of H^p under the randomization by $\{X_n\}_{n\geq 0}$ and $\{Y_n\}_{n\geq 0}$, respectively.

Functional Hilbert spaces. By Lemma 2.2, Claim B admits the following generalization.

Corollary 2.4. *Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. random variables. Let $\mathcal{H}$ be a Hilbert space of analytic functions over $\mathbb{D}$ with $\{z^n\}_{n\geq 0}$ being an orthogonal basis. Then $(\mathcal{H})_\star = \mathcal{H}$ if and only if $X_0 \in L^2(\Omega)$.*

Non-tangential boundary values. The original Littlewood theorem for non-square-summable coefficients ([5, Theorem A.5, p. 228], [20]) actually states that if $\{a_n\}_{n\geq 0} \notin \ell^2$, then almost surely $\sum_{n=0}^{\infty} a_n \epsilon_n z^n$ has a radial limit almost nowhere. The following follows from the proof of Theorem 1.1 and the device of symmetrization, and improves Littlewood's conclusion. It is perhaps satisfactory to notice that there is essentially no restriction on the distribution of X_0 at all.

Corollary 2.5. *Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. non-zero random variables. If $\{a_n\}_{n\geq 0} \notin \ell^2$, then*

$$\mathbb{P}\Big(\lim_{r\to 1^-}\Big(\sum_{n=0}^{\infty} a_n X_n r^n \xi^n\Big) \textit{ does not exist a.e. } \xi \in \mathbb{T}\Big) = 1.$$

It follows readily that

Corollary 2.6. *Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. non-zero random variables. Then, for any $0 < p < \infty$, one has $(H^p)_\star \subset H^2$.*

3. Integrability

This section discusses three types of integrability results. We first prove Theorem 1.2, which may be viewed as an L^p version of the Fernique theorem ([11, Corollary 3.2, p. 59] or [12, Theorem V.26, p. 255]). More precisely, let $\{X_n\}_{n\geq 0}$ be a standard Gaussian sequence and $\varphi_q(x) = e^{x^q} - 1$ $(0 < q < \infty)$. Note that $X_0 \in L^{\varphi_2}(\Omega)$, but $X_0 \notin L^{\varphi_q}(\Omega)$ if $q > 2$. The Fernique theorem, in our setting, implies that, for all $\{a_n\}_{n\geq 0} \in \ell^2$ and $1 \leq p < \infty$,

$$\Big\| \sum_{n=0}^{\infty} a_n X_n z^n \Big\|_{H^p} \in L^{\varphi_2}(\Omega).$$

On the other hand, one can argue, say, by the contraction principle [7, Theorem 6.1.13, p. 9], that for any non-zero sequence $\{b_n\}_{n\geq 0}$ and $q > 2$, one has

$$\Big\| \sum_{n=0}^{\infty} b_n X_n z^n \Big\|_{H^p} \notin L^{\varphi_q}(\Omega).$$

By this perspective, Theorem 1.2 delves into the implication of

$$\Big\| \sum_{n=0}^{\infty} a_n X_n z^n \Big\|_{H^p} \in L^{\varphi}(\Omega)$$

with $\varphi(x) = x^q$, viewed as another class of Young functions.

We need the following lemma for the proof of Theorem 1.2, whose proof follows from the arguments similar to those of Proposition 2.5 in [10], hence skipped.

Lemma 3.1. *Let $\{\zeta_n\}_{n\geq 1}$ be a sequence of i.i.d. non-zero symmetric random variables and let E be a p-Banach space $(0 < p \leq 1)$. If $0 < q < \infty$, then*

(i) *for all sequence $\{e_n\}_{1\leq n\leq N} \subset E$,*

$$\left(\mathbb{E}\Big\| \sum_{n=1}^{N} \epsilon_n e_n \Big\|_E^q\right)^{\frac{1}{q}} \lesssim \left(\mathbb{E}\Big\| \sum_{n=1}^{N} \zeta_n e_n \Big\|_E^q\right)^{\frac{1}{q}};$$

(ii) *for all sequence $\{e_n\}_{1\leq n\leq N} \subset E$ and $\{\lambda\}_{1\leq n\leq N} \subset \mathbb{C}$,*

$$\left(\mathbb{E}\Big\| \sum_{n=1}^{N} \lambda_n \zeta_n e_n \Big\|_E^q\right)^{\frac{1}{q}} \lesssim \sup_{1\leq n\leq N} |\lambda_n| \left(\mathbb{E}\Big\| \sum_{n=1}^{N} \zeta_n e_n \Big\|_E^q\right)^{\frac{1}{q}}.$$

Proof of Theorem 1.2. (i) Let $\{\widetilde{X}_n\}_{n\geq 0}$ be an independent copy of $\{X_n\}_{n\geq 0}$, and $Y_n = \widetilde{X}_n - X_n$. Then $\mathbb{E}\big\| \sum_{n=0}^{\infty} a_n X_n z^n \big\|_{H^p}^q < \infty$ implies $\mathbb{E}\big\| \sum_{n=0}^{\infty} a_n Y_n z^n \big\|_{H^p}^q < \infty$. By the contraction principle and Lemma 3.1, for $a_k \neq 0$,

$$|a_k|^q \cdot \mathbb{E}|X_k|^q = \mathbb{E}\|a_k X_k z^k\|_{H^p}^q \lesssim \mathbb{E}\Big\| \sum_{n=0}^{\infty} a_n Y_n z^n \Big\|_{H^p}^q < \infty.$$

So $X_0 \in L^q(\Omega)$, as desired.

