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Young type integral inequalities for tensorial and Hadamard products of continuous fields of operators in Hilbert spaces

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ABSTRACT. Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_\Omega 1d\mu(t)=1$. In this paper we show among others that, if $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\sigma(A_\tau), \sigma(B_\tau)\subseteq[m,M]\subset(0,\infty)$ for each $\tau\in\Omega$, then for all $\nu\in[0,1]$

$$\begin{aligned}&\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right]\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\otimes\int_\Omega B_\tau^{\nu}d\mu(\tau)\\&\le(1-\nu)\int_\Omega A_\tau d\mu(\tau)\otimes 1+\nu 1\otimes\int_\Omega B_\tau d\mu(\tau)\\&\le\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right]\int_\Omega A_\tau^{1-\nu}d\mu(\tau)\otimes\int_\Omega B_\tau^{\nu}d\mu(\tau).\end{aligned}$$

We also have similar inequalities for the Hadamard product ε∘ε.

CONTENTS

1. Introduction

The famous *Young inequality* for scalars says that if $x, y>0$ and $\lambda\in[0,1]$, then

$$x^{1-\lambda}y^\lambda\le(1-\lambda)x+\lambda y \tag{1.1}$$

with equality if and only if $x=y$. The inequality (1.1) is also called *λ-weighted arithmetic-geometric mean inequality.*

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We recall that *Specht's ratio* is defined by [19]

$$S(h):=\begin{cases}\dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} & \text{if } h\in(0,1)\cup(1,\infty)\\ 1 & \text{if } h=1.\end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1}S(h)=1$, $S(h)=S\left(\frac{1}{h}\right)>1$ for $h>0$, $h\neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{x}{y}\right)^r\right)x^{1-\lambda}y^{\lambda}\leq(1-\lambda)x+\lambda y\leq S\left(\frac{x}{y}\right)x^{1-\lambda}y^{\lambda}, \tag{1.3}$$

where $x,y>0$, $\lambda\in[0,1]$, $r=\min\{1-\lambda,\lambda\}$.

The second inequality in (1.3) is due to Tominaga [20] while the first one is due to Furuichi [10].

Kittaneh and Manasrah [15], [16] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{x}-\sqrt{y}\right)^2\leq(1-\lambda)x+\lambda y-x^{1-\lambda}y^{\lambda}\leq R\left(\sqrt{x}-\sqrt{y}\right)^2 \tag{1.4}$$

where $x,y>0$, $\lambda\in[0,1]$, $r=\min\{1-\lambda,\lambda\}$ and $R=\max\{1-\lambda,\lambda\}$.

We also consider the *Kantorovich's ratio* defined by

$$K(h):=\frac{(h+1)^2}{4h},\ h>0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h)\geq 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{x}{y}\right)x^{1-\lambda}y^{\lambda}\leq(1-\lambda)x+\lambda y\leq K^R\left(\frac{x}{y}\right)x^{1-\lambda}y^{\lambda} \tag{1.6}$$

where $x,y>0$, $\lambda\in[0,1]$, $r=\min\{1-\lambda,\lambda\}$ and $R=\max\{1-\lambda,\lambda\}$.

The first inequality in (1.6) was obtained by Zou et al. in [22] while the second by Liao et al. [18].

In [22] the authors also showed that $K^r(h)\geq S(h^r)$ for $h>0$ and $r\in\left[0,\frac{1}{2}\right]$ implying that the lower bound in (1.6) is better than the lower bound from (1.3).

In the recent paper [5] we obtained the following reverses of Young's inequality as well:

$$0\leq(1-\lambda)x+\lambda y-x^{1-\lambda}y^{\lambda}\leq\lambda(1-\lambda)(x-y)(\ln x-\ln y) \tag{1.7}$$

and

$$1 \le \frac{(1-\lambda)x + \lambda y}{x^{1-\lambda}y^{\lambda}} \le \exp\left[4\lambda(1-\lambda)\left(K\left(\frac{x}{y}\right)-1\right)\right], \tag{1.8}$$

where $x, y > 0$, $\lambda \in [0,1]$.

In [6] we obtained the following Young related inequalities:

Theorem 1.1. *For any $x, y > 0$ and $\lambda \in [0,1]$ we have*

$$\begin{aligned}\frac{1}{2}\lambda(1-\lambda)(\ln x - \ln y)^2 \min\{x,y\} &\le (1-\lambda)x + \lambda y - x^{1-\lambda}y^{\lambda} \\ &\le \frac{1}{2}\lambda(1-\lambda)(\ln x - \ln y)^2 \max\{x,y\}\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}\exp\left[\frac{1}{2}\lambda(1-\lambda)\frac{(y-x)^2}{\max^2\{x,y\}}\right] &\le \frac{(1-\lambda)x+\lambda y}{x^{1-\lambda}y^{\lambda}} \\ &\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\frac{(y-x)^2}{\min^2\{x,y\}}\right].\end{aligned} \tag{1.10}$$

For an equivalent form and a different approach in proving the results (1.9) and (1.10) see [1].

The second inequalities in (1.9) and (1.10) are better than the corresponding results obtained by Furuichi and Minculete in [13] where instead of constant $\frac{1}{2}$ they had the constant 1.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [3], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.11}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [3] extends the definition of Korányi [17] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [12, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.12}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st)\, dE(t) \otimes dF(s) \tag{1.13}$$

for the continuous function f on $[0,\infty)$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\left\langle (A \circ B) e_j, e_j \right\rangle = \left\langle A e_j, e_j \right\rangle \left\langle B e_j, e_j \right\rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j \in \mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [11], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B)\, \mathcal{U} \tag{1.14}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If *f is super-multiplicative and operator concave(sub-multiplicative and operator convex)* on $[0,\infty)$, then also [12, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.15}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2}\right) \circ 1 \text{ for } A,\ B \geq 0 \tag{1.16}$$

and *Fiedler inequality*

$$A \circ A^{-1} \geq 1 \text{ for } A > 0. \tag{1.17}$$

Alternatively, as extension of Kadison's Schwarz inequality on the Hadamard product, Ando [2] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [4] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [14] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

For some recent results concerning new inequalities for tensorial and Hadamard products, see [7], [8] and [21].

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \varphi(A_t)\, d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

Motivated by the above results, in this paper we show among others that, if $(P_\tau)_{\tau\in\Omega}$ and $(Q_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m, M] \subset (0,\infty)$ for each $\tau \in \Omega$, then for all $\lambda \in [0,1]$

$$\begin{aligned}
&\exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right]\int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\leq (1-\lambda)\int_\Omega P_\tau d\mu(\tau)\otimes 1 + \lambda 1 \otimes \int_\Omega Q_\tau d\mu(\tau) \\
&\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right]\int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau)
\end{aligned}$$

We also have similar inequalities for the Hadamard product εοε.

