New York Journal of Mathematics

New York J. Math. **31** (2025) 792–817.

Young type integral inequalities for tensorial and Hadamard products of continuous fields of operators in Hilbert spaces

S. S. Dragomir and **M. T. Garayev**

ABSTRACT. Let H be a Hilbert space and Ω a locally compact Hausdorff space endowed with a Radon measure μ with $\int_\Omega 1 d\mu(t) = 1$. In this paper we show among others that, if $(A_\tau)_{\tau\in\Omega}$ and $(B_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\sigma(A_\tau), \sigma(B_\tau) \subseteq [m, M] \subset (0, \infty)$ for each $\tau \in \Omega$, then for all $\nu \in [0, 1]$

$$\exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{M}\right)^2\right]\int_\Omega A_\tau^{1-\nu} d\mu(\tau) \otimes \int_\Omega B_\tau^{\nu} d\mu(\tau)$$
$$\leq (1-\nu)\int_\Omega A_\tau d\mu(\tau) \otimes 1 + \nu 1 \otimes \int_\Omega B_\tau d\mu(\tau)$$
$$\leq \exp\left[\frac{1}{2}\nu(1-\nu)\left(\frac{M-m}{m}\right)^2\right]\int_\Omega A_\tau^{1-\nu} d\mu(\tau) \otimes \int_\Omega B_\tau^{\nu} d\mu(\tau).$$

We also have similar inequalities for the Hadamard product εοε.

CONTENTS

1. Introduction

The famous *Young inequality* for scalars says that if $x, y > 0$ and $\lambda \in [0, 1]$, then

$$x^{1-\lambda} y^{\lambda} \leq (1-\lambda)x + \lambda y \tag{1.1}$$

with equality if and only if $x = y$. The inequality (1.1) is also called *λ-weighted arithmetic-geometric mean inequality.*

Received April 24, 2024.

2010 *Mathematics Subject Classification.* 47A63; 47A99.

Key words and phrases. Tensorial product, Hadamard Product, selfadjoint operators, convex functions, compact Hausdorff space, Radon measure, Young operator inequality.

ISSN 1076-9803/2025

We recall that *Specht's ratio* is defined by [19]

$$S(h) := \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e\ln\left(h^{\frac{1}{h-1}}\right)} \text{ if } h \in (0,1) \cup (1,\infty) \\ \\ 1 \text{ if } h = 1. \end{cases} \tag{1.2}$$

It is well known that $\lim_{h\to 1} S(h) = 1$, $S(h) = S\left(\frac{1}{h}\right) > 1$ for $h > 0$, $h \neq 1$. The function is decreasing on $(0,1)$ and increasing on $(1,\infty)$.

The following inequality provides a refinement and a multiplicative reverse for Young's inequality

$$S\left(\left(\frac{x}{y}\right)^r\right) x^{1-\lambda} y^{\lambda} \le (1-\lambda)x + \lambda y \le S\left(\frac{x}{y}\right) x^{1-\lambda} y^{\lambda}, \tag{1.3}$$

where $x, y > 0$, $\lambda \in [0,1]$, $r = \min\{1-\lambda, \lambda\}$.

The second inequality in (1.3) is due to Tominaga [20] while the first one is due to Furuichi [10].

Kittaneh and Manasrah [15], [16] provided a refinement and an additive reverse for Young inequality as follows:

$$r\left(\sqrt{x} - \sqrt{y}\right)^2 \le (1-\lambda)x + \lambda y - x^{1-\lambda} y^{\lambda} \le R\left(\sqrt{x} - \sqrt{y}\right)^2 \tag{1.4}$$

where $x, y > 0$, $\lambda \in [0,1]$, $r = \min\{1-\lambda, \lambda\}$ and $R = \max\{1-\lambda, \lambda\}$.

We also consider the *Kantorovich's ratio* defined by

$$K(h) := \frac{(h+1)^2}{4h},\ h > 0. \tag{1.5}$$

The function K is decreasing on $(0,1)$ and increasing on $[1,\infty)$, $K(h) \ge 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

The following multiplicative refinement and reverse of Young inequality in terms of Kantorovich's ratio holds

$$K^r\left(\frac{x}{y}\right) x^{1-\lambda} y^{\lambda} \le (1-\lambda)x + \lambda y \le K^R\left(\frac{x}{y}\right) x^{1-\lambda} y^{\lambda} \tag{1.6}$$

where $x, y > 0$, $\lambda \in [0,1]$, $r = \min\{1-\lambda, \lambda\}$ and $R = \max\{1-\lambda, \lambda\}$.

The first inequality in (1.6) was obtained by Zou et al. in [22] while the second by Liao et al. [18].

In [22] the authors also showed that $K^r(h) \ge S(h^r)$ for $h > 0$ and $r \in \left[0, \frac{1}{2}\right]$ implying that the lower bound in (1.6) is better than the lower bound from (1.3).

In the recent paper [5] we obtained the following reverses of Young's inequality as well:

$$0 \le (1-\lambda)x + \lambda y - x^{1-\lambda} y^{\lambda} \le \lambda(1-\lambda)(x-y)(\ln x - \ln y) \tag{1.7}$$

and

$$1 \le \frac{(1-\lambda)x + \lambda y}{x^{1-\lambda}y^{\lambda}} \le \exp\left[4\lambda(1-\lambda)\left(K\left(\frac{x}{y}\right)-1\right)\right], \tag{1.8}$$

where $x, y > 0$, $\lambda \in [0,1]$.

In [6] we obtained the following Young related inequalities:

Theorem 1.1. *For any $x, y > 0$ and $\lambda \in [0,1]$ we have*

$$\frac{1}{2}\lambda(1-\lambda)(\ln x - \ln y)^2 \min\{x,y\} \le (1-\lambda)x + \lambda y - x^{1-\lambda}y^{\lambda} \le \frac{1}{2}\lambda(1-\lambda)(\ln x - \ln y)^2 \max\{x,y\} \tag{1.9}$$

and

$$\exp\left[\frac{1}{2}\lambda(1-\lambda)\frac{(y-x)^2}{\max^2\{x,y\}}\right] \le \frac{(1-\lambda)x+\lambda y}{x^{1-\lambda}y^{\lambda}} \le \exp\left[\frac{1}{2}\lambda(1-\lambda)\frac{(y-x)^2}{\min^2\{x,y\}}\right]. \tag{1.10}$$

For an equivalent form and a different approach in proving the results (1.9) and (1.10) see [1].

The second inequalities in (1.9) and (1.10) are better than the corresponding results obtained by Furuichi and Minculete in [13] where instead of constant $\frac{1}{2}$ they had the constant 1.

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [3], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \tag{1.11}$$

as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [3] extends the definition of Korányi [17] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0,\infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s,t \in [0,\infty)$$

and if f is continuous on $[0,\infty)$, then [12, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{1.12}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) dE(t) \otimes dF(s) \tag{1.13}$$

for the continuous function f on $[0,\infty)$.

Recall that the *Hadamard product* of A and B in $B(H)$ is defined to be the operator $A \circ B \in B(H)$ satisfying

$$\langle (A \circ B) e_j, e_j \rangle = \langle A e_j, e_j \rangle \langle B e_j, e_j \rangle$$

for all $j \in \mathbb{N}$, where $\{e_j\}_{j\in\mathbb{N}}$ is an *orthonormal basis* for the separable Hilbert space H.

It is known that, see [11], we have the representation

$$A \circ B = \mathcal{U}^* (A \otimes B) \mathcal{U} \tag{1.14}$$

where $\mathcal{U} : H \to H \otimes H$ is the isometry defined by $\mathcal{U} e_j = e_j \otimes e_j$ for all $j \in \mathbb{N}$.