(ii) We shall show that $\big(\mathbb{E}\big\| \sum_{n=0}^{\infty} a_n X_n z^n \big\|_{H^p}^t\big)^{1/t} \approx \big(\sum_{n=0}^{\infty} |a_n|^2\big)^{1/2}$ if $1 \leq t \leq q$. We first consider the case when X_0 is symmetric. Since $X_0 \in L^q(\Omega)$ and $\{a_n\}_{n\geq 0} \in \ell^2$, by Theorem 1.1, we have $\sum_{n=0}^{\infty} a_n X_n z^n \in H^p$ a.s. Then, the Marcinkiewicz–Zygmund–Kahane theorem [13, Theorem II.4, p. 240] implies that $\sum_{n=0}^{N} a_n X_n z^n$ converges in H^p a.s. Let

$$M = \sup_{N\geq 0} \Big\| \sum_{n=0}^{N} a_n X_n z^n \Big\|_{H^p}^q,$$

which is pointwisely finite now. If we prove that $\sup_{N\geq 0} \mathbb{E}\big\| \sum_{n=0}^{N} a_n X_n z^n \big\|_{H^p}^q < \infty$, then by the Lévy maximal inequality [12, Theorem III.2, p. 129] and [4, Theorem 27], one has $\mathbb{E}M < \infty$ and

$$\mathbb{E}\Big\| \sum_{n=0}^{\infty} a_n X_n z^n \Big\|_{H^p}^q = \lim_{N\to\infty} \mathbb{E}\Big\| \sum_{n=0}^{N} a_n X_n z^n \Big\|_{H^p}^q.$$

So it is sufficient to consider $\mathbb{E}\big\| \sum_{n=0}^{N} a_n X_n z^n \big\|_{H^p}^q$. Let $\{\epsilon_n\}_{n\geq 0}$ be a standard Rademacher sequence independent of $\{X_n\}_{n\geq 0}$. The proof is further divided into two cases.

Case 1: $0 < p \leq 2$.

For the lower bound, we need only consider $t = 1$. Then

$$\begin{aligned}
\mathbb{E}\Big\| \sum_{n=0}^{N} a_n X_n z^n \Big\|_{H^p} &\gtrsim (\mathbb{E}|X_0|)\mathbb{E}_\epsilon \Big\| \sum_{n=0}^{N} a_n \epsilon_n z^n \Big\|_{H^p} \\
&\gtrsim (\mathbb{E}|X_0|)\Big(\mathbb{E}_\epsilon \Big\| \sum_{n=0}^{N} a_n \epsilon_n z^n \Big\|_{H^p}^2\Big)^{1/2} \\
&\gtrsim (\mathbb{E}|X_0|)\Big(\sum_{n=0}^{N} |a_n|^2\Big)^{1/2},
\end{aligned}$$

where the first "$\gtrsim$" holds by the comparison principle [7, Proposition 6.1.15, p. 10] for Banach spaces and by Lemma 3.1 for p-Banach spaces when $0 < p < 1$; the second one by the Kahane-Khintchine inequality [7, Theorem 6.2.4, p. 21] and [4, Proposition 32], and the last one by the fact that H^p has cotype 2 [12, Theorem IV.9, p. 188].

For the upper bound, it suffices to take $t = q$. By the Hölder inequality for $p \leq 2$ and the Minkowski inequality,

$$\Big(\mathbb{E}\Big\|\sum_{n=0}^{N} a_n X_n z^n\Big\|_{H^p}^q\Big)^{1/q} \quad\leq\quad \Big(\sum_{n=0}^{N}\big(\mathbb{E}|a_n|^q|X_n|^q\big)^{2/q}\Big)^{1/2},$$

which is $(\mathbb{E}|X_0|^q)^{1/q}\big(\sum_{n=0}^N |a_n|^2\big)^{1/2}$, as desired.

Case 2: $2 < p < \infty$.

If $t = 1$, then by the comparison principle, the Kahane-Khintchine inequality and the Hölder inequality for $p \geq 2$,

$$\mathbb{E}\Big\|\sum_{n=0}^{N} a_n X_n z^n\Big\|_{H^p} \quad\gtrsim\quad \mathbb{E}|X_0|\Big(\mathbb{E}_\epsilon\Big\|\sum_{n=0}^{N} a_n \epsilon_n z^n\Big\|_{H^2}^2\Big)^{1/2},$$

which is equivalent to $\big(\sum_{n=0}^N |a_n|^2\big)^{1/2}$.