2. Main results

We start with the following tensorial refinement and reverse of Young's inequality:

Lemma 2.1. *Assume that the selfadjoint operators P and Q satisfy the condition $0 < m \leq P, Q \leq M$, then*

$$\begin{aligned}
0 &\leq m\lambda(1-\lambda)\left[\frac{\left(\ln^2 P\right)\otimes 1 + 1 \otimes \left(\ln^2 Q\right)}{2} - \ln P \otimes \ln Q\right] \quad (2.1) \\
&\leq (1-\lambda) P \otimes 1 + \lambda 1 \otimes Q - P^{1-\lambda} \otimes Q^\lambda \\
&\leq M\lambda(1-\lambda)\left[\frac{\left(\ln^2 P\right)\otimes 1 + 1 \otimes \left(\ln^2 Q\right)}{2} - \ln P \otimes \ln Q\right] \\
&\leq \frac{1}{2}\lambda(1-\lambda) M(\ln M - \ln m)^2
\end{aligned}$$

for all $\lambda \in [0,1]$.

In particular,

$$
\begin{aligned}
0 &\leq \frac{1}{4}m\left[\frac{\left(\ln^2 P\right)\otimes 1 + 1\otimes\left(\ln^2 Q\right)}{2} - \ln P \otimes \ln Q\right] \qquad (2.2)\\
&\leq \frac{P\otimes 1 + 1\otimes Q}{2} - P^{1/2}\otimes Q^{1/2}\\
&\leq \frac{1}{4}M\left[\frac{\left(\ln^2 P\right)\otimes 1 + 1\otimes\left(\ln^2 Q\right)}{2} - \ln P \otimes \ln Q\right]\\
&\leq \frac{1}{8}M\left(\ln M - \ln m\right)^2.
\end{aligned}
$$

Proof. If $t, s \in [m, M] \subset (0, \infty)$, then by (1.9) we get

$$
\begin{aligned}
0 &\leq \frac{1}{2}m\lambda(1-\lambda)(\ln t - \ln s)^2 \leq (1-\lambda)t + \lambda s - t^{1-\lambda}s^{\lambda} \qquad (2.3)\\
&\leq \frac{1}{2}M\lambda(1-\lambda)(\ln t - \ln s)^2 \leq \frac{1}{2}M\lambda(1-\lambda)(\ln M - \ln m)^2.
\end{aligned}
$$

If

$$
P = \int_m^M t dE(t) \text{ and } Q = \int_m^M s dF(s)
$$

are the spectral resolutions of P and Q, then by taking in (2.3) the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, we get

$$
\begin{aligned}
0 &\leq \frac{1}{2}m\lambda(1-\lambda)\int_m^M \int_m^M (\ln t - \ln s)^2 dE(t)\otimes dF(s) \qquad (2.4)\\
&\leq \int_m^M \int_m^M \left[(1-\lambda)t + \lambda s - t^{1-\lambda}s^{\lambda}\right] dE(t)\otimes dF(s)\\
&\leq \frac{1}{2}M\lambda(1-\lambda)\int_m^M \int_m^M (\ln t - \ln s)^2 dE(t)\otimes dF(s)\\
&\leq \frac{1}{8}M(\ln M - \ln m)^2 \int_m^M \int_m^M dE(t)\otimes dF(s).
\end{aligned}
$$

Now, observe that, by (1.11)

$$\int_m^M \int_m^M (\ln t - \ln s)^2 \, dE(t) \otimes dF(s)$$
$$= \int_m^M \int_m^M \left(\ln^2 t - 2\ln t \ln s + \ln^2 s\right) dE(t) \otimes dF(s)$$
$$= \int_m^M \int_m^M \ln^2 t dE(t) \otimes dF(s) + \int_m^M \int_m^M \ln^2 s dE(t) \otimes dF(s)$$
$$- 2\int_m^M \int_m^M \ln t \ln s dE(t) \otimes dF(s)$$
$$= \left(\ln^2 P\right) \otimes 1 + 1 \otimes \left(\ln^2 Q\right) - 2\ln P \otimes \ln Q,$$

$$\int_m^M \int_m^M \left[(1-\lambda) t + \lambda s - t^{1-\lambda} s^{\lambda}\right] dE(t) \otimes dF(s)$$
$$= (1-\lambda) \int_m^M \int_m^M t dE(t) \otimes dF(s) + \lambda \int_m^M \int_m^M s dE(t) \otimes dF(s)$$
$$- \int_m^M \int_m^M t^{1-\lambda} s^{\lambda} dE(t) \otimes dF(s)$$
$$= (1-\lambda) P \otimes 1 + \lambda 1 \otimes Q - P^{1-\lambda} \otimes Q^{\lambda}$$

and

$$\int_m^M \int_m^M dE(t) \otimes dF(s) = 1 \otimes 1 = 1.$$

By employing (2.4) we then get the desired result (2.1). □

We have the representation

$$X \circ Y = \mathcal{U}^* (X \otimes Y) \mathcal{U}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If we take $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in Lemma 2.1, then we can also state the following result for the Hadamard product:

Corollary 2.2. *With the assumptions of Lemma* 2.1*,*

$$0 \le m\lambda (1-\lambda) \left[\left(\frac{\ln^2 P + \ln^2 Q}{2}\right) \circ 1 - \ln P \circ \ln Q\right] \tag{2.5}$$
$$\le [(1-\lambda) P + \lambda Q] \circ 1 - P^{1-\lambda} \circ Q^{\lambda}$$
$$\le M\lambda (1-\lambda) \left[\left(\frac{\ln^2 P + \ln^2 Q}{2}\right) \circ 1 - \ln P \circ \ln Q\right]$$
$$\le \frac{1}{2} \lambda (1-\lambda) M (\ln M - \ln m)^2$$

for all $\lambda \in [0,1]$.
In particular,

$$\begin{aligned}0 &\le \frac{1}{4}m\left[\left(\frac{\ln^2 P+\ln^2 Q}{2}\right)\circ 1-\ln P\circ\ln Q\right] \qquad (2.6)\\ &\le \frac{P+Q}{2}\circ 1-P^{1/2}\circ Q^{1/2}\\ &\le \frac{1}{4}M\left[\left(\frac{\ln^2 P+\ln^2 Q}{2}\right)\circ 1-2\ln P\circ\ln Q\right]\\ &\le \frac{1}{8}M\left(\ln M-\ln m\right)^2.\end{aligned}$$

The inequality (2.5) provides a weighted refinement and reverse inequality for (1.16) that is obtained for $\lambda = 1/2$.