If *f is super-multiplicative and operator concave(sub-multiplicative and operator convex)* on $[0,\infty)$, then also [12, p. 173]

$$f(A \circ B) \geq (\leq) f(A) \circ f(B) \text{ for all } A,\ B \geq 0. \tag{1.15}$$

We recall the following elementary inequalities for the Hadamard product

$$A^{1/2} \circ B^{1/2} \leq \left(\frac{A+B}{2} \right) \circ 1 \text{ for } A,\ B \geq 0 \tag{1.16}$$

and *Fiedler inequality*

$$A \circ A^{-1} \geq 1 \text{ for } A > 0. \tag{1.17}$$

Alternatively, as extension of Kadison's Schwarz inequality on the Hadamard product, Ando [2] showed that

$$A \circ B \leq \left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2} \text{ for } A,\ B \geq 0$$

and Aujla and Vasudeva [4] gave an alternative upper bound

$$A \circ B \leq \left(A^2 \circ B^2\right)^{1/2} \text{ for } A,\ B \geq 0.$$

It has been shown in [14] that $\left(A^2 \circ 1\right)^{1/2} \left(B^2 \circ 1\right)^{1/2}$ and $\left(A^2 \circ B^2\right)^{1/2}$ are incomparable for 2-square positive definite matrices A and B.

For some recent results concerning new inequalities for tensorial and Hadamard products, see [7], [8] and [21].

Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. A field $(A_t)_{t\in\Omega}$ of operators in $B(H)$ is called a continuous field of operators if the parametrization $t \longmapsto A_t$ is norm continuous on $B(H)$. If, in addition, the norm function $t \longmapsto \|A_t\|$ is Lebesgue integrable on Ω, we can form the Bochner integral $\int_\Omega A_t d\mu(t)$, which is the unique operator in $B(H)$ such that $\varphi\left(\int_\Omega A_t d\mu(t)\right) = \int_\Omega \varphi(A_t)\, d\mu(t)$ for every bounded linear functional φ on $B(H)$. Assume also that, $\int_\Omega 1 d\mu(t) = 1$.

Motivated by the above results, in this paper we show among others that, if $(P_\tau)_{\tau\in\Omega}$ and $(Q_\tau)_{\tau\in\Omega}$ are continuous fields of positive operators in $B(H)$ such that $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m, M] \subset (0, \infty)$ for each $\tau \in \Omega$, then for all $\lambda \in [0,1]$

$$\begin{aligned}
&\exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right]\int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau)\\
&\le (1-\lambda)\int_\Omega P_\tau d\mu(\tau) \otimes 1 + \lambda 1 \otimes \int_\Omega Q_\tau d\mu(\tau)\\
&\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right]\int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau)
\end{aligned}$$

We also have similar inequalities for the Hadamard product $\varepsilon\circ\varepsilon$.

2. Main results

We start with the following tensorial refinement and reverse of Young's inequality:

Lemma 2.1. *Assume that the selfadjoint operators P and Q satisfy the condition $0 < m \le P, Q \le M$, then*

$$\begin{aligned}
0 &\le m\lambda(1-\lambda)\left[\frac{\left(\ln^2 P\right)\otimes 1 + 1 \otimes \left(\ln^2 Q\right)}{2} - \ln P \otimes \ln Q\right] \qquad (2.1)\\
&\le (1-\lambda) P \otimes 1 + \lambda 1 \otimes Q - P^{1-\lambda} \otimes Q^{\lambda}\\
&\le M\lambda(1-\lambda)\left[\frac{\left(\ln^2 P\right)\otimes 1 + 1 \otimes \left(\ln^2 Q\right)}{2} - \ln P \otimes \ln Q\right]\\
&\le \frac{1}{2}\lambda(1-\lambda) M (\ln M - \ln m)^2
\end{aligned}$$

for all $\lambda \in [0,1]$.

In particular,

$$\begin{aligned} 0 &\leq \frac{1}{4}m\left[\frac{\left(\ln^2 P\right)\otimes 1 + 1\otimes\left(\ln^2 Q\right)}{2} - \ln P\otimes\ln Q\right] \\ &\leq \frac{P\otimes 1 + 1\otimes Q}{2} - P^{1/2}\otimes Q^{1/2} \\ &\leq \frac{1}{4}M\left[\frac{\left(\ln^2 P\right)\otimes 1 + 1\otimes\left(\ln^2 Q\right)}{2} - \ln P\otimes\ln Q\right] \\ &\leq \frac{1}{8}M\left(\ln M - \ln m\right)^2. \end{aligned} \tag{2.2}$$

Proof. If $t, s \in [m, M] \subset (0, \infty)$, then by (1.9) we get

$$\begin{aligned} 0 &\leq \frac{1}{2}m\lambda(1-\lambda)(\ln t - \ln s)^2 \leq (1-\lambda)t + \lambda s - t^{1-\lambda}s^{\lambda} \\ &\leq \frac{1}{2}M\lambda(1-\lambda)(\ln t - \ln s)^2 \leq \frac{1}{2}M\lambda(1-\lambda)(\ln M - \ln m)^2. \end{aligned} \tag{2.3}$$

If

$$P = \int_m^M t\,dE(t) \text{ and } Q = \int_m^M s\,dF(s)$$

are the spectral resolutions of P and Q, then by taking in (2.3) the double integral $\int_m^M \int_m^M$ over $dE(t)\otimes dF(s)$, we get

$$\begin{aligned} 0 &\leq \frac{1}{2}m\lambda(1-\lambda)\int_m^M \int_m^M (\ln t - \ln s)^2\, dE(t)\otimes dF(s) \\ &\leq \int_m^M \int_m^M \left[(1-\lambda)t + \lambda s - t^{1-\lambda}s^{\lambda}\right] dE(t)\otimes dF(s) \\ &\leq \frac{1}{2}M\lambda(1-\lambda)\int_m^M \int_m^M (\ln t - \ln s)^2\, dE(t)\otimes dF(s) \\ &\leq \frac{1}{8}M(\ln M - \ln m)^2 \int_m^M \int_m^M dE(t)\otimes dF(s). \end{aligned} \tag{2.4}$$

Now, observe that, by (1.11)

$$\begin{aligned}&\int_m^M\int_m^M (\ln t-\ln s)^2\,dE(t)\otimes dF(s)\\&=\int_m^M\int_m^M\left(\ln^2 t-2\ln t\ln s+\ln^2 s\right)dE(t)\otimes dF(s)\\&=\int_m^M\int_m^M \ln^2 t dE(t)\otimes dF(s)+\int_m^M\int_m^M \ln^2 s dE(t)\otimes dF(s)\\&-2\int_m^M\int_m^M \ln t\ln s dE(t)\otimes dF(s)\\&=\left(\ln^2 P\right)\otimes 1+1\otimes\left(\ln^2 Q\right)-2\ln P\otimes\ln Q,\end{aligned}$$

$$\begin{aligned}&\int_m^M\int_m^M\left[(1-\lambda)t+\lambda s-t^{1-\lambda}s^{\lambda}\right]dE(t)\otimes dF(s)\\&=(1-\lambda)\int_m^M\int_m^M t dE(t)\otimes dF(s)+\lambda\int_m^M\int_m^M s dE(t)\otimes dF(s)\\&-\int_m^M\int_m^M t^{1-\lambda}s^{\lambda}dE(t)\otimes dF(s)\\&=(1-\lambda)P\otimes 1+\lambda 1\otimes Q-P^{1-\lambda}\otimes Q^{\lambda}\end{aligned}$$

and

$$\int_m^M\int_m^M dE(t)\otimes dF(s)=1\otimes 1=1.$$

By employing (2.4) we then get the desired result (2.1). □

We have the representation

$$X\circ Y=\mathcal{U}^*(X\otimes Y)\,\mathcal{U}$$

where $\mathcal{U}:H\to H\otimes H$ is the isometry defined by $\mathcal{U}e_j=e_j\otimes e_j$ for all $j\in\mathbb{N}$.