If $t = q$, then

$$\mathbb{E}\Big\|\sum_{n=0}^{N} a_n X_n z^n\Big\|_{H^p}^q \approx \mathbb{E}\Big(\sum_{n=0}^{N}|a_n|^2|X_n|^2\Big)^{q/2} \leq (\mathbb{E}|X_0|^q)\Big(\sum_{n=0}^{N}|a_n|^2\Big)^{q/2}.$$

If X_0 is non-symmetric, then (12) in Section 4, the hypothesis $\mathbb{E}X_0 = 0$, and [12, Proposition II.13, p. 128] allow one to reduce the problem to the symmetric case by considering $\sum_{n=0}^\infty a_n Y_n z^n$ with $Y_n = \widetilde{X}_n - X_n$.

Lastly, the proof that $(H^p)_\star$ admits a natural (p-)Banach space structure is routine, hence omitted here. □

Proof of Theorem 1.3. We consider only the case when X_0 is symmetric since the device of symmetrization works here. Let $S = \|\sum_{n=0}^\infty a_n X_n z^n\|_{H^p}$ and $S_N = \|\sum_{n=0}^N a_n X_n z^n\|_{H^p}$. The Marcinkiewicz–Zygmund–Kahane Theorem implies that $\lim_{N\to\infty} S_N = S$ a.s. Next, we shall prove that $C = \sup_{N\geq 0}\|S_N\|_{L^{\varphi_q}(\Omega)} < \infty$. Then by Fatou's lemma, one has $\mathbb{E}\big(\varphi_q\big(\frac{|S|}{C}\big)\big) \leq 1$ and

$$\|S\|_{L^{\varphi_q}(\Omega)} \leq \sup_{N\geq 0}\|S_N\|_{L^{\varphi_q}(\Omega)} < \infty,$$

which implies (6). By [11, Theorem 6.21, p. 172],

$$\begin{aligned}\|S_N\|_{L^{\varphi_q}(\Omega)} &\lesssim \|S_N\|_{L^1(\Omega)} + \Big(\sum_{n=0}^{N} |a_n|^{q'} \|X_0\|^{q'}_{L^{\varphi_q}(\Omega)}\Big)^{\frac{1}{q'}} \\ &\lesssim \|S\|_{L^1(\Omega)} + \|X_0\|_{L^{\varphi_q}(\Omega)} \Big(\sum_{n=0}^{\infty} |a_n|^2\Big)^{\frac{1}{2}}\end{aligned}$$

which is finite. Here $\frac{1}{q} + \frac{1}{q'} = 1$. The proof is complete now. □

Remark. The remaining cases in Theorem 1.3 are $0 < q < 1$ and $2 < q < \infty$. If $0 < q < 1$ and $\{a_n\}_{n\ge 0} \in \ell^2$, by [23, Theorem 3] (or [11, Theorem 6.21, p. 172]),

$$\|S\|_{L^{\varphi_q}(\Omega)} \lesssim \|S\|_{L^1(\Omega)} + \Big\| \sup_{n\ge 0} |a_n X_n| \Big\|_{L^{\varphi_q}(\Omega)},$$

where $S = \big\| \sum_{n=0}^{\infty} a_n X_n z^n \big\|_{H^p}$. For $q \in (2, \infty)$, by [11, p. 174],

$$\|S\|_{L^{\varphi_q}(\Omega)} \lesssim \|S\|_{L^1(\Omega)} + \Big\| \{\|a_n X_n\|_{L^\infty(\Omega)}\}_{n\ge 0} \Big\|_{q',\infty} \tag{11}$$

with $\frac{1}{q} + \frac{1}{q'} = 1$. Recall that for a sequence $\{\lambda_n\}_{n\ge 0}$,

$$\|\{\lambda_n\}_{n\ge 0}\|_{p,\infty} = \Big(\sup_{t>0} t^p \mathrm{card}\{n : |\lambda_n| > t\}\Big)^{1/p}.$$

Note that (11) should be compared to the conjecture below.

For a general Young function, we offer the following:

Conjecture. Let φ be a Young function with $\lim_{x\to\infty} \frac{\varphi(x)}{x^2} = \infty$. Let $\{X_n\}_{n\ge 0}$ be a sequence of i.i.d. random variables with $X_0 \in L^\varphi(\Omega)$, and let $1 \le p < \infty$. Then for each $\{a_n\}_{n\ge 0} \in \ell^2$, $\big\| \sum_{n=0}^{\infty} a_n X_n z^n \big\|_{H^p} \in L^\varphi(\Omega)$.

On the other hand, for any Young function ψ such that $\lim_{x\to\infty} \frac{\psi(x)}{\varphi(x)} = \infty$ and $X_0 \in L^\varphi(\Omega) \setminus L^\psi(\Omega)$, by considering a monomial, one sees easily that there exists $\{b_n\}_{n\ge 0} \in \ell^2$ such that $\big\| \sum_{n=0}^{\infty} b_n X_n z^n \big\|_{H^p} \notin L^\psi(\Omega)$. So, in this sense, the conjecture offers a much sharper form of the Fernique theorem in the setting of H^p, which is, desirably, to be generalized to other (p-)Banach spaces as well.