Remark 2.3. *If we take* $Q = P$ *in Corollary* 2.2*, then we get*

$$\begin{aligned}0 &\le m\lambda\left(1-\lambda\right)\left[\left(\ln^2 P\right)\circ 1-\ln P\circ\ln P\right]\le P\circ 1-P^{1-\lambda}\circ P^{\lambda} \qquad (2.7)\\ &\le M\lambda\left(1-\lambda\right)\left[\left(\ln^2 P\right)\circ 1-\ln P\circ\ln P\right]\\ &\le \frac{1}{2}\lambda\left(1-\lambda\right)M\left(\ln M-\ln m\right)^2\end{aligned}$$

for all $\lambda \in [0,1]$.
In particular,

$$\begin{aligned}0 &\le \frac{1}{4}m\left[\left(\ln^2 P\right)\circ 1-\ln P\circ\ln P\right]\le P\circ 1-P^{1/2}\circ P^{1/2} \qquad (2.8)\\ &\le \frac{1}{4}M\left[\left(\ln^2 P\right)\circ 1-\ln P\circ\ln P\right]\le \frac{1}{8}M\left(\ln M-\ln m\right)^2.\end{aligned}$$

Our first main result is as follows:

Theorem 2.4. *Let* Ω *be a locally compact Hausdorff space endowed with a Radon measure* μ. *Let* $(P_\tau)_{\tau\in\Omega}$ *and* $(Q_\tau)_{\tau\in\Omega}$ *be continuous fields of positive operators in* $B(H)$ *such that* $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m, M] \subset (0,\infty)$ *for each* $\tau \in \Omega$. *Then for all*

$\lambda \in [0,1]$ *we have*

$$0 \leq m\lambda(1-\lambda)\left[\frac{\int_\Omega \ln^2 P_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega \ln^2 Q_\tau d\mu(\tau)}{2}\right. \tag{2.9}$$
$$\left. - \int_\Omega \ln P_\tau d\mu(\tau) \otimes \int_\Omega \ln Q_\tau d\mu(\tau)\right]$$
$$\leq (1-\lambda)\int_\Omega P_\tau d\mu(\tau) \otimes 1 + \lambda 1 \otimes \int_\Omega Q_\tau d\mu(\tau)$$
$$- \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau)$$
$$\leq M\lambda(1-\lambda)\left[\frac{\int_\Omega \ln^2 P_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega \ln^2 Q_\tau d\mu(\tau)}{2}\right.$$
$$\left. - \int_\Omega \ln P_\tau d\mu(\tau) \otimes \int_\Omega \ln Q_\tau d\mu(\tau)\right]$$
$$\leq \frac{1}{2}\lambda(1-\lambda) M (\ln M - \ln m)^2 .$$

In particular,

$$0 \leq \frac{1}{4} m\left[\frac{\int_\Omega \ln^2 P_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega \ln^2 Q_\tau d\mu(\tau)}{2}\right. \tag{2.10}$$
$$\left. - \int_\Omega \ln P_\tau d\mu(\tau) \otimes \int_\Omega \ln Q_\tau d\mu(\tau)\right]$$
$$\leq \frac{1}{2}\left[\int_\Omega P_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega Q_\tau d\mu(\tau)\right]$$
$$- \int_\Omega P_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau)$$
$$\leq \frac{1}{4} M\left[\frac{\int_\Omega \ln^2 P_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega \ln^2 Q_\tau d\mu(\tau)}{2}\right.$$
$$\left. - \int_\Omega \ln P_\tau d\mu(\tau) \otimes \int_\Omega \ln Q_\tau d\mu(\tau)\right]$$
$$\leq \frac{1}{8} M (\ln M - \ln m)^2 .$$

Proof. From (2.1) we get

$$\begin{aligned}
0 &\le \frac{1}{2}m\lambda(1-\lambda)\left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right] \qquad (2.11)\\
&\le (1-\lambda)P_\tau\otimes 1 + \lambda 1\otimes Q_\gamma - P_\tau^{1-\lambda}\otimes Q_\gamma^\lambda\\
&\le \frac{1}{2}M\lambda(1-\lambda)\left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right]\\
&\le \frac{1}{2}\lambda(1-\lambda)M(\ln M - \ln m)^2
\end{aligned}$$

for all $\tau, \gamma \in \Omega$.

If we take the integral $\int_\Omega$ over $d\mu(\tau)$, then we get

$$\begin{aligned}
0 &\le \frac{1}{2}m\lambda(1-\lambda) \qquad (2.12)\\
&\times \int_\Omega \left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right] d\mu(\tau)\\
&\le \int_\Omega \left[(1-\lambda)P_\tau\otimes 1 + \lambda 1\otimes Q_\gamma - P_\tau^{1-\lambda}\otimes Q_\gamma^\lambda\right] d\mu(\tau)\\
&\le \frac{1}{2}M\lambda(1-\lambda)\\
&\times \int_\Omega \left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right] d\mu(\tau)\\
&\le \frac{1}{2}\lambda(1-\lambda)M(\ln M - \ln m)^2 \int_\Omega 1 d\mu(\tau).
\end{aligned}$$

By using the properties of integrals and tensorial products, we have

$$\begin{aligned}
&\int_\Omega \left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right] d\mu(\tau)\\
&= \left(\int_\Omega \ln^2 P_\tau d\mu(\tau)\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right)\\
&- 2\int_\Omega \ln P_\tau d\mu(\tau)\otimes \ln Q_\gamma d\mu(\tau)
\end{aligned}$$

and

$$\begin{aligned}
&\int_\Omega \left[(1-\lambda)P_\tau\otimes 1 + \lambda 1\otimes Q_\gamma - P_\tau^{1-\lambda}\otimes Q_\gamma^\lambda\right] d\mu(\tau)\\
&= (1-\lambda)\int_\Omega P_\tau d\mu(\tau)\otimes 1 + \lambda 1\otimes Q_\gamma - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau)\otimes Q_\gamma^\lambda
\end{aligned}$$

for all $\gamma \in \Omega$.