If we take $\mathcal{U}^*$ to the left and $\mathcal{U}$ to the right in Lemma 2.1, then we can also state the following result for the Hadamard product:

Corollary 2.2. *With the assumptions of Lemma 2.1,*

$$\begin{aligned}0&\le m\lambda(1-\lambda)\left[\left(\frac{\ln^2 P+\ln^2 Q}{2}\right)\circ 1-\ln P\circ\ln Q\right]\qquad(2.5)\\&\le[(1-\lambda)P+\lambda Q]\circ 1-P^{1-\lambda}\circ Q^{\lambda}\\&\le M\lambda(1-\lambda)\left[\left(\frac{\ln^2 P+\ln^2 Q}{2}\right)\circ 1-\ln P\circ\ln Q\right]\\&\le\frac{1}{2}\lambda(1-\lambda)M(\ln M-\ln m)^2\end{aligned}$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \leq \frac{1}{4}m\left[\left(\frac{\ln^2 P + \ln^2 Q}{2}\right)\circ 1 - \ln P \circ \ln Q\right] \tag{2.6}$$
$$\leq \frac{P+Q}{2}\circ 1 - P^{1/2}\circ Q^{1/2}$$
$$\leq \frac{1}{4}M\left[\left(\frac{\ln^2 P + \ln^2 Q}{2}\right)\circ 1 - 2\ln P \circ \ln Q\right]$$
$$\leq \frac{1}{8}M\left(\ln M - \ln m\right)^2.$$

The inequality (2.5) provides a weighted refinement and reverse inequality for (1.16) that is obtained for $\lambda = 1/2$.

Remark 2.3. *If we take* $Q = P$ *in Corollary* 2.2*, then we get*

$$0 \leq m\lambda(1-\lambda)\left[\left(\ln^2 P\right)\circ 1 - \ln P \circ \ln P\right] \leq P\circ 1 - P^{1-\lambda}\circ P^{\lambda} \tag{2.7}$$
$$\leq M\lambda(1-\lambda)\left[\left(\ln^2 P\right)\circ 1 - \ln P \circ \ln P\right]$$
$$\leq \frac{1}{2}\lambda(1-\lambda)M\left(\ln M - \ln m\right)^2$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \leq \frac{1}{4}m\left[\left(\ln^2 P\right)\circ 1 - \ln P \circ \ln P\right] \leq P\circ 1 - P^{1/2}\circ P^{1/2} \tag{2.8}$$
$$\leq \frac{1}{4}M\left[\left(\ln^2 P\right)\circ 1 - \ln P \circ \ln P\right] \leq \frac{1}{8}M\left(\ln M - \ln m\right)^2.$$

Our first main result is as follows:

Theorem 2.4. *Let* Ω *be a locally compact Hausdorff space endowed with a Radon measure* μ*. Let* $(P_\tau)_{\tau\in\Omega}$ *and* $(Q_\tau)_{\tau\in\Omega}$ *be continuous fields of positive operators in* $B(H)$ *such that* $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m, M] \subset (0,\infty)$ *for each* $\tau \in \Omega$*. Then for all*

$\lambda \in [0,1]$ *we have*

$$\begin{aligned}0 &\leq m\lambda(1-\lambda)\left[\frac{\int_\Omega \ln^2 P_\tau d\mu(\tau)\otimes 1 + 1\otimes \int_\Omega \ln^2 Q_\tau d\mu(\tau)}{2}\right.\\&\left.-\int_\Omega \ln P_\tau d\mu(\tau)\otimes\int_\Omega \ln Q_\tau d\mu(\tau)\right]\\&\leq (1-\lambda)\int_\Omega P_\tau d\mu(\tau)\otimes 1 + \lambda 1\otimes\int_\Omega Q_\tau d\mu(\tau)\\&-\int_\Omega P_\tau^{1-\lambda} d\mu(\tau)\otimes\int_\Omega Q_\tau^{\lambda} d\mu(\tau)\\&\leq M\lambda(1-\lambda)\left[\frac{\int_\Omega \ln^2 P_\tau d\mu(\tau)\otimes 1 + 1\otimes \int_\Omega \ln^2 Q_\tau d\mu(\tau)}{2}\right.\\&\left.-\int_\Omega \ln P_\tau d\mu(\tau)\otimes\int_\Omega \ln Q_\tau d\mu(\tau)\right]\\&\leq \frac{1}{2}\lambda(1-\lambda)M(\ln M-\ln m)^2.\end{aligned}\tag{2.9}$$

In particular,

$$\begin{aligned}0 &\leq \frac{1}{4}m\left[\frac{\int_\Omega \ln^2 P_\tau d\mu(\tau)\otimes 1 + 1\otimes \int_\Omega \ln^2 Q_\tau d\mu(\tau)}{2}\right.\\&\left.-\int_\Omega \ln P_\tau d\mu(\tau)\otimes\int_\Omega \ln Q_\tau d\mu(\tau)\right]\\&\leq \frac{1}{2}\left[\int_\Omega P_\tau d\mu(\tau)\otimes 1 + 1\otimes\int_\Omega Q_\tau d\mu(\tau)\right]\\&-\int_\Omega P_\tau^{1/2} d\mu(\tau)\otimes\int_\Omega Q_\tau^{1/2} d\mu(\tau)\\&\leq \frac{1}{4}M\left[\frac{\int_\Omega \ln^2 P_\tau d\mu(\tau)\otimes 1 + 1\otimes \int_\Omega \ln^2 Q_\tau d\mu(\tau)}{2}\right.\\&\left.-\int_\Omega \ln P_\tau d\mu(\tau)\otimes\int_\Omega \ln Q_\tau d\mu(\tau)\right]\\&\leq \frac{1}{8}M(\ln M-\ln m)^2.\end{aligned}\tag{2.10}$$

Proof. From (2.1) we get

$$
\begin{aligned}
0 &\leq \frac{1}{2}m\lambda(1-\lambda)\left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right] \\
&\leq (1-\lambda)P_\tau \otimes 1 + \lambda 1 \otimes Q_\gamma - P_\tau^{1-\lambda}\otimes Q_\gamma^\lambda \\
&\leq \frac{1}{2}M\lambda(1-\lambda)\left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right] \\
&\leq \frac{1}{2}\lambda(1-\lambda)M(\ln M - \ln m)^2
\end{aligned}
\tag{2.11}
$$

for all $\tau, \gamma \in \Omega$.

If we take the integral $\int_\Omega$ over $d\mu(\tau)$, then we get

$$
\begin{aligned}
0 &\leq \frac{1}{2}m\lambda(1-\lambda) \\
&\times \int_\Omega \left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right] d\mu(\tau) \\
&\leq \int_\Omega \left[(1-\lambda)P_\tau \otimes 1 + \lambda 1 \otimes Q_\gamma - P_\tau^{1-\lambda}\otimes Q_\gamma^\lambda\right] d\mu(\tau) \\
&\leq \frac{1}{2}M\lambda(1-\lambda) \\
&\times \int_\Omega \left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right] d\mu(\tau) \\
&\leq \frac{1}{2}\lambda(1-\lambda)M(\ln M - \ln m)^2 \int_\Omega 1 d\mu(\tau).
\end{aligned}
\tag{2.12}
$$

By using the properties of integrals and tensorial products, we have

$$
\begin{aligned}
&\int_\Omega \left[\left(\ln^2 P_\tau\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) - 2\ln P_\tau \otimes \ln Q_\gamma\right] d\mu(\tau) \\
&= \left(\int_\Omega \ln^2 P_\tau d\mu(\tau)\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right) \\
&- 2\int_\Omega \ln P_\tau d\mu(\tau) \otimes \ln Q_\gamma d\mu(\tau)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\Omega \left[(1-\lambda)P_\tau \otimes 1 + \lambda 1 \otimes Q_\gamma - P_\tau^{1-\lambda}\otimes Q_\gamma^\lambda\right] d\mu(\tau) \\
&= (1-\lambda)\int_\Omega P_\tau d\mu(\tau)\otimes 1 + \lambda 1\otimes Q_\gamma - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau)\otimes Q_\gamma^\lambda
\end{aligned}
$$

for all $\gamma \in \Omega$.