Recall that the following proof is motivated by [9, Proposition 2, p. 122].

Proof of Theorem 1.4. By Lemma 2.1, we have $\sum_{n=0}^{\infty} |a_n|^2 |X_n|^2 = \infty$ a.s. Let $\{r_n\}_{n\ge 0}$ be a non-decreasing sequence with $\lim_{n\to\infty} r_n = 1$. Let

$$\rho_n \doteq \Big(\sum_{j=0}^{\infty} |a_j|^2 r_n^{2j} |X_j|^2\Big)^{1/2}.$$

Then $\lim_{n\to\infty} \rho_n = \infty$ a.s. So there exists a non-decreasing sequence τ_n such that

$$\mathbb{P}\big(\rho_n \geq \tau_n \ :\ n = 1, 2, \cdots\big) \geq \frac{9}{10}.$$

Let $\{\epsilon_n\}_{n\geq 0}$ be a standard Rademacher sequence independent with $\{X_n\}_{n\geq 0}$. By the Paley-Zygmund inequality [9, Theorem 3.3, p. 31], for $\lambda \in (0, 1)$ and $\eta = \frac{1}{3}(1-\lambda^2)^2$,

$$\mathbb{P}_\epsilon\Big(\Big|\sum_{j=0}^{\infty} \epsilon_j a_j r_n^j X_j e^{ij\theta}\Big| > \lambda\Big(\sum_{j=0}^{\infty} |a_j|^2 r_n^{2j} |X_j|^2\Big)^{1/2}\Big) > \eta.$$

Note that X_j is symmetry, hence

$$\mathbb{P}\Big(\Big|\sum_{j=0}^{\infty} a_j r_n^j X_j e^{ij\theta}\Big| > \lambda\Big(\sum_{j=0}^{\infty} |a_j|^2 r_n^{2j} |X_j|^2\Big)^{1/2}\Big) > \eta.$$

Combine with $\mathbb{P}(\rho_n \geq \tau_n \ :\ n = 1, 2, \cdots) \geq \frac{9}{10}$, we can take $\lambda_0 \in (0, 1)$ such that

$$\mathbb{P}\Big(\Big|\sum_{j=0}^{\infty} a_j r_n^j X_j e^{ij\theta}\Big| \geq \lambda_0 \tau_n \ :\ n = 1, 2, \cdots\Big) \geq \frac{\eta_0}{2} > 0,$$

where $\eta_0 = \frac{1}{3}(1-\lambda_0^2)^2$. For each n, define

$$W_n = \Big\{(\omega, \theta) \ :\ \Big|\sum_{j=0}^{\infty} a_j r_n^j X_j e^{ij\theta}\Big| > \lambda_0 \tau_n\Big\},$$

$E^\theta = \{\omega \ :\ (\omega, \theta) \in W_n\}$, $E_\omega = \{\theta \ :\ (\omega, \theta) \in W_n\}$, and $F_n = \{\omega \ :\ |E_\omega| > \eta_0\}$. Since

$$\begin{aligned}
\pi\eta_0 &\leq \int_0^{2\pi} \mathbb{P}(E^\theta) d\theta \\
&= |W_n| = \int_{F_n} |E_\omega| d\mathbb{P} + \int_{\Omega\setminus F_n} |E_\omega| d\mathbb{P} \leq 2\pi\mathbb{P}(F_n) + \eta_0(1 - \mathbb{P}(F_n)),
\end{aligned}$$

where $|\cdot|$ denotes the Lebesgue measure. This implies that $\mathbb{P}(F_n) \geq \frac{(\pi-1)\eta_0}{2\pi-\eta_0}$. Hence $\mathbb{P}\big(\limsup_{n\to\infty} F_n\big) \geq \frac{(\pi-1)\eta_0}{2\pi-\eta_0}$. Therefore, if $\omega \in F_n$,

$$\begin{aligned}
\int_0^{2\pi} \varphi\Big(\Big|\sum_{j=0}^{\infty} a_j r_n^j X_j e^{ij\theta}\Big|\Big) d\theta &\geq \int_{E_\omega} \varphi\Big(\Big|\sum_{j=0}^{\infty} a_j r_n^j X_j e^{ij\theta}\Big|\Big) d\theta \\
&\geq \int_{E_\omega} \varphi(\lambda_0 \tau_n) d\theta \\
&\geq \varphi(\lambda_0 \tau_n) \cdot \eta_0.
\end{aligned}$$