By (2.12) we get

$$
\begin{aligned}
0 &\leq \frac{1}{2}m\lambda(1-\lambda)\left[\left(\int_{\Omega}\ln^2 P_\tau d\mu(\tau)\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right)\right. \\
&\left.-2\int_{\Omega}\ln P_\tau d\mu(\tau)\otimes \ln Q_\gamma\right] \\
&\leq (1-\lambda)\int_{\Omega}P_\tau d\mu(\tau)\otimes 1 + \lambda 1\otimes Q_\gamma - \int_{\Omega}P_\tau^{1-\lambda}d\mu(\tau)\otimes Q_\gamma^{\lambda} \\
&\leq \frac{1}{2}M\lambda(1-\lambda)\left[\left(\int_{\Omega}\ln^2 P_\tau d\mu(\tau)\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right)\right. \\
&\left.-2\int_{\Omega}\ln P_\tau d\mu(\tau)\otimes \ln Q_\gamma\right] \\
&\leq \frac{1}{2}\lambda(1-\lambda)M(\ln M - \ln m)^2
\end{aligned} \tag{2.13}
$$

for all $\gamma \in \Omega$.

If we take the integral $\int_{\Omega}$ over $d\mu(\gamma)$, then we get the desired result (2.9). □

Corollary 2.5. *With the assumptions of Theorem* 2.4*, we have the following inequalities for the Hadamard product*

$$
\begin{aligned}
0 &\leq m\lambda(1-\lambda)\left[\int_{\Omega}\frac{\ln^2 P_\tau + \ln^2 Q_\tau}{2}d\mu(\tau)\circ 1\right. \\
&\left.-\int_{\Omega}\ln P_\tau d\mu(\tau)\circ\int_{\Omega}\ln Q_\tau d\mu(\tau)\right] \\
&\leq \int_{\Omega}[(1-\lambda)P_\tau + \lambda Q_\tau]\, d\mu(\tau)\circ 1 - \int_{\Omega}P_\tau^{1-\lambda}d\mu(\tau)\circ\int_{\Omega}Q_\tau^{\lambda}d\mu(\tau) \\
&\leq \frac{1}{2}M\lambda(1-\lambda)\left[\int_{\Omega}\frac{\ln^2 P_\tau + \ln^2 Q_\tau}{2}d\mu(\tau)\circ 1\right. \\
&\left.-\int_{\Omega}\ln P_\tau d\mu(\tau)\circ\int_{\Omega}\ln Q_\tau d\mu(\tau)\right] \\
&\leq \frac{1}{2}\lambda(1-\lambda)M(\ln M - \ln m)^2
\end{aligned} \tag{2.14}
$$

for all $\lambda \in [0,1]$.

In particular,

$$
\begin{aligned}
0 &\leq \frac{1}{4} m\left[\int_{\Omega} \frac{\ln^2 P_\tau + \ln^2 Q_\tau}{2} d\mu(\tau) \circ 1\right. \\
&\left.- \int_{\Omega} \ln P_\tau d\mu(\tau) \circ \int_{\Omega} \ln Q_\tau d\mu(\tau)\right] \\
&\leq \int_{\Omega} \frac{P_\tau + Q_\tau}{2} d\mu(\tau) \circ 1 - \int_{\Omega} P_\tau^{1/2} d\mu(\tau) \circ \int_{\Omega} Q_\tau^{1/2} d\mu(\tau) \\
&\leq \frac{1}{4} M\left[\int_{\Omega} \frac{\ln^2 P_\tau + \ln^2 Q_\tau}{2} d\mu(\tau) \circ 1\right. \\
&\left.- \int_{\Omega} \ln P_\tau d\mu(\tau) \circ \int_{\Omega} \ln Q_\tau d\mu(\tau)\right] \\
&\leq \frac{1}{8} M(\ln M - \ln m)^2 .
\end{aligned} \tag{2.15}
$$

Remark 2.6. *If we take $Q_\tau = P_\tau$, $\tau \in \Omega$ in Corollary* 2.5*, then we get*

$$
\begin{aligned}
0 &\leq m\lambda(1-\lambda)\left[\int_{\Omega} \ln^2 P_\tau d\mu(\tau) \circ 1 - \int_{\Omega} \ln P_\tau d\mu(\tau) \circ \int_{\Omega} \ln P_\tau d\mu(\tau)\right] \\
&\leq \int_{\Omega} P_\tau d\mu(\tau) \circ 1 - \int_{\Omega} P_\tau^{1-\lambda} d\mu(\tau) \circ \int_{\Omega} P_\tau^{\lambda} d\mu(\tau) \\
&\leq \frac{1}{2} M\lambda(1-\lambda)\left[\int_{\Omega} \ln^2 P_\tau d\mu(\tau) \circ 1 - \int_{\Omega} \ln P_\tau d\mu(\tau) \circ \int_{\Omega} \ln P_\tau d\mu(\tau)\right] \\
&\leq \frac{1}{2} \lambda(1-\lambda) M(\ln M - \ln m)^2
\end{aligned} \tag{2.16}
$$

for all $\lambda \in [0,1]$.

In particular,

$$
\begin{aligned}
0 &\leq \frac{1}{4} m\left[\int_{\Omega} \ln^2 P_\tau d\mu(\tau) \circ 1 - \int_{\Omega} \ln P_\tau d\mu(\tau) \circ \int_{\Omega} \ln P_\tau d\mu(\tau)\right] \\
&\leq \int_{\Omega} P_\tau d\mu(\tau) \circ 1 - \int_{\Omega} P_\tau^{1/2} d\mu(\tau) \circ \int_{\Omega} P_\tau^{1/2} d\mu(\tau) \\
&\leq \frac{1}{4} M\left[\int_{\Omega} \ln^2 P_\tau d\mu(\tau) \circ 1 - \int_{\Omega} \ln P_\tau d\mu(\tau) \circ \int_{\Omega} \ln P_\tau d\mu(\tau)\right] \\
&\leq \frac{1}{8} M(\ln M - \ln m)^2 .
\end{aligned} \tag{2.17}
$$

Lemma 2.7. *With the assumptions of Lemma* 2.1 *we have*

$$0 \le \frac{m}{M^2}\lambda(1-\lambda)\left(\frac{P^2 \otimes 1 + 1 \otimes Q^2}{2} - P \otimes Q\right) \tag{2.18}$$
$$\le (1-\lambda) P \otimes 1 + \lambda 1 \otimes Q - P^{1-\lambda} \otimes Q^{\lambda}$$
$$\le \frac{M}{m^2}\lambda(1-\lambda)\left(\frac{P^2 \otimes 1 + 1 \otimes Q^2}{2} - P \otimes Q\right) \le \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \le \frac{m}{4M^2}\left(\frac{P^2 \otimes 1 + 1 \otimes Q^2}{2} - P \otimes Q\right) \tag{2.19}$$
$$\le \frac{P \otimes 1 + 1 \otimes Q}{2} - P^{1/2} \otimes Q^{1/2}$$
$$\le \frac{M}{4m^2}\left(\frac{P^2 \otimes 1 + 1 \otimes Q^2}{2} - P \otimes Q\right) \le \frac{M}{8m^2}(M-m)^2.$$

Proof. We observe that

$$0 < \frac{1}{\max\{x,y\}} \le \frac{\ln x - \ln y}{x-y} \le \frac{1}{\min\{x,y\}},$$

which implies that

$$0 < \frac{1}{\max^2\{x,y\}} \le \left(\frac{\ln x - \ln y}{x-y}\right)^2 \le \frac{1}{\min^2\{x,y\}}$$

for all $x, y > 0$.