By (2.12) we get

$$\begin{aligned}
0 &\le \frac{1}{2} m\lambda(1-\lambda)\left[\left(\int_\Omega \ln^2 P_\tau d\mu(\tau)\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right)\right. \\
&\left. -2\int_\Omega \ln P_\tau d\mu(\tau)\otimes \ln Q_\gamma\right] \\
&\le (1-\lambda)\int_\Omega P_\tau d\mu(\tau)\otimes 1 + \lambda 1\otimes Q_\gamma - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau)\otimes Q_\gamma^\lambda \\
&\le \frac{1}{2} M\lambda(1-\lambda)\left[\left(\int_\Omega \ln^2 P_\tau d\mu(\tau)\right)\otimes 1 + 1\otimes\left(\ln^2 Q_\gamma\right)\right. \\
&\left. -2\int_\Omega \ln P_\tau d\mu(\tau)\otimes \ln Q_\gamma\right] \\
&\le \frac{1}{2}\lambda(1-\lambda) M(\ln M - \ln m)^2
\end{aligned} \tag{2.13}$$

for all $\gamma \in \Omega$.

If we take the integral $\int_\Omega$ over $d\mu(\gamma)$, then we get the desired result (2.9). □

Corollary 2.5. *With the assumptions of Theorem 2.4, we have the following inequalities for the Hadamard product*

$$\begin{aligned}
0 &\le m\lambda(1-\lambda)\left[\int_\Omega \frac{\ln^2 P_\tau + \ln^2 Q_\tau}{2} d\mu(\tau)\circ 1\right. \\
&\left. - \int_\Omega \ln P_\tau d\mu(\tau)\circ\int_\Omega \ln Q_\tau d\mu(\tau)\right] \\
&\le \int_\Omega \left[(1-\lambda)P_\tau + \lambda Q_\tau\right] d\mu(\tau)\circ 1 - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau)\circ\int_\Omega Q_\tau^\lambda d\mu(\tau) \\
&\le \frac{1}{2} M\lambda(1-\lambda)\left[\int_\Omega \frac{\ln^2 P_\tau + \ln^2 Q_\tau}{2} d\mu(\tau)\circ 1\right. \\
&\left. - \int_\Omega \ln P_\tau d\mu(\tau)\circ\int_\Omega \ln Q_\tau d\mu(\tau)\right] \\
&\le \frac{1}{2}\lambda(1-\lambda) M(\ln M - \ln m)^2
\end{aligned} \tag{2.14}$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \le \frac{1}{4} m \left[\int_\Omega \frac{\ln^2 P_\tau + \ln^2 Q_\tau}{2} d\mu(\tau) \circ 1 - \int_\Omega \ln P_\tau d\mu(\tau) \circ \int_\Omega \ln Q_\tau d\mu(\tau) \right] \le \int_\Omega \frac{P_\tau + Q_\tau}{2} d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega Q_\tau^{1/2} d\mu(\tau) \le \frac{1}{4} M \left[\int_\Omega \frac{\ln^2 P_\tau + \ln^2 Q_\tau}{2} d\mu(\tau) \circ 1 - \int_\Omega \ln P_\tau d\mu(\tau) \circ \int_\Omega \ln Q_\tau d\mu(\tau) \right] \le \frac{1}{8} M (\ln M - \ln m)^2 . \tag{2.15}$$

Remark 2.6. *If we take $Q_\tau = P_\tau$, $\tau \in \Omega$ in Corollary 2.5, then we get*

$$0 \le m\lambda(1-\lambda) \left[\int_\Omega \ln^2 P_\tau d\mu(\tau) \circ 1 - \int_\Omega \ln P_\tau d\mu(\tau) \circ \int_\Omega \ln P_\tau d\mu(\tau) \right] \le \int_\Omega P_\tau d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega P_\tau^{\lambda} d\mu(\tau) \le \frac{1}{2} M\lambda(1-\lambda) \left[\int_\Omega \ln^2 P_\tau d\mu(\tau) \circ 1 - \int_\Omega \ln P_\tau d\mu(\tau) \circ \int_\Omega \ln P_\tau d\mu(\tau) \right] \le \frac{1}{2} \lambda(1-\lambda) M (\ln M - \ln m)^2 \tag{2.16}$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \le \frac{1}{4} m \left[\int_\Omega \ln^2 P_\tau d\mu(\tau) \circ 1 - \int_\Omega \ln P_\tau d\mu(\tau) \circ \int_\Omega \ln P_\tau d\mu(\tau) \right] \le \int_\Omega P_\tau d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega P_\tau^{1/2} d\mu(\tau) \le \frac{1}{4} M \left[\int_\Omega \ln^2 P_\tau d\mu(\tau) \circ 1 - \int_\Omega \ln P_\tau d\mu(\tau) \circ \int_\Omega \ln P_\tau d\mu(\tau) \right] \le \frac{1}{8} M (\ln M - \ln m)^2 . \tag{2.17}$$

Lemma 2.7. *With the assumptions of Lemma 2.1 we have*

$$0 \le \frac{m}{M^2}\lambda(1-\lambda)\left(\frac{P^2\otimes 1+1\otimes Q^2}{2}-P\otimes Q\right) \tag{2.18}$$
$$\le (1-\lambda)P\otimes 1+\lambda 1\otimes Q-P^{1-\lambda}\otimes Q^{\lambda}$$
$$\le \frac{M}{m^2}\lambda(1-\lambda)\left(\frac{P^2\otimes 1+1\otimes Q^2}{2}-P\otimes Q\right)\le \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2$$

for all $\lambda\in[0,1]$.

In particular,

$$0 \le \frac{m}{4M^2}\left(\frac{P^2\otimes 1+1\otimes Q^2}{2}-P\otimes Q\right) \tag{2.19}$$
$$\le \frac{P\otimes 1+1\otimes Q}{2}-P^{1/2}\otimes Q^{1/2}$$
$$\le \frac{M}{4m^2}\left(\frac{P^2\otimes 1+1\otimes Q^2}{2}-P\otimes Q\right)\le \frac{M}{8m^2}(M-m)^2.$$

Proof. We observe that

$$0<\frac{1}{\max\{x,y\}}\le\frac{\ln x-\ln y}{x-y}\le\frac{1}{\min\{x,y\}},$$

which implies that

$$0<\frac{1}{\max^2\{x,y\}}\le\left(\frac{\ln x-\ln y}{x-y}\right)^2\le\frac{1}{\min^2\{x,y\}}$$

for all $x,y>0$.