It implies that for all $\omega \in \limsup\limits_{n\to\infty} F_n$, $\limsup\limits_{n\to\infty} \int_0^{2\pi} \varphi\big(\big|\sum_{j=0}^{\infty} a_j r_n^j X_j e^{in\theta}\big|\big)d\theta = \infty$. Then by the Kolmogorov zero-one law,

$$\limsup_{n\to\infty} \int_0^{2\pi} \varphi\Big(\Big|\sum_{j=0}^{\infty} a_j r_n^j X_j e^{ij\theta}\Big|\Big)d\theta = \infty \quad \text{a.s.}$$

That is

$$\limsup_{r\to 1^-} \int_0^{2\pi} \varphi\Big(\Big|\sum_{j=0}^{\infty} a_j r^j X_j e^{ij\theta}\Big|\Big)d\theta = \infty \quad \text{a.s.}$$

□

4. The range of $(H^p)_\star$

In this section various examples are presented to illustrate the possible realizations of $(H^p)_\star$; that is, what vector spaces can arise as a version of $(H^p)_\star$ for an arbitrarily chosen X_0? Clearly, by Theorem 1.1, only the case when $X_0 \notin L^2(\Omega)$ is meaningful. Recall that $(H^p)_\star$ is always contained in H^2 if X_0 is non-zero. We start with two reductions which are of independent interests.

Lemma 4.1. *Let E be a separable Banach space, and let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables with $X_0 \in L^1(\Omega)$ and $\mathbb{E}X_0 = 0$. Then for all $N \geq 1$ and all sequence $\{u_k\}_{1\leq k\leq N} \subset E$, we have*

$$\mathbb{E}\Big\|\sum_{k=1}^{N} X_k u_k\Big\|_E \leq \mathbb{E}\Big\|\sum_{k=1}^{N} Y_k u_k\Big\|_E \leq 2\mathbb{E}\Big\|\sum_{k=1}^{N} X_k u_k\Big\|_E,$$

where $Y_k = \widetilde{X}_k - X_k$ and $\{\widetilde{X}_n\}_{n\geq 1}$ is an independent copy of $\{X_n\}_{n\geq 1}$.

This lemma suggests that, when the symbol space carries a natural Banach space structure, the symmetric and non-symmetric randomization methods are equivalent. The proof of Lemma 4.1 follows from

$$\mathbb{E}\Big\|\sum_{k=1}^{N} Y_k u_k\Big\|_E \leq 2\mathbb{E}\Big\|\sum_{K=1}^{N} X_k u_k\Big\|_E$$

and [12, Proposition II.13, p. 128]. In particular, let $E = H^p$ with $1 \leq p < \infty$ and $\{a_n\}_{n\geq 0} \in \ell^2$, we have

$$\mathbb{E}\Big\|\sum_{n=0}^{\infty} a_n X_n z^n\Big\|_{H^p} \approx \mathbb{E}\Big\|\sum_{n=0}^{\infty} a_n Y_n z^n\Big\|_{H^p}. \tag{12}$$

Lemma 4.2. *Let $\{X_n\}$ be a sequence of i.i.d. symmetric random variables. Then, for $0 < p < \infty$, one has*

$$(H^p)_\star = (H^2)_\star.$$

Proof. Let $\{\epsilon_n\}_{n\geq 0}$ be a standard Rademacher sequence independent of $\{X_n\}_{n\geq 0}$ over $(\Omega', \mathcal{F}', \mathbb{P}')$. If $\sum_{n=0}^{\infty} a_n z^n \in (H^p)_\star$, then, by symmetry, for almost surely $\omega' \in \Omega'$,

$$\sum_{n=0}^{\infty} a_n X_n(\omega)\epsilon_n(\omega')z^n \in H^p \text{ a.s. } \omega \in \Omega.$$

Equivalently, for almost surely $\omega \in \Omega$,

$$\sum_{n=0}^{\infty} a_n X_n(\omega)\epsilon_n(\omega')z^n \in H^p \text{ a.s. } \omega' \in \Omega'.$$

The Littlewood theorem implies that $\sum_{n=0}^{\infty} |a_n X_n(\omega)|^2 < \infty$ a.s. $\omega \in \Omega$. That is, $\sum_{n=0}^{\infty} a_n z^n \in (H^2)_\star$, as desired. □

The above two lemmas suggest that, to understand the range problem, a good starting place is to look at $(H^2)_\star$ for a non-square integrable, symmetric X_0. Our first example is:

Cauchy-type distributions. Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. random variables such that the density function of X_0 is

$$f_t(x) = \frac{C_t}{(1+x^2)^t}, \tag{13}$$

where $x \in \mathbb{R}$, $t \in (\frac{1}{2}, \infty)$ and $C_t = \frac{\Gamma(t)}{\sqrt{\pi}\Gamma(t-\frac{1}{2})}$. In particular, $f_1(x)$ is the density of the classical Cauchy distribution, and $X_0 \notin L^2(\Omega)$ if $t \in (\frac{1}{2}, \frac{3}{2}]$.