By making use of (1.9) we derive

$$\frac{1}{2}\lambda(1-\lambda)(y-x)^2 \frac{\min\{x,y\}}{\max^2\{x,y\}} \tag{2.20}$$
$$\le \frac{1}{2}\lambda(1-\lambda)(\ln x - \ln y)^2 \min\{x,y\} \le (1-\lambda)x + \lambda y - x^{1-\lambda}y^{\lambda}$$
$$\le \frac{1}{2}\lambda(1-\lambda)(y-x)^2 \frac{\max\{x,y\}}{\min^2\{x,y\}}.$$

If $t, s \in [m, M] \subset (0, \infty)$, then by (2.20) we get

$$0 \le \frac{m}{2M^2}\lambda(1-\lambda)(t-s)^2 \le (1-\lambda)t + \lambda s - t^{1-\lambda}s^{\lambda} \tag{2.21}$$
$$\le \frac{M}{2m^2}\lambda(1-\lambda)(t-s)^2.$$

If

$$P = \int_m^M t dE(t) \text{ and } Q = \int_m^M s dF(s)$$

are the spectral resolutions of P and Q, then by taking in (2.21) the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, we get

$$0 \leq \frac{m}{2M^2}\lambda(1-\lambda)\int_m^M \int_m^M (t-s)^2 E(t) \otimes dF(s) \tag{2.22}$$
$$\leq \int_m^M \int_m^M \left[(1-\lambda)t + \lambda s - t^{1-\lambda}s^\lambda\right] E(t) \otimes dF(s)$$
$$\leq \frac{M}{2m^2}\lambda(1-\lambda)\int_m^M \int_m^M (t-s)^2 E(t) \otimes dF(s).$$

Since, by (1.11)

$$\int_m^M \int_m^M (t-s)^2 E(t) \otimes dF(s)$$
$$= \int_m^M \int_m^M \left(t^2 - 2ts + s^2\right) E(t) \otimes dF(s)$$
$$= \int_m^M \int_m^M t^2 E(t) \otimes dF(s) + \int_m^M \int_m^M s^2 E(t) \otimes dF(s)$$
$$- \int_m^M \int_m^M 2tsE(t) \otimes dF(s)$$
$$= P^2 \otimes 1 + 1 \otimes Q^2 - 2P \otimes Q,$$

then by (2.22) we derive the first part of (2.18).

The last part follows by the fact that

$$(t-s)^2 \leq (M-m)^2$$

for all $t, s \in [m, M]$. □

Corollary 2.8. *With the assumptions of Lemma* 2.7*, we have the following inequalities for the Hadamard product*

$$0 \leq \frac{m}{M^2}\lambda(1-\lambda)\left(\frac{P^2+Q^2}{2}\circ 1 - P\circ Q\right) \tag{2.23}$$
$$\leq [(1-\lambda)P + \lambda Q]\circ 1 - P^{1-\lambda}\circ Q^\lambda$$
$$\leq \frac{M}{m^2}\lambda(1-\lambda)\left(\frac{P^2+Q^2}{2}\circ 1 - P\circ Q\right) \leq \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2$$

for all $\lambda \in [0, 1]$.

In particular,

$$\begin{aligned} 0 &\le \frac{m}{4M^2}\left(\frac{P^2+Q^2}{2}\circ 1 - P\circ Q\right) \le \frac{P+Q}{2}\circ 1 - P^{1/2}\circ Q^{1/2} \qquad (2.24)\\ &\le \frac{M}{4m^2}\left(\frac{P^2+Q^2}{2}\circ 1 - P\circ Q\right) \le \frac{M}{8m^2}(M-m)^2 . \end{aligned}$$

The inequality (2.23) provides another weighted refinement and reverse inequality for (1.16) that is obtained for $\lambda = 1/2$.

Remark 2.9. *If we take $Q = P$ in Corollary 2.8, then we get*

$$\begin{aligned} 0 &\le \frac{m}{M^2}\lambda(1-\lambda)\left(P^2\circ 1 - P\circ P\right) \le P - P^{1-\lambda}\circ P^{\lambda} \qquad (2.25)\\ &\le \frac{M}{m^2}\lambda(1-\lambda)\left(P^2\circ 1 - P\circ P\right) \le \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2 \end{aligned}$$

for all $\lambda \in [0,1]$.

In particular,

$$\begin{aligned} 0 &\le \frac{m}{4M^2}\left(P^2\circ 1 - P\circ P\right) \le P\circ 1 - P^{1/2}\circ P^{1/2} \qquad (2.26)\\ &\le \frac{M}{4m^2}\left(P^2\circ 1 - P\circ P\right) \le \frac{M}{8m^2}(M-m)^2 . \end{aligned}$$

The following integral inequalities also hold:

Theorem 2.10. *Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. Let $(P_\tau)_{\tau\in\Omega}$ and $(Q_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m,M] \subset (0,\infty)$ for each $\tau \in \Omega$. Then for all $\lambda \in [0,1]$ we have*

$$\begin{aligned} 0 &\le \frac{m}{M^2}\lambda(1-\lambda) \qquad (2.27)\\ &\times\left(\frac{\int_\Omega P_\tau^2 d\mu(\tau)\otimes 1 + 1\otimes\int_\Omega Q_\tau^2 d\mu(\tau)}{2} - \int_\Omega P_\tau d\mu(\tau)\otimes\int_\Omega Q_\tau d\mu(\tau)\right)\\ &\le (1-\lambda)\int_\Omega P_\tau d\mu(\tau)\otimes 1 + \lambda 1\otimes\int_\Omega Q_\tau d\mu(\tau)\\ &- \int_\Omega P_\tau^{1-\lambda} d\mu(\tau)\otimes\int_\Omega Q_\tau^{\lambda} d\mu(\tau)\\ &\le \frac{M}{m^2}\lambda(1-\lambda)\\ &\times\left(\frac{\int_\Omega P_\tau^2 d\mu(\tau)\otimes 1 + 1\otimes\int_\Omega Q_\tau^2 d\mu(\tau)}{2} - \int_\Omega P_\tau d\mu(\tau)\otimes\int_\Omega Q_\tau d\mu(\tau)\right)\\ &\le \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2 \end{aligned}$$

for all $\lambda \in [0,1]$.