By making use of (1.9) we derive

$$\frac{1}{2}\lambda(1-\lambda)(y-x)^2\frac{\min\{x,y\}}{\max^2\{x,y\}} \tag{2.20}$$
$$\le\frac{1}{2}\lambda(1-\lambda)(\ln x-\ln y)^2\min\{x,y\}\le(1-\lambda)x+\lambda y-x^{1-\lambda}y^{\lambda}$$
$$\le\frac{1}{2}\lambda(1-\lambda)(y-x)^2\frac{\max\{x,y\}}{\min^2\{x,y\}}.$$

If $t,s\in[m,M]\subset(0,\infty)$, then by (2.20) we get

$$0\le\frac{m}{2M^2}\lambda(1-\lambda)(t-s)^2\le(1-\lambda)t+\lambda s-t^{1-\lambda}s^{\lambda} \tag{2.21}$$
$$\le\frac{M}{2m^2}\lambda(1-\lambda)(t-s)^2.$$

If

$$P=\int_m^M t\,dE(t)\text{ and }Q=\int_m^M s\,dF(s)$$

are the spectral resolutions of P and Q, then by taking in (2.21) the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, we get

$$0 \leq \frac{m}{2M^2}\lambda(1-\lambda)\int_m^M \int_m^M (t-s)^2 E(t) \otimes dF(s) \tag{2.22}$$
$$\leq \int_m^M \int_m^M \left[(1-\lambda)t + \lambda s - t^{1-\lambda}s^{\lambda}\right] E(t) \otimes dF(s)$$
$$\leq \frac{M}{2m^2}\lambda(1-\lambda)\int_m^M \int_m^M (t-s)^2 E(t) \otimes dF(s).$$

Since, by (1.11)

$$\int_m^M \int_m^M (t-s)^2 E(t) \otimes dF(s)$$
$$= \int_m^M \int_m^M \left(t^2 - 2ts + s^2\right) E(t) \otimes dF(s)$$
$$= \int_m^M \int_m^M t^2 E(t) \otimes dF(s) + \int_m^M \int_m^M s^2 E(t) \otimes dF(s)$$
$$- \int_m^M \int_m^M 2tsE(t) \otimes dF(s)$$
$$= P^2 \otimes 1 + 1 \otimes Q^2 - 2P \otimes Q,$$

then by (2.22) we derive the first part of (2.18).

The last part follows by the fact that

$$(t-s)^2 \leq (M-m)^2$$

for all $t, s \in [m, M]$. □

Corollary 2.8. *With the assumptions of Lemma 2.7, we have the following inequalities for the Hadamard product*

$$0 \leq \frac{m}{M^2}\lambda(1-\lambda)\left(\frac{P^2+Q^2}{2}\circ 1 - P\circ Q\right) \tag{2.23}$$
$$\leq [(1-\lambda)P + \lambda Q]\circ 1 - P^{1-\lambda}\circ Q^{\lambda}$$
$$\leq \frac{M}{m^2}\lambda(1-\lambda)\left(\frac{P^2+Q^2}{2}\circ 1 - P\circ Q\right) \leq \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \le \frac{m}{4M^2}\left(\frac{P^2+Q^2}{2}\circ 1 - P\circ Q\right) \le \frac{P+Q}{2}\circ 1 - P^{1/2}\circ Q^{1/2} \le \frac{M}{4m^2}\left(\frac{P^2+Q^2}{2}\circ 1 - P\circ Q\right) \le \frac{M}{8m^2}(M-m)^2. \tag{2.24}$$

The inequality (2.23) provides another weighted refinement and reverse inequality for (1.16) that is obtained for $\lambda = 1/2$.

Remark 2.9. *If we take $Q = P$ in Corollary 2.8, then we get*

$$0 \le \frac{m}{M^2}\lambda(1-\lambda)\left(P^2\circ 1 - P\circ P\right) \le P - P^{1-\lambda}\circ P^{\lambda} \le \frac{M}{m^2}\lambda(1-\lambda)\left(P^2\circ 1 - P\circ P\right) \le \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2 \tag{2.25}$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \le \frac{m}{4M^2}\left(P^2\circ 1 - P\circ P\right) \le P\circ 1 - P^{1/2}\circ P^{1/2} \le \frac{M}{4m^2}\left(P^2\circ 1 - P\circ P\right) \le \frac{M}{8m^2}(M-m)^2. \tag{2.26}$$

The following integral inequalities also hold:

Theorem 2.10. *Let Ω be a locally compact Hausdorff space endowed with a Radon measure μ. Let $(P_\tau)_{\tau\in\Omega}$ and $(Q_\tau)_{\tau\in\Omega}$ be continuous fields of positive operators in $B(H)$ such that $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m,M] \subset (0,\infty)$ for each $\tau \in \Omega$. Then for all $\lambda \in [0,1]$ we have*

$$\begin{aligned} 0 &\le \frac{m}{M^2}\lambda(1-\lambda) \\ &\times\left(\frac{\int_\Omega P_\tau^2 d\mu(\tau)\otimes 1 + 1\otimes\int_\Omega Q_\tau^2 d\mu(\tau)}{2} - \int_\Omega P_\tau d\mu(\tau)\otimes\int_\Omega Q_\tau d\mu(\tau)\right) \\ &\le (1-\lambda)\int_\Omega P_\tau d\mu(\tau)\otimes 1 + \lambda 1\otimes\int_\Omega Q_\tau d\mu(\tau) \\ &- \int_\Omega P_\tau^{1-\lambda} d\mu(\tau)\otimes\int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\ &\le \frac{M}{m^2}\lambda(1-\lambda) \\ &\times\left(\frac{\int_\Omega P_\tau^2 d\mu(\tau)\otimes 1 + 1\otimes\int_\Omega Q_\tau^2 d\mu(\tau)}{2} - \int_\Omega P_\tau d\mu(\tau)\otimes\int_\Omega Q_\tau d\mu(\tau)\right) \\ &\le \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2 \end{aligned} \tag{2.27}$$

for all $\lambda \in [0,1]$.

In particular,

$$
\begin{aligned}
0 &\leq \frac{m}{4M^2} \\
&\times\left(\frac{\int_\Omega P_\tau^2 d\mu(\tau)\otimes 1+1\otimes\int_\Omega Q_\tau^2 d\mu(\tau)}{2}-\int_\Omega P_\tau d\mu(\tau)\otimes\int_\Omega Q_\tau d\mu(\tau)\right) \\
&\leq \frac{\int_\Omega P_\tau d\mu(\tau)\otimes 1+1\otimes\int_\Omega Q_\tau d\mu(\tau)}{2}-\int_\Omega P_\tau^{1/2} d\mu(\tau)\otimes\int_\Omega Q_\tau^{1/2} d\mu(\tau) \\
&\leq \frac{M}{4m^2} \\
&\times\left(\frac{\int_\Omega P_\tau^2 d\mu(\tau)\otimes 1+1\otimes\int_\Omega Q_\tau^2 d\mu(\tau)}{2}-\int_\Omega P_\tau d\mu(\tau)\otimes\int_\Omega Q_\tau d\mu(\tau)\right) \\
&\leq \frac{M}{8m^2}(M-m)^2
\end{aligned}
\tag{2.28}
$$

The proof follows by Lemma 2.7 by a similar argument to the one in the proof of Theorem 2.4.

Corollary 2.11. *With the assumptions of Theorem 2.10, we have the following inequalities for the Hadamard product*

$$
\begin{aligned}
0 &\leq \frac{m}{M^2}\lambda(1-\lambda) \\
&\times\left(\int_\Omega \frac{P_\tau^2+Q_\tau^2}{2} d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega Q_\tau d\mu(\tau)\right) \\
&\leq \int_\Omega [(1-\lambda)P_\tau+\lambda Q_\tau]\,d\mu(\tau)\circ 1-\int_\Omega P_\tau^{1-\lambda} d\mu(\tau)\circ\int_\Omega Q_\tau^{\lambda} d\mu(\tau) \\
&\leq \frac{M}{m^2}\lambda(1-\lambda) \\
&\times\left(\int_\Omega \frac{P_\tau^2+Q_\tau^2}{2} d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega Q_\tau d\mu(\tau)\right) \\
&\leq \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2
\end{aligned}
\tag{2.29}
$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \leq \frac{m}{4M^2}\left(\int_\Omega \frac{P_\tau^2+Q_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega Q_\tau d\mu(\tau)\right) \tag{2.30}$$
$$\leq \int_\Omega \frac{P_\tau+Q_\tau}{2}d\mu(\tau)\circ 1-\int_\Omega P_\tau^{1/2}d\mu(\tau)\circ\int_\Omega Q_\tau^{1/2}d\mu(\tau)$$
$$\leq \frac{M}{4m^2}\left(\int_\Omega \frac{P_\tau^2+Q_\tau^2}{2}d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega Q_\tau d\mu(\tau)\right)$$
$$\leq \frac{M}{8m^2}(M-m)^2$$

Remark 2.12. *If we take $Q_\tau = P_\tau$, $\tau\in\Omega$ in Corollary 2.11, then we get*

$$0 \leq \frac{m}{M^2}\lambda(1-\lambda)\left(\int_\Omega P_\tau^2 d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega P_\tau d\mu(\tau)\right) \tag{2.31}$$
$$\leq \int_\Omega P_\tau d\mu(\tau)\circ 1-\int_\Omega P_\tau^{1-\lambda}d\mu(\tau)\circ\int_\Omega P_\tau^{\lambda}d\mu(\tau)$$
$$\leq \frac{M}{m^2}\lambda(1-\lambda)\left(\int_\Omega P_\tau^2 d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega P_\tau d\mu(\tau)\right)$$
$$\leq \frac{M}{2m^2}\lambda(1-\lambda)(M-m)^2$$

for all $\lambda\in[0,1]$.