The following follows from Lemma 4.2 and the three series theorem: *Let $\{X_n\}_{n\geq 0}$ be a sequence of i.i.d. random variables with the density function (13), $0 < p < \infty$ and $\frac{1}{2} < t < \infty$.*

(i) *If $t \in (\frac{1}{2}, \frac{3}{2})$, then*

$$(H^p)_\star = \ell^{2t-1};$$

(ii) *If $t = \frac{3}{2}$, then*

$$(H^p)_\star = \Big\{ \sum_{n=0}^{\infty} a_n z^n \in H^2 \, : \, \sum_{n=0}^{\infty} |a_n|^2 \log^+ \frac{1}{|a_n|} < \infty \Big\};$$

(iii) *If $t \in (\frac{3}{2}, \infty)$, then*

$$(H^p)_\star = H^2.$$

It is not a coincidence that all the above $(H^p)_\star$ spaces take the form of an ℓ^p-type sequence space. Indeed, a general description of $(H^2)_\star$ for any symmetric X_0 is as follows.

For a right continuous, decreasing function $k(x)$ defined on $\mathbb{R}^+$ with $\lim_{x\to\infty} k(x) = 0$, we introduce a sequence space, which resembles a generalized ℓ^p space:

$$K = \Big\{\{a_n\}_{n\ge 0} : \sum_{n=0}^{\infty} k\Big(\frac{1}{|a_n|}\Big) < \infty\Big\}.$$

There exists an associated $\tilde{k}(x)$, given by

$$\tilde{k}(x) = \frac{2}{x^2}\int_0^x \big(k(u) - k(x)\big)u\,du,$$

for which we define

$$\widetilde{K} = \Big\{\{a_n\}_{n\ge 0} : \sum_{n=0}^{\infty} \tilde{k}\Big(\frac{1}{|a_n|}\Big) < \infty\Big\}.$$

Note that $\lim_{x\to\infty} \tilde{k}(x) = 0$ and $\lim_{x\to\infty} x^2\tilde{k}(x) \in (0,\infty]$. Now we introduce a family of sequence spaces, ranging over all possible choices of $k \doteq k(x)$ as above:

$$\mathfrak{E} = \Big\{E_k : E_k = K \cap \widetilde{K}\Big\}.$$

Then, we claim that the possible realizations of $(H^2)_\star$ for any symmetric X_0 is precisely the family $\mathfrak{E}$. Indeed, one has $(H^2)_\star = E_k$ if we choose $k(x) = \mathbb{P}(|X_1| > x)$. In details, one may verify by using the three series theorem that

$$(H^2)_\star = \Big\{\sum_{n=0}^{\infty} a_n z^n \in H(\mathbb{D}) : \textstyle\sum_{n=0}^{\infty}\Big(1 - \displaystyle\int_{-\frac{1}{|a_n|}}^{\frac{1}{|a_n|}} dF(x)\Big) < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} |a_n|^2 \int_{-\frac{1}{|a_n|}}^{\frac{1}{|a_n|}} x^2 dF(x) < \infty\Big\}.$$

Now a little calculus manipulation yields the above description.

Non-symmetric randomization. A general description of $(H^p)_\star$ is elusive to us, although that of $(H^2)_\star$ is possible; it is just more awkward than the symmetric case, hence skipped. Instead, we present three examples:

(i) The uniform distribution over $[0,1]$. By Theorem 1.1 and Corollary 2.3, we have

- $(H^p)_\star = \ell^2$ for $0 < p \le 2$, and
- $(H^p)_\star = H^p$ for $2 < p \le \infty$.

The same conclusion holds true if $\mathbb{P}(X_0 = 0) = \mathbb{P}(X_0 = 1) = \frac{1}{2}$.

(ii) The non-symmetric Cauchy distribution. Let X_0 be a random variable with a density function:

$$f(z) = \begin{cases} \frac{2}{\pi(1+x^2)}, & x \geq 0; \\ 0, & x < 0. \end{cases}$$

Then, for $0 < p < \infty$, one has

$$(H^p)_\star = \ell^1.$$

The proof is divided into three cases, with $p = 2$ treated first, since it is needed for the other two cases.

Let $p = 2$. Then one can verify easily, with the help of the three series theorem, that $(H^2)_\star = \ell^1$.

Let $2 < p < \infty$. Since $(H^p)_\star \subset (H^2)_\star = \ell^1$, it suffices to show that $\ell^1 \subset (H^p)_\star$. By the three series theorem again, $\sum_{n=0}^\infty |a_n X_n|^{p'} < \infty$ a.s. if and only if $\{a_n\}_{n\geq 0} \in \ell^1$, where $\frac{1}{p} + \frac{1}{p'} = 1$. On the other hand, by the Hausdorff-Young inequality [8, Theorem 6.B, p. 113], $\sum_{n=0}^\infty |a_n X_n|^{p'} < \infty$ a.s. implies that $\sum_{n=0}^\infty a_n X_n z^n \in H^p$ a.s. So $\ell^1 \subset (H^p)_\star$, as desired.

