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Jørgensen's inequality for bicomplex Möbius groups

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ABSTRACT. In this paper, we investigate Möbius groups theory in the framework of bicomplex numbers, which are pairs of complex numbers making up a commutative ring with zero-divisors. We shall generalize classical Jørgensen's inequality to bicomplex analysis and obtain an analogue of Jørgensen type inequality for discrete non-elementary Möbius groups of bicomplex setting generated by two elements.

CONTENTS

1. Introduction

Throughout this paper, we will adopt the same notations and definitions as in [7, 8, 22, 28] such as PU(2, 1), PSL(2, ℂ), elementary groups, discrete groups, bicomplex Möbius transformations and so on. For example, the collection of all complex Möbius transformations for which $ad - bc$ takes the value 1 forms a group which can be identified with PSL(2, ℂ).

In 1976, Jørgensen[28] obtained the following Jørgensen' inequality, which is important in the study of the geometry of discrete groups.

Theorem 1.1. *Suppose that $f, g \in \mathrm{SL}(2,\mathbb{C})$ generate a group. If*

$$|\mathrm{tr}^2(f) - 4| + |\mathrm{tr}(fgf^{-1}g^{-1}) - 2| < 1. \tag{1}$$

Then the group $\langle f, g\rangle$ is either elementary or not discrete.

It is well known that to study Jørgensen' inequality of Möbius transformation is very important and there has been an active research in this area. Gehring and Martin[24] discussed quantified forms of Jørgensen' inequality for the norms

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$m(f), d(f), \varrho(f)$ and obtained the well known fact that axis(f) and axis(g) are disjoint whenever $\langle f, g\rangle$ is nonelementary. Alaqad, Gong and Martin [2] used certain shifted Chebyshev polynomials and trace identities to determine new families of Jørgensen type inequalities. In 1989, Jørgensen' inequality was generalized by Martin [39] to nonelementary discrete groups of Möbius transformations of any dimension $n > 2$. Waterman[44] proved a generalization of Jørgensen' inequality to Möbius transformations in high dimension by using Clifford algebras. In 1998, Basmajian[7] generalized Jørgensen' s inequality to the two generator subgroup of PU(2, 1) when the generators are loxodromic or boundary elliptic. Friedland and Hersonsky [23] extended Jørgensen' inequality to certain discrete nonelementary subgroups $\langle A, B\rangle$ of the symplectic group. In [14, 17, 29, 30, 40], Jørgensen' inequality was bulit in complex hyperbolic space. See [1, 16, 31, 42, 43, 44, 45, 46] etc. for more discussions about Jørgensen' inequality.

In recent years, more and more people try to generalize the classical results of complex analysis to bicomplex analysis. Luna-Elizarrará obtained some classical bicomplex analysis results, for example: bicomplex Riemann mapping theorem [36, Theorem 8.6.2], bicomplex Weierstrass' theorem [36, Theorem 10.2.3], bicomplex Abel's theorem [36, Theorem 10.4.1], bicomplex Cauchy integral theorem [36, Theorem 11.1.1], bicomplex Borel-Pompeiu formula [36, Theorem 11.2.2], bicomplex Cauchy integral representation [36, Theorem 11.2.3], bicomplex residue theorem [37] and bicomplex Laurent theorem [38]. Agarwal discussed bicomplex Maxwell's equations [3] and bicomplex Mittag-Leffler function [4]. Banerjee studied bicomplex Fourier transform [9], bicomplex Laplace transform [10], bicomplex indefinite inner product modules [11] and bicomplex harmonic and isotonic oscillators [12]. In addition, Emanuello [21] studied the Möbius transformation in $\mathbb{R}^{1,1}$ and discussed its conformality, transitivity, and fixed points. Golberg [25] defined the $\mathbb{D}-$Möbius transformation in $\mathbb{D}^n$ and came to the conclusion that $\mathbb{D}-$Möbius transformation can be expressed as a product of the six elementary Möbius transformations. In 2022, Dai[15] et al. studied a conjugacy classification according to the number of fixed points in SL(2, $\mathbb{BC}$), and detailedly prove that the cross-ratio is invariant under hyperbolic Möbius transformations. See [5, 6, 13, 33, 34] etc. for more details of bicomplex analysis.