In particular,

$$0 \leq \frac{m}{4M^2}\left(\int_\Omega P_\tau^2 d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega P_\tau d\mu(\tau)\right) \tag{2.32}$$
$$\leq \int_\Omega P_\tau d\mu(\tau)\circ 1-\int_\Omega P_\tau^{1/2}d\mu(\tau)\circ\int_\Omega P_\tau^{1/2}d\mu(\tau)$$
$$\leq \frac{M}{4m^2}\left(\int_\Omega P_\tau^2 d\mu(\tau)\circ 1-\int_\Omega P_\tau d\mu(\tau)\circ\int_\Omega P_\tau d\mu(\tau)\right)$$
$$\leq \frac{M}{8m^2}(M-m)^2.$$

3. Some related results

Further, we also have:

Lemma 3.1. *Assume that the selfadjoint operators P and Q satisfy the condition $0<P,Q\leq M$, then*

$$0 \leq (1-\lambda)P\otimes 1+\lambda 1\otimes Q-P^{1-\lambda}\otimes Q^{\lambda} \tag{3.1}$$
$$\leq M\lambda(1-\lambda)\left(\frac{P^{-1}\otimes Q+P\otimes Q^{-1}}{2}-1\right)$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \leq \frac{P \otimes 1 + 1 \otimes Q}{2} - P^{1/2} \otimes Q^{1/2} \leq \frac{1}{4} M \left(\frac{P^{-1} \otimes Q + P \otimes Q^{-1}}{2} - 1 \right). \quad (3.2)$$

Proof. Recall that if $x, y > 0$ and

$$L(x,y) := \begin{cases} \frac{y-x}{\ln y - \ln x} \text{ if } x \neq y, \\ \\ y \text{ if } x = y \end{cases}$$

is the *logarithmic mean* and $G(x,y) := \sqrt{xy}$ is the *geometric mean*, then $L(x,y) \geq G(x,y)$ for all $x, y > 0$.

Then from (1.9) we have for $x \neq y$ that

$$\begin{aligned} (1-\lambda)x + \lambda y - x^{1-\lambda} y^{\lambda} &\leq \frac{1}{2} \lambda (1-\lambda) (\ln x - \ln y)^2 \max\{x,y\} \\ &= \frac{1}{2} \lambda (1-\lambda) (y-x)^2 \left(\frac{\ln x - \ln y}{y-x} \right)^2 \max\{x,y\} \\ &\leq \frac{1}{2} \lambda (1-\lambda) \frac{(y-x)^2}{xy} \max\{x,y\} \\ &= \frac{1}{2} \lambda (1-\lambda) \left(\frac{y}{x} + \frac{x}{y} - 2 \right) \max\{x,y\}, \end{aligned}$$

which implies that

$$(1-\lambda)x + \lambda y - x^{1-\lambda} y^{\lambda} \leq \frac{1}{2} \lambda (1-\lambda) \left(\frac{y}{x} + \frac{x}{y} - 2 \right) \max\{x,y\} \quad (3.3)$$

for all $x, y > 0$.

If $t, s \in (0, M] \subset (0, \infty)$, then by (3.3) we get on taking the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, that

$$\begin{aligned} &\int_m^M \int_m^M \left[(1-\lambda)t + \lambda s - t^{1-\lambda} s^{\lambda} \right] dE(t) \otimes dF(s) \quad (3.4) \\ &\leq \frac{1}{2} M \lambda (1-\lambda) \int_m^M \int_m^M \left(\frac{s}{t} + \frac{t}{s} - 2 \right) dE(t) \otimes dF(s). \end{aligned}$$

Since

$$\begin{aligned}
&\int_m^M \int_m^M \left(\frac{s}{t} + \frac{t}{s} - 2\right) dE(t) \otimes dF(s) \\
&= \int_m^M \int_m^M t^{-1} sE(t) \otimes dF(s) + \int_m^M \int_m^M ts^{-1} dE(t) \otimes dF(s) \\
&- \int_m^M \int_m^M dE(t) \otimes dF(s) \\
&= P^{-1} \otimes Q + P \otimes Q^{-1} - 2,
\end{aligned}$$

hence by (3.4) we derive (3.1). □

Corollary 3.2. *With the assumptions of Lemma 3.1, we have the inequalities for the Hadamard product*

$$\begin{aligned}
0 &\leq [(1-\lambda) P + \lambda Q] \circ 1 - P^{1-\lambda} \circ Q^{\lambda} \\
&\leq M\lambda (1-\lambda) \left(\frac{P^{-1} \circ Q + P \circ Q^{-1}}{2} - 1 \right)
\end{aligned} \tag{3.5}$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \leq \frac{P+Q}{2} \circ 1 - P^{1/2} \circ Q^{1/2} \leq \frac{1}{4} M \left(\frac{P^{-1} \circ Q + P \circ Q^{-1}}{2} - 1 \right). \tag{3.6}$$

We observe that, if we take $Q = P$ in Corollary 3.2, then we get

$$0 \leq P \circ 1 - P^{1-\lambda} \circ P^{\lambda} \leq M\lambda (1-\lambda) \left(P^{-1} \circ P - 1 \right) \tag{3.7}$$

for all $\lambda \in [0,1]$.

In particular,

$$0 \leq P \circ 1 - P^{1/2} \circ P^{1/2} \leq \frac{1}{8} M \left(P^{-1} \circ P - 1 \right). \tag{3.8}$$

The inequality (3.7) provides a weighted refinement of *Fiedler inequality (1.17).* Moreover, we also have the integral inequalities:

Theorem 3.3. *Let* Ω *be a locally compact Hausdorff space endowed with a Radon measure* μ*. Let* $(P_\tau)_{\tau \in \Omega}$ *and* $(Q_\tau)_{\tau \in \Omega}$ *be continuous fields of positive operators in* $B(H)$ *such that* $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m, M] \subset (0, \infty)$ *for each* $\tau \in \Omega$*. Then for all*

$\lambda \in [0,1]$ *we have*

$$0 \le (1-\lambda)\int_\Omega P_\tau d\mu(\tau) \otimes 1 + \lambda 1 \otimes \int_\Omega Q_\tau d\mu(\tau) - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \le M\lambda(1-\lambda) \times \left(\frac{\int_\Omega P_\tau^{-1} d\mu(\tau) \otimes \int_\Omega Q_\tau d\mu(\tau) + \int_\Omega P_\tau d\mu(\tau) \otimes \int_\Omega Q_\tau^{-1} d\mu(\tau)}{2} - 1 \right) \tag{3.9}$$

and

$$0 \le \frac{\int_\Omega P_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega Q_\tau d\mu(\tau)}{2} - \int_\Omega P_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau) \le \frac{1}{4}M \times \left(\frac{\int_\Omega P_\tau^{-1} d\mu(\tau) \otimes \int_\Omega Q_\tau d\mu(\tau) + \int_\Omega P_\tau d\mu(\tau) \otimes \int_\Omega Q_\tau^{-1} d\mu(\tau)}{2} - 1 \right). \tag{3.10}$$