In particular,

$$\begin{aligned}
0 &\le \frac{m}{4M^2} \\
&\times\left(\frac{\int_\Omega P_\tau^2 d\mu(\tau)\otimes 1 + 1\otimes \int_\Omega Q_\tau^2 d\mu(\tau)}{2} - \int_\Omega P_\tau d\mu(\tau)\otimes \int_\Omega Q_\tau d\mu(\tau)\right) \\
&\le \frac{\int_\Omega P_\tau d\mu(\tau)\otimes 1 + 1\otimes \int_\Omega Q_\tau d\mu(\tau)}{2} - \int_\Omega P_\tau^{1/2} d\mu(\tau)\otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau) \\
&\le \frac{M}{4m^2} \\
&\times\left(\frac{\int_\Omega P_\tau^2 d\mu(\tau)\otimes 1 + 1\otimes \int_\Omega Q_\tau^2 d\mu(\tau)}{2} - \int_\Omega P_\tau d\mu(\tau)\otimes \int_\Omega Q_\tau d\mu(\tau)\right) \\
&\le \frac{M}{8m^2}(M-m)^2
\end{aligned} \tag{2.28}$$

The proof follows by Lemma 2.7 by a similar argument to the one in the proof of Theorem 2.4.

Corollary 2.11. *With the assumptions of Theorem* 2.10*, we have the following inequalities for the Hadamard product*

$$\begin{aligned}
0 &\le \frac{m}{M^2}\lambda(1-\lambda) \\
&\times\left(\int_\Omega \frac{P_\tau^2 + Q_\tau^2}{2} d\mu(\tau)\circ 1 - \int_\Omega P_\tau d\mu(\tau)\circ \int_\Omega Q_\tau d\mu(\tau)\right) \\
&\le \int_\Omega [(1-\lambda)P_\tau + \lambda Q_\tau]\, d\mu(\tau)\circ 1 - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau)\circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\le \frac{M}{m^2}\lambda(1-\lambda) \\
&\times\left(\int_\Omega \frac{P_\tau^2 + Q_\tau^2}{2} d\mu(\tau)\circ 1 - \int_\Omega P_\tau d\mu(\tau)\circ \int_\Omega Q_\tau d\mu(\tau)\right) \\
&\le \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2
\end{aligned} \tag{2.29}$$

for all $\lambda \in [0,1]$.

In particular,

$$\begin{aligned} 0 &\leq \frac{m}{4M^2}\left(\int_\Omega \frac{P_\tau^2+Q_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega Q_\tau d\mu(\tau)\right) \\ &\leq \int_\Omega \frac{P_\tau+Q_\tau}{2}d\mu(\tau)\circ 1-\int_\Omega P_\tau^{1/2}d\mu(\tau)\circ\int_\Omega Q_\tau^{1/2}d\mu(\tau) \\ &\leq \frac{M}{4m^2}\left(\int_\Omega \frac{P_\tau^2+Q_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega Q_\tau d\mu(\tau)\right) \\ &\leq \frac{M}{8m^2}(M-m)^2 \end{aligned} \tag{2.30}$$

Remark 2.12. *If we take $Q_\tau = P_\tau$, $\tau \in \Omega$ in Corollary 2.11, then we get*

$$\begin{aligned} 0 &\leq \frac{m}{M^2}\lambda(1-\lambda)\left(\int_\Omega P_\tau^2 d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega P_\tau d\mu(\tau)\right) \\ &\leq \int_\Omega P_\tau d\mu(\tau)\circ 1-\int_\Omega P_\tau^{1-\lambda}d\mu(\tau)\circ\int_\Omega P_\tau^{\lambda}d\mu(\tau) \\ &\leq \frac{M}{m^2}\lambda(1-\lambda)\left(\int_\Omega P_\tau^2 d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega P_\tau d\mu(\tau)\right) \\ &\leq \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2 \end{aligned} \tag{2.31}$$

for all $\lambda \in [0,1]$.

In particular,

$$\begin{aligned} 0 &\leq \frac{m}{4M^2}\left(\int_\Omega P_\tau^2 d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega P_\tau d\mu(\tau)\right) \\ &\leq \int_\Omega P_\tau d\mu(\tau)\circ 1-\int_\Omega P_\tau^{1/2}d\mu(\tau)\circ\int_\Omega P_\tau^{1/2}d\mu(\tau) \\ &\leq \frac{M}{4m^2}\left(\int_\Omega P_\tau^2 d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega P_\tau d\mu(\tau)\right) \\ &\leq \frac{M}{8m^2}(M-m)^2. \end{aligned} \tag{2.32}$$

3. Some related results

Further, we also have:

Lemma 3.1. *Assume that the selfadjoint operators P and Q satisfy the condition $0 < P, Q \leq M$, then*

$$\begin{aligned} 0 &\leq (1-\lambda)P\otimes 1+\lambda 1\otimes Q-P^{1-\lambda}\otimes Q^{\lambda} \\ &\leq M\lambda(1-\lambda)\left(\frac{P^{-1}\otimes Q+P\otimes Q^{-1}}{2}-1\right) \end{aligned} \tag{3.1}$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \leq \frac{P \otimes 1 + 1 \otimes Q}{2} - P^{1/2} \otimes Q^{1/2} \leq \frac{1}{4} M \left(\frac{P^{-1} \otimes Q + P \otimes Q^{-1}}{2} - 1 \right). \quad (3.2)$$

Proof. Recall that if $x, y > 0$ and

$$L(x,y) := \begin{cases} \frac{y-x}{\ln y - \ln x} \text{ if } x \neq y, \\ \\ y \text{ if } x = y \end{cases}$$

is the *logarithmic mean* and $G(x,y) := \sqrt{xy}$ is the *geometric mean*, then $L(x,y) \geq G(x,y)$ for all $x, y > 0$.

Then from (1.9) we have for $x \neq y$ that

$$\begin{aligned} (1-\lambda)x + \lambda y - x^{1-\lambda}y^{\lambda} &\leq \frac{1}{2}\lambda(1-\lambda)(\ln x - \ln y)^2 \max\{x,y\} \\ &= \frac{1}{2}\lambda(1-\lambda)(y-x)^2 \left(\frac{\ln x - \ln y}{y-x}\right)^2 \max\{x,y\} \\ &\leq \frac{1}{2}\lambda(1-\lambda)\frac{(y-x)^2}{xy} \max\{x,y\} \\ &= \frac{1}{2}\lambda(1-\lambda)\left(\frac{y}{x} + \frac{x}{y} - 2\right) \max\{x,y\}, \end{aligned}$$

which implies that

$$(1-\lambda)x + \lambda y - x^{1-\lambda}y^{\lambda} \leq \frac{1}{2}\lambda(1-\lambda)\left(\frac{y}{x} + \frac{x}{y} - 2\right) \max\{x,y\} \quad (3.3)$$

for all $x, y > 0$.