Let $0 < p < 2$. Since $\ell^1 = (H^2)_\star \subset (H^p)_\star$, we shall prove that $(H^p)_\star \subset \ell^1$. Let $\{\widetilde{X}_n\}_{n\geq 0}$ be an independent copy of $\{X_n\}_{n\geq 0}$ and $Y_n = X_n - \widetilde{X}_n$. By Lemma 4.2, $(H^p)_\star^Y = (H^2)_\star^Y$. Then by the three series theorem, $\sum_{n=0}^\infty |a_n Y_n|^2 < \infty$ a.s. implies that $\{a_n\}_{n\geq 0} \in \ell^1$. So $(H^p)_\star^Y \subset \ell^1$. Together with (10), one has $(H^p)_\star \subset \ell^1$, as desired.

(iii) The Lévy distribution. Let $\{X_n\}_{n\geq 0}$ be an i.i.d. sequence of Lévy random variables; that is, the density function of X_0 [1, p. 36] is given by:

$$g(x) = \begin{cases} \frac{1}{\sqrt{2\pi} x^{\frac{3}{2}} e^{\frac{1}{2x}}}, & x > 0; \\ 0, & x \leq 0. \end{cases}$$

Then, for $0 < p \leq \infty$,

$$(H^p)_\star = \ell^{\frac{1}{2}}.$$

The proof is similar to that of the non-symmetric Cauchy distribution.

We end this paper with the following:

Problem: For any $p > 0$, how to characterize those X_0's such that

$$(L_a^p(\mathbb{D}))_\star = H^{2,p,\frac{1}{p}}(\mathbb{D}),$$

where $L_a^p(\mathbb{D})$ is the Bergman space? Or, more generally, how to characterize X_0 such that $(H^{p,q,\alpha}(\mathbb{D}))_\star = H^{2,q,\alpha}(\mathbb{D})$, where $0 < p, q, \alpha < \infty$?

Acknowledgement

G. Cheng is supported by NSFC (12371126). X. Fang is supported by NSTC of Taiwan (112-2115-M-008-010-MY2). C. Liu is supported by NSFC (12461028) and Yunnan Fundamental Research Projects (202501CF070076).

References

[1] Applebaum, David. *Lévy Processes and Stochastic Calculus*. Cambridge Studies in Advanced Mathematics, 116, Second edition. Cambridge University Press, Cambridge, 2009. xxx+460 pp. ISBN: 978-0-521-73865-1. MR2512800, Zbl 1200.60001, doi: 10.1017/CBO9780511809781. 687

[2] Borel, E. Sur les séries de Taylor. *C.r. hebd. Séanc. Acad. Sci., Paris* **123** (1896), 1051–1052. MR3736496, doi: 10.1007/s11854-017-0037-0. 669

[3] Borichev, Alexander; Nishry, Alon; Sodin, Mikhail. Entire functions of exponential type represented by pseudo-random and random Taylor series. *J. Anal. Math.* **133** (2017), 361–396. MR3736496, Zbl 1387.30031, doi: 10.1007/s11854-017-0037-0. 669, 670

[4] Cheng, Guozheng; Fang, Xiang; Liu, Chao. A Littlewood-type theorem for random Bergman functions. *Int. Math. Res. Not. IMRN* (2022), no. 14, 11056–11091. MR4452447, Zbl 1494.30003, doi: 10.1093/imrn/rnab018. 671, 680, 681

[5] Duren, Peter L. *Theory of H^p Spaces*. Pure and Applied Mathematics, vol. 38. Academic Press, New York-London, 1970. xii+258 pp. MR0268655, Zbl 0215.20203. 678

[6] Fang, Xiang; Tien, Pham Trong. Two problems on random analytic functions in Fock spaces. *Canad. J. Math.* **75** (2023), no. 4, 1176–1198. MR4620319, Zbl 1520.30005, doi: 10.4153/s0008414x22000372. 671

[7] Hytönen, Tuomas; van Neerven, Jan; Veraar, Mark; Weis, Lutz. *Analysis in Banach Spaces. Vol. II: Probabilistic methods and operator theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 67. Springer, Cham, 2017. xxi+616 pp. ISBN: 978-3-319-69807-6; 978-3-319-69808-3. MR3752640, Zbl 1402.46002, doi: 10.1007/978-3-319-69808-3. 677, 679, 680

[8] Jevtić, Miroljub; Vukotić, Dragan; Arsenović, Miloš. *Taylor Coefficients and Coefficient Multipliers of Hardy and Bergman-type Spaces*. RSME Springer Series, 2. Springer, Cham, 2016. xvi+323 pp. ISBN: 978-3-319-45643-0; 978-3-319-45644-7. MR3587910, Zbl 1368.30001, doi: 10.1007/978-3-319-45644-7. 677, 687

[9] Kahane, Jean-Pierre. *Some Random Series of Functions*. Cambridge Studies in Advanced Mathematics, 5, Second edition. Cambridge University Press, Cambridge, 1985. xiv+305 pp. ISBN: 0-521-24966-X; 0-521-45602-9. MR833073, Zbl 0571.60002. 669, 670, 673, 674, 675, 682, 683