We also can state the corresponding inequalities for the Hadamard product

$$0 \le \int_\Omega [(1-\lambda)P_\tau + \lambda Q_\tau]\, d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \le M\lambda(1-\lambda) \times \left(\frac{\int_\Omega P_\tau^{-1} d\mu(\tau) \circ \int_\Omega Q_\tau d\mu(\tau) + \int_\Omega P_\tau d\mu(\tau) \circ \int_\Omega Q_\tau^{-1} d\mu(\tau)}{2} - 1 \right)$$

and

$$0 \le \int_\Omega \frac{P_\tau + Q_\tau}{2} d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \le \frac{1}{4}M \times \left(\frac{\int_\Omega P_\tau^{-1} d\mu(\tau) \circ \int_\Omega Q_\tau d\mu(\tau) + \int_\Omega P_\tau d\mu(\tau) \circ \int_\Omega Q_\tau^{-1} d\mu(\tau)}{2} - 1 \right).$$

In the case when $Q_\tau = P_\tau$, $\tau \in \Omega$, we derive

$$0 \le \int_\Omega P_\tau d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega P_\tau^{\lambda} d\mu(\tau) \le M\lambda(1-\lambda)\left(\int_\Omega P_\tau^{-1} d\mu(\tau) \circ \int_\Omega P_\tau d\mu(\tau) - 1 \right)$$

for all $\lambda \in [0,1]$ and

$$\begin{aligned} 0 &\leq \int_\Omega P_\tau d\mu(\tau) \circ 1 - \int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega P_\tau^{1/2} d\mu(\tau) \\ &\leq \frac{1}{4} M \left(\int_\Omega P_\tau^{-1} d\mu(\tau) \circ \int_\Omega P_\tau d\mu(\tau) - 1 \right). \end{aligned}$$

We also have the following multiplicative results:

Lemma 3.4. *Assume that the selfadjoint operators P and Q satisfy the condition* $0 < m \leq P, Q \leq M$, *then*

$$\begin{aligned} P^{1-\lambda} \otimes Q^\lambda &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] P^{1-\lambda} \otimes Q^\lambda \\ &\leq (1-\lambda) P \otimes 1 + \lambda 1 \otimes Q \\ &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] P^{1-\lambda} \otimes Q^\lambda \end{aligned} \tag{3.11}$$

for all $\lambda \in [0,1]$.

In particular,

$$\begin{aligned} P^{1-\lambda} \otimes Q^\lambda &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] P^{1/2} \otimes Q^{1/2} \\ &\leq \frac{P \otimes 1 + 1 \otimes Q}{2} \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] P^{1/2} \otimes Q^{1/2}. \end{aligned} \tag{3.12}$$

Proof. Since

$$\frac{(y-x)^2}{\max^2\{x,y\}} = \left(\frac{\max\{x,y\} - \min\{x,y\}}{\max\{x,y\}}\right)^2 = \left(1 - \frac{\min\{x,y\}}{\max\{x,y\}}\right)^2$$

and

$$\frac{(y-x)^2}{\min^2\{x,y\}} = \left(\frac{\max\{x,y\} - \min\{x,y\}}{\min\{x,y\}}\right)^2 = \left(\frac{\max\{x,y\}}{\min\{x,y\}} - 1\right)^2,$$

hence by (1.10) we derive

$$\begin{aligned} &\exp\left[\frac{1}{2}\lambda(1-\lambda)\left(1 - \frac{\min\{x,y\}}{\max\{x,y\}}\right)^2\right] \\ &\leq \frac{(1-\lambda)x + \lambda y}{x^{1-\lambda} y^\lambda} \\ &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{\max\{x,y\}}{\min\{x,y\}} - 1\right)^2\right]. \end{aligned} \tag{3.13}$$

If $t, s \in [m, M] \subset (0, \infty)$, then by (3.13) we get, if we take the double integral $\int_m^M \int_m^M$ over $dE(t) \otimes dF(s)$, the desired result (3.11). □

Corollary 3.5. *With the assumptions of Lemma 3.4, we have the inequalities for Hadamard product*

$$\begin{aligned} P^{1-\lambda} \circ Q^{\lambda} &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] P^{1-\lambda} \circ Q^{\lambda} \\ &\leq (1-\lambda)P + \lambda Q \\ &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] P^{1-\lambda} \circ Q^{\lambda} \end{aligned} \tag{3.14}$$

for all $\lambda \in [0, 1]$.

In particular,

$$\begin{aligned} P^{1/2} \circ Q^{1/2} &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] P^{1/2} \circ Q^{1/2} \\ &\leq \frac{P+Q}{2} \circ 1 \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] P^{1/2} \circ Q^{1/2}. \end{aligned} \tag{3.15}$$

If we take $Q = P$ in Corollary 3.5, then we get the following inequalities for one operator P satisfying the condition $0 < m \leq P \leq M$,

$$\begin{aligned} P^{1-\lambda} \circ P^{\lambda} &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] P^{1-\lambda} \circ P^{\lambda} \\ &\leq P \circ 1 \\ &\leq \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] P^{1-\lambda} \circ P^{\lambda} \end{aligned} \tag{3.16}$$

for all $\lambda \in [0, 1]$.

In particular,

$$\begin{aligned} P^{1/2} \circ P^{1/2} &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] P^{1/2} \circ P^{1/2} \\ &\leq P \circ 1 \\ &\leq \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] P^{1/2} \circ P^{1/2}. \end{aligned} \tag{3.17}$$

Finally, we can state the following multiplicative result:

Theorem 3.6. *Let $(P_\tau)_{\tau \in \Omega}$ and $(Q_\tau)_{\tau \in \Omega}$ be continuous fields of positive operators in $B(H)$ such that $\sigma(P_\tau), \sigma(Q_\tau) \subseteq [m, M] \subset (0, \infty)$ for each $\tau \in \Omega$. Then for all*

$\lambda \in [0,1]$ *we have*

$$\begin{aligned}
&\int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \qquad (3.18)\\
&\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau)\\
&\le (1-\lambda)\int_\Omega P_\tau d\mu(\tau) \otimes 1 + \lambda 1 \otimes \int_\Omega Q_\tau d\mu(\tau)\\
&\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \otimes \int_\Omega Q_\tau^{\lambda} d\mu(\tau)
\end{aligned}$$

and, in particular

$$\begin{aligned}
&\int_\Omega P_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau) \qquad (3.19)\\
&\le \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] \int_\Omega P_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau)\\
&\le \frac{1}{2}\left[\int_\Omega P_\tau d\mu(\tau) \otimes 1 + 1 \otimes \int_\Omega Q_\tau d\mu(\tau)\right]\\
&\le \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] \int_\Omega P_\tau^{1/2} d\mu(\tau) \otimes \int_\Omega Q_\tau^{1/2} d\mu(\tau).
\end{aligned}$$

We can state the following results for the Hadamard product as well:

$$\begin{aligned}
&\int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \qquad (3.20)\\
&\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau)\\
&\le \int_\Omega \left[(1-\lambda)P_\tau + \lambda Q_\tau\right] d\mu(\tau) \circ 1\\
&\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau)
\end{aligned}$$

and, in particular

$$\begin{aligned}
&\int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega Q_\tau^{1/2} d\mu(\tau) \qquad (3.21)\\
&\le \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] \int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega Q_\tau^{1/2} d\mu(\tau)\\
&\le \int_\Omega \frac{P_\tau + Q_\tau}{2} d\mu(\tau) \circ 1\\
&\le \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] \int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega Q_\tau^{1/2} d\mu(\tau).
\end{aligned}$$