If $t, s \in (0, M] \subset (0, \infty)$, then by (3.3) we get on taking the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, that

$$\begin{aligned} &\int_m^M \int_m^M \left[(1-\lambda)t + \lambda s - t^{1-\lambda}s^{\lambda}\right] dE(t) \otimes dF(s) \quad (3.4) \\ &\leq \frac{1}{2}M\lambda(1-\lambda)\int_m^M \int_m^M \left(\frac{s}{t} + \frac{t}{s} - 2\right) dE(t) \otimes dF(s). \end{aligned}$$

Since

$$\begin{aligned}
&\int_m^M \int_m^M \left(\frac{s}{t} + \frac{t}{s} - 2\right) dE(t) \otimes dF(s) \\
&= \int_m^M \int_m^M t^{-1} sE(t) \otimes dF(s) + \int_m^M \int_m^M ts^{-1} dE(t) \otimes dF(s) \\
&- \int_m^M \int_m^M dE(t) \otimes dF(s) \\
&= P^{-1} \otimes Q + P \otimes Q^{-1} - 2,
\end{aligned}$$

hence by (3.4) we derive (3.1). □

Corollary 3.2. *With the assumptions of Lemma* 3.1*, we have the inequalities for the Hadamard product*

$$\begin{aligned}
0 &\le [(1-\lambda) P + \lambda Q] \circ 1 - P^{1-\lambda} \circ Q^{\lambda} \\
&\le M\lambda (1-\lambda) \left(\frac{P^{-1} \circ Q + P \circ Q^{-1}}{2} - 1\right)
\end{aligned} \tag{3.5}$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \le \frac{P+Q}{2} \circ 1 - P^{1/2} \circ Q^{1/2} \le \frac{1}{4} M \left(\frac{P^{-1} \circ Q + P \circ Q^{-1}}{2} - 1\right). \tag{3.6}$$

We observe that, if we take $Q = P$ in Corollary 3.2, then we get

$$0 \le P \circ 1 - P^{1-\lambda} \circ P^{\lambda} \le M\lambda (1-\lambda) \left(P^{-1} \circ P - 1\right) \tag{3.7}$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \le P \circ 1 - P^{1/2} \circ P^{1/2} \le \frac{1}{8} M \left(P^{-1} \circ P - 1\right). \tag{3.8}$$

The inequality (3.7) provides a weighted refinement of *Fiedler inequality (*1.17*).*

Moreover, we also have the integral inequalities:

Theorem 3.3. *Let* Ω *be a locally compact Hausdorff space endowed with a Radon measure* μ*. Let* $(P_\tau)_{\tau\in\Omega}$ *and* $(Q_\tau)_{\tau\in\Omega}$ *be continuous fields of positive operators in* $B(H)$ *such that* $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m, M] \subset (0, \infty)$ *for each* $\tau \in \Omega$*. Then for all*

$\lambda \in [0,1]$ *we have*

$$
\begin{aligned}
0 &\le (1-\lambda)\int_\Omega P_\tau d\mu(\tau) \otimes 1 + \lambda 1 \otimes \int_\Omega Q_\tau d\mu(\tau) \\
&- \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\le M\lambda(1-\lambda) \\
&\times \left(\frac{\int_\Omega P_\tau^{-1} d\mu(\tau) \otimes \int_\Omega Q_\tau d\mu(\tau) + \int_\Omega P_\tau d\mu(\tau) \otimes \int_\Omega Q_\tau^{-1} d\mu(\tau)}{2} - 1 \right)
\end{aligned} \tag{3.9}
$$

and

$$
\begin{aligned}
0 &\le \frac{\int_\Omega P_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega Q_\tau d\mu(\tau)}{2} - \int_\Omega P_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau) \\
&\le \frac{1}{4} M \\
&\times \left(\frac{\int_\Omega P_\tau^{-1} d\mu(\tau) \otimes \int_\Omega Q_\tau d\mu(\tau) + \int_\Omega P_\tau d\mu(\tau) \otimes \int_\Omega Q_\tau^{-1} d\mu(\tau)}{2} - 1 \right).
\end{aligned} \tag{3.10}
$$

We also can state the corresponding inequalities for the Hadamard product

$$
\begin{aligned}
0 &\le \int_\Omega [(1-\lambda)P_\tau + \lambda Q_\tau]\, d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\le M\lambda(1-\lambda) \\
&\times \left(\frac{\int_\Omega P_\tau^{-1} d\mu(\tau) \circ \int_\Omega Q_\tau d\mu(\tau) + \int_\Omega P_\tau d\mu(\tau) \circ \int_\Omega Q_\tau^{-1} d\mu(\tau)}{2} - 1 \right)
\end{aligned}
$$

and

$$
\begin{aligned}
0 &\le \int_\Omega \frac{P_\tau + Q_\tau}{2} d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\le \frac{1}{4} M \\
&\times \left(\frac{\int_\Omega P_\tau^{-1} d\mu(\tau) \circ \int_\Omega Q_\tau d\mu(\tau) + \int_\Omega P_\tau d\mu(\tau) \circ \int_\Omega Q_\tau^{-1} d\mu(\tau)}{2} - 1 \right).
\end{aligned}
$$

In the case when $Q_\tau = P_\tau$, $\tau \in \Omega$, we derive

$$
\begin{aligned}
0 &\le \int_\Omega P_\tau d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega P_\tau^{\lambda} d\mu(\tau) \\
&\le M\lambda(1-\lambda) \left(\int_\Omega P_\tau^{-1} d\mu(\tau) \circ \int_\Omega P_\tau d\mu(\tau) - 1 \right)
\end{aligned}
$$

for all $\lambda \in [0,1]$ and

$$0 \le \int_\Omega P_\tau d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega P_\tau^{1/2} d\mu(\tau)$$
$$\le \frac{1}{4} M \left(\int_\Omega P_\tau^{-1} d\mu(\tau) \circ \int_\Omega P_\tau d\mu(\tau) - 1 \right).$$

We also have the following multiplicative results:

Lemma 3.4. *Assume that the selfadjoint operators P and Q satisfy the condition $0 < m \le P, Q \le M$, then*

$$\begin{aligned} P^{1-\lambda} \otimes Q^\lambda &\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] P^{1-\lambda} \otimes Q^\lambda \\ &\le (1-\lambda) P \otimes 1 + \lambda 1 \otimes Q \\ &\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] P^{1-\lambda} \otimes Q^\lambda \end{aligned} \tag{3.11}$$

for all $\lambda \in [0,1]$.