[10] Kalton, N. J. Rademacher series and decoupling. *New York J. Math.* **11** (2005), 563–595. MR2188256, Zbl 1107.46002. 679

[11] Ledoux, Michel; Talagrand, Michel. *Probability in Banach Spaces. Isoperimetry and processes*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 23. Springer-Verlag, Berlin, 1991. xii+480 pp. ISBN: 3-540-52013-9. MR1102015, Zbl 0748.60004, doi: 10.1007/978-3-642-20212-4. 672, 679, 682

[12] Li, Daniel; Queffélec, Hervé. *Introduction to Banach spaces: analysis and probability. Vol. 1*. Cambridge Studies in Advanced Mathematics, 166, Translated from the French by Danièle Gibbons and Greg Gibbons. Cambridge University Press, Cambridge, 2018. xxx+431 pp. ISBN: 978-1-107-16051-4; 978-1-107-16263-1. MR3729312, Zbl 1404.46002. 669, 672, 676, 677, 679, 680, 681, 684

[13] Li, Daniel; Queffélec, Hervé. *Introduction to Banach spaces: analysis and probability. Vol. 2*. Cambridge Studies in Advanced Mathematics, 167, Translated from the French by Danièle Gibbons and Greg Gibbons, With appendices by Gilles Godefroy, Olivier Guédon, Gilles Pisier and Luis Rodríguez-Piazza. Cambridge University Press, Cambridge, 2018. xxx+374 pp. ISBN: 978-1-107-16262-4; 978-1-107-16263-1. MR3729311, Zbl 1404.46003. 680

[14] Littlewood, J. E. On the mean values of power series. *Proc. London Math. Soc. (2)* **25** (1926), 328–337. MR1575287, doi: 10.1112/plms/s2-25.1.328. 670

[15] Littlewood, J. E. Mathematical notes (13): On mean values of power series (II). *J. London Math. Soc.* **5** (1930), no. 3, 179–182. MR1574067, doi: 10.1112/jlms/s1-5.3.179. 670

[16] Littlewood, J. E.; Offord, A. C. On the distribution of zeros and a-values of a random integral function. II. *Ann. of Math. (2)* **49** (1948), 885–952; errata 50, 990–991. MR29981, Zbl 0034.34305, doi: 10.2307/1969404. 669

[17] Offord, A. C. The distribution of the values of an entire function whose coefficients are independent random variables. *Proc. London Math. Soc. (3)* **14a** (1965), 199–238. MR177117, Zbl 0134.29204, doi: 10.1112/plms/s3-14A.1.199. 669, 670

[18] Offord, A. C. The distribution of the values of an entire function whose coefficients are independent random variables. II. *Math. Proc. Cambridge Philos. Soc.* **118** (1995), no. 3, 527–542. MR1342969, Zbl 0846.60037, doi: 10.1017/S0305004100073849. 669, 670

[19] Paley, R. E. A. C.; Wiener, N.; Zygmund, A. Notes on random functions. *Math. Z.* **37** (1933), no. 1, 647–668. MR1545426, Zbl 0007.35402, doi: 10.1007/BF01474606. 670

[20] Paley, R. E. A. C.; Zygmund, A. On some series of functions (1). *Proc. Camb. Phil. Soc.* **26** (1930), 337–357. 678

[21] Paley, R. E. A. C.; Zygmund, A. On some series of functions (2). *Proc. Camb. Phil. Soc.* **26** (1930), 458–474. 670

[22] Rao, M. M.; Ren, Z. D. *Theory of Orlicz Spaces*. Monographs and Textbooks in Pure and Applied Mathematics, 146. Marcel Dekker, Inc., New York, 1991. xii+449 pp. ISBN: 0-8247-8478-2. MR1113700, Zbl 0724.46032. 673

[23] Talagrand, Michel. Isoperimetry and integrability of the sum of independent Banach-space valued random variables. *Ann. Probab.* **17** (1989), no. 4, 1546–1570. MR1048946, Zbl 0692.60016. 672, 682

[24] Vershynin, Roman. *High-Dimensional Probability: An Introduction with Applications in Data Science*. Cambridge Series in Statistical and Probabilistic Mathematics, 47. Cambridge University Press, Cambridge, 2018. xiv+284 pp. ISBN: 978-1-108-41519-4. MR3837109, Zbl 1430.60005, doi: 10.1017/9781108231596. 669

(Yin Cai) School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
cy-math@mail.dlut.edu.cn

(Guozheng Cheng) School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
gzhcheng@dlut.edu.cn

(Xiang Fang) Department of Applied Mathematics, National Yang Ming Chiao Tung University, Hsinchu, Taiwan
xfang@nycu.edu.tw

(Chao Liu) School of Mathematics and Statistics, Yunnan University, Kunming 650091, China
chaoliu@ynu.edu.cn

This paper is available via http://nyjm.albany.edu/j/2025/31-24.html.