In this paper, motivated by these developments, we shall give an analogue of Jørgensen type inequality for discrete non-elementary Möbius groups of bicomplex setting generated by two elements. Our main result is the following theorem.

Theorem 1.2. *Let $f, g \in \mathrm{Aut}(\overline{\mathbb{BC}})$ generate a group. If*

$$|\mathrm{tr}^2(f) - 4|_{\mathbf{k}} + |\mathrm{tr}(fgf^{-1}g^{-1}) - 2|_{\mathbf{k}} \prec 1, \qquad (2)$$

then the group $\langle f, g\rangle$ is either elementary or not discrete.

The structure of the paper is as follows. Section 2 preliminaries present the basic and necessary concepts and results about the bicomplex numbers. In Section 3, we give the necessary background material for bicomplex Möbius transformations: Section 3.1 describes the fixed point of bicomplex Möbius transformations; Section 3.2 give necessary condition for a bicomplex Möbius group to be non-elementary. In Section 3.3, we present the geometric representations of bicomplex Möbius transformations. In Section 4 we obtain the proof of Theorem 1.2.

2. Preliminaries

2.1. Bicomplex and hyperbolic numbers. The commutative ring $\mathbb{BC}$ of bicomplex numbers is defined as

$$\mathbb{BC} := \{Z = z_1 + \mathbf{j} z_2 \mid z_1, z_2 \in \mathbb{C}(\mathbf{i})\}.$$

Here $\mathbb{C}(\mathbf{i})$ and $\mathbb{C}(\mathbf{j})$ are different commutative imaginary units, that is

$$\mathbf{i} \neq \mathbf{j}, \quad \mathbf{ij} = \mathbf{ji} = \mathbf{k}, \quad \mathbf{i}^2 = \mathbf{j}^2 = -1,$$

and $\mathbb{C}(\mathbf{i})$ is the set of complex numbers with imaginary unit $\mathbf{i}$. If $z_1 = x_1$ is a real number and $z_2 = y_2\mathbf{i}$ is a purely imaginary complex number, then $Z = x_1 + y_2\mathbf{k}$ is an element of the ring of hyperbolic numbers $\mathbb{D}$.

$$\mathbb{D} := \{x + y\mathbf{k} \mid x, y \in \mathbb{R}\}$$

Both the ring $\mathbb{BC}$ and the ring $\mathbb{D}$ have zero-divisors. In the case of bicomplex numbers, the set of zero-divisors is

$$\mathfrak{S} := \{Z | Z \neq 0, z_1^2 + z_2^2 = 0\}$$

and by definition

$$\mathfrak{S}_0 := \mathfrak{S} \cup \{0\}.$$

There are two special zero-divisors $\mathbf{e} := \frac{1+\mathbf{k}}{2}$ and $\mathbf{e}^\dagger := \frac{1-\mathbf{k}}{2}$, which have the properties:

$$\mathbf{e} + \mathbf{e}^\dagger = 1; \quad \mathbf{e} - \mathbf{e}^\dagger = \mathbf{k};$$
$$\mathbf{e}\mathbf{e}^\dagger = 0; \quad \mathbf{e}\mathbf{e} = \mathbf{e}; \text{ and } \mathbf{e}^\dagger\mathbf{e}^\dagger = \mathbf{e}^\dagger. \tag{3}$$

Consider the sets:

$$\mathbb{BC}_{\mathbf{e}} = \{\beta_1\mathbf{e} | \beta_1 \in \mathbb{C}(\mathbf{i})\} \text{ and } \mathbb{BC}_{\mathbf{e}^\dagger} = \left\{\beta_2\mathbf{e}^\dagger | \beta_2 \in \mathbb{C}(\mathbf{i})\right\}.$$

Obviously, the set of zero-divisors in $\mathbb{BC}$ is given by

$$\mathfrak{S}_0 = \mathbb{BC}_{\mathbf{e}} \cup \mathbb{BC}_{\mathbf{e}^\dagger}.$$

Each bicomplex number $Z = z_1 + \mathbf