In particular,

$$\begin{aligned} P^{1-\lambda} \otimes Q^\lambda &\le \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] P^{1/2} \otimes Q^{1/2} \\ &\le \frac{P \otimes 1 + 1 \otimes Q}{2} \\ &\le \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] P^{1/2} \otimes Q^{1/2}. \end{aligned} \tag{3.12}$$

Proof. Since

$$\frac{(y-x)^2}{\max^2\{x,y\}} = \left(\frac{\max\{x,y\} - \min\{x,y\}}{\max\{x,y\}}\right)^2 = \left(1 - \frac{\min\{x,y\}}{\max\{x,y\}}\right)^2$$

and

$$\frac{(y-x)^2}{\min^2\{x,y\}} = \left(\frac{\max\{x,y\} - \min\{x,y\}}{\min\{x,y\}}\right)^2 = \left(\frac{\max\{x,y\}}{\min\{x,y\}} - 1\right)^2,$$

hence by (1.10) we derive

$$\begin{aligned} &\exp\left[\frac{1}{2}\lambda(1-\lambda)\left(1 - \frac{\min\{x,y\}}{\max\{x,y\}}\right)^2\right] \\ &\le \frac{(1-\lambda)x + \lambda y}{x^{1-\lambda} y^\lambda} \\ &\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{\max\{x,y\}}{\min\{x,y\}} - 1\right)^2\right]. \end{aligned} \tag{3.13}$$

If $t, s \in [m, M] \subset (0, \infty)$, then by (3.13) we get, if we take the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, the desired result (3.11). □

Corollary 3.5. *With the assumptions of Lemma 3.4, we have the inequalities for Hadamard product*

$$\begin{aligned} P^{1-\lambda} \circ Q^{\lambda} &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] P^{1-\lambda} \circ Q^{\lambda} \\ &\leq (1-\lambda) P + \lambda Q \\ &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] P^{1-\lambda} \circ Q^{\lambda} \end{aligned} \tag{3.14}$$

for all $\lambda \in [0, 1]$.

In particular,

$$\begin{aligned} P^{1/2} \circ Q^{1/2} &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] P^{1/2} \circ Q^{1/2} \\ &\leq \frac{P+Q}{2} \circ 1 \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] P^{1/2} \circ Q^{1/2}. \end{aligned} \tag{3.15}$$

If we take $Q = P$ in Corollary 3.5, then we get the following inequalities for one operator P satisfying the condition $0 < m \leq P \leq M$,

$$\begin{aligned} P^{1-\lambda} \circ P^{\lambda} &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] P^{1-\lambda} \circ P^{\lambda} \\ &\leq P \circ 1 \\ &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] P^{1-\lambda} \circ P^{\lambda} \end{aligned} \tag{3.16}$$

for all $\lambda \in [0, 1]$.

In particular,

$$\begin{aligned} P^{1/2} \circ P^{1/2} &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] P^{1/2} \circ P^{1/2} \\ &\leq P \circ 1 \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] P^{1/2} \circ P^{1/2}. \end{aligned} \tag{3.17}$$

Finally, we can state the following multiplicative result:

Theorem 3.6. *Let $(P_\tau)_{\tau\in\Omega}$ and $(Q_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m, M] \subset (0, \infty)$ for each $\tau \in \Omega$. Then for all*

$\lambda \in [0,1]$ *we have*

$$\begin{aligned}
&\int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\leq (1-\lambda)\int_\Omega P_\tau d\mu(\tau) \otimes 1 + \lambda 1 \otimes \int_\Omega Q_\tau d\mu(\tau) \\
&\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau)
\end{aligned} \tag{3.18}$$

and, in particular

$$\begin{aligned}
&\int_\Omega P_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau) \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] \int_\Omega P_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau) \\
&\leq \frac{1}{2}\left[\int_\Omega P_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega Q_\tau d\mu(\tau)\right] \\
&\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] \int_\Omega P_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau).
\end{aligned} \tag{3.19}$$

We can state the following results for the Hadamard product as well:

$$\begin{aligned}
&\int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\leq \int_\Omega [(1-\lambda)P_\tau + \lambda Q_\tau]\, d\mu(\tau) \circ 1 \\
&\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau)
\end{aligned} \tag{3.20}$$

and, in particular

$$\begin{aligned}
&\int_{\Omega} P_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} Q_{\tau}^{1/2} d\mu(\tau) \qquad (3.21)\\
&\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] \int_{\Omega} P_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} Q_{\tau}^{1/2} d\mu(\tau)\\
&\leq \int_{\Omega} \frac{P_{\tau}+Q_{\tau}}{2} d\mu(\tau) \circ 1\\
&\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] \int_{\Omega} P_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} Q_{\tau}^{1/2} d\mu(\tau).
\end{aligned}$$

When $Q_{\tau} = P_{\tau}, \tau \in \Omega$, we obtain for $\lambda \in [0,1]$ that

$$\begin{aligned}
&\int_{\Omega} P_{\tau}^{1-\lambda} d\mu(\tau) \circ \int_{\Omega} Q_{\tau}^{\lambda} d\mu(\tau) \qquad (3.22)\\
&\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] \int_{\Omega} P_{\tau}^{1-\lambda} d\mu(\tau) \circ \int_{\Omega} Q_{\tau}^{\lambda} d\mu(\tau)\\
&\leq \int_{\Omega} \left[(1-\lambda) P_{\tau} + \lambda Q_{\tau}\right] d\mu(\tau) \circ 1\\
&\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] \int_{\Omega} P_{\tau}^{1-\lambda} d\mu(\tau) \circ \int_{\Omega} Q_{\tau}^{\lambda} d\mu(\tau)
\end{aligned}$$

and, in particular,

$$\begin{aligned}
&\int_{\Omega} P_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} P_{\tau}^{1/2} d\mu(\tau) \qquad (3.23)\\
&\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] \int_{\Omega} P_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} P_{\tau}^{1/2} d\mu(\tau)\\
&\leq \int_{\Omega} P_{\tau} d\mu(\tau) \circ 1\\
&\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] \int_{\Omega} P_{\tau}^{1/2} d\mu(\tau) \circ \int_{\Omega} P_{\tau}^{1/2} d\mu(\tau).
\end{aligned}$$

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