When $Q_\tau = P_\tau$, $\tau \in \Omega$, we obtain for $\lambda \in [0,1]$ that

$$\begin{aligned}
&\int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau) \qquad (3.22)\\
&\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{M}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau)\\
&\le \int_\Omega \left[(1-\lambda) P_\tau + \lambda Q_\tau\right] d\mu(\tau) \circ 1\\
&\le \exp\left[\frac{1}{2}\lambda(1-\lambda)\left(\frac{M-m}{m}\right)^2\right] \int_\Omega P_\tau^{1-\lambda} d\mu(\tau) \circ \int_\Omega Q_\tau^{\lambda} d\mu(\tau)
\end{aligned}$$

and, in particular,

$$\begin{aligned}
&\int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega P_\tau^{1/2} d\mu(\tau) \qquad (3.23)\\
&\le \exp\left[\frac{1}{8}\left(\frac{M-m}{M}\right)^2\right] \int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega P_\tau^{1/2} d\mu(\tau)\\
&\le \int_\Omega P_\tau d\mu(\tau) \circ 1\\
&\le \exp\left[\frac{1}{8}\left(\frac{M-m}{m}\right)^2\right] \int_\Omega P_\tau^{1/2} d\mu(\tau) \circ \int_\Omega P_\tau^{1/2} d\mu(\tau).
\end{aligned}$$

4. Acknowledgement.

The second author would like to extend his appreciation to the Distinguished Scientist Fellowship Program at King Saud University, Riyadh, Saudi Arabia, for funding this work through Researchers Supporting Project number RSPD2025R1056.

References

[1] ALZER, H.; DA FONSECA, C. M.; KOVAČEC, A. Young-type inequalities and their matrix analogues. *Linear and Multilinear Algebra* **63** (2015), Issue 3, 622-635. MR3274000, Zbl 1316.15023, doi: 10.1080/03081087.2014.891588 . 794

[2] ANDO, T. Concavity of certain maps on positive definite matrices and applications to Hadamard products. *Lin. Alg. & Appl.* **26** (1979), 203-241. MR0535686, Zbl 0495.15018, doi: 10.1016/0024-3795(79)90179-4 . 795

[3] ARAKI, H.; HANSEN, F. Jensen's operator inequality for functions of several variables. *Proc. Amer. Math. Soc.* **128** (2000), No. 7, 2075-2084. MR1670414, Zbl 0956.47009, doi: 10.1090/s0002-9939-00-05371-5 . 794

[4] AUJILA, J. S.; VASUDEVA, H. L. Inequalities involving Hadamard product and operator means. *Math. Japon.* **42** (1995), 265-272. MR1356386, Zbl 0859.15007. 795

[5] DRAGOMIR, S. S. A note on Young's inequality. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas* **111** (2017), no. 2, 349-354. MR3623844, Zbl 1360.26020, doi: 10.1007/s13398-016-0300-8 . 793

[6] DRAGOMIR, S. S. A note on new refinements and reverses of Young's inequality. *Transyl. J. Math. Mec.* **8** (2016), No.1, 45-49. MR3531966. 794

[7] DRAGOMIR, S. S. Tensorial and Hadamard product inequalities for functions of selfadjoint operators in Hilbert spaces in terms of Kantorovich ratio. *Extr. Math.* **38** (2023) No. 2, 237-250. MR4699075, Zbl 1546.47025, doi: 10.17398/2605-5686.38.2.237 . 796

[8] DRAGOMIR, S. S. Tensorial and Hadamard product integral inequalities for convex functions of continuous fields of operators in Hilbert spaces. *Hacettepe Journal of Mathematics and Statistics* **54** (2025), No.1, 115-124. MR4875157, doi: 10.15672/hujms.1362698 . 796

[9] FURUICHI, S. On refined Young inequalities and reverse inequalities. *J. Math. Inequal.* **5** (2011), 21-31. MR2799055, Zbl 1213.15017 , doi: 10.7153/jmi-05-03 .

[10] FURUICHI, S. Refined Young inequalities with Specht's ratio. *Journal of the Egyptian Mathematical Society* **20**(2012) , 46-49. MR2975333, Zbl 1287.47031, doi: 10.1016/j.joems.2011.12.010 . 793

[11] FUJII, J. I. The Marcus-Khan theorem for Hilbert space operators. *Math. Jpn.* **41** (1995), 531-535. MR1339012, Zbl 0826.47016. 795

[12] FURUTA, T.; MIĆIĆ HOT, J.; PEČARIĆ, J.; SEO, Y.) MOND-PEČARIĆ Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space. *Element, Zagreb*, 2005. MR3026316 Zbl 1135.47012. 795

[13] FURUICHI, S.; MINCULETE, N. Alternative reverse inequalities for Young's inequality. *J. Math Inequal.* **5** (2011), No. 4, 595–600. MR2908115, Zbl 1236.15046, doi: 10.7153/jmi-05-51 . 794

[14] KITAMURA, K.; SEO, Y. Operator inequalities on Hadamard product associated with Kadison's Schwarz inequalities. *Scient. Math.* **1** (1998), No. 2, 237-241. MR1686236, Zbl 0947.47015. 795

[15] KITTANEH, F.; MANASRAH, Y. Improved Young and Heinz inequalities for matrices. *J. Math. Anal. Appl.* **361** (2010), 262-269. MR2567300, Zbl 1180.15021, doi: 10.1016/j.jmaa.2009.08.059 . 793

[16] KITTANEH, F.; MANASRAH, Y. Reverse Young and Heinz inequalities for matrices, *Linear Multilinear Algebra* **59** (2011), 1031-1037. MR2826070, Zbl 1225.15022, doi: 10.1080/03081087.2010.551661 . 793

[17] KORÁNYI, A. On some classes of analytic functions of several variables. *Trans. Amer. Math. Soc.* **101** (1961), 520–554. MR0136765, Zbl 0111.11501, doi: 10.1090/s0002-9947-1961-0136765-6 . 794

[18] LIAO, W.; WU, J.; ZHAO, J. New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant. *Taiwanese J. Math.* **19** (2015), No. 2, pp. 467-479. MR3332308, Zbl 1357.26048, doi: 10.11650/tjm.19.2015.4548 . 793

[19] SPECHT, W. Zer Theorie der elementaren Mittel. *Math. Z.* **74** (1960), pp. 91-98. 793 Zbl 0095.03801, doi: 10.1007/bf01180475 .

[20] TOMINAGA, M. Specht's ratio in the Young inequality. *Sci. Math. Japon.* **55** (2002), 583-588. MR1901045, Zbl 1021.47010, doi: 10.7153/mia-07-13 . 793

[21] STOJILJKOVIC, V.; DRAGOMIR, S. S. Differentiable Ostrowski type tensorial norm inequality for continuous functions of selfadjoint operators in Hilbert spaces. *Gulf J. Math.* **15** (2023) No. 2, 40-55 . MR4680182, Zbl 7781237, doi: 10.56947/gjom.v15i2.1247 . 796

[22] ZUO, G.; SHI, G.; FUJII, M. Refined Young inequality with Kantorovich constant. *J. Math. Inequal.* **5** (2011), 551-556. MR2908111, Zbl 1242.47017, doi: 10.7153/jmi-05-47 . 793

(S. S. Dragomir) APPLIED MATHEMATICS RESEARCH GROUP, ISILC, VICTORIA UNIVERSITY, MELBOURNE , VIC, AUSTRALIA, AND DEPARTMENT OF MATHEMATICAL AND PHYSICAL SCIENCES, LA TROBE UNIVERSITY, MELBOURNE, VIC, AUSTRALIA.
sever.dragomir@vu.edu.au

(M. T. Garayev) DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE, KING SAUD UNIVERSITY, P.O.BOX 2455, RIYADH 11451, SAUDI ARABIA.
mgarayev@ksu.edu.sa

This paper is available via http://nyjm.albany.edu/j/2025/31-29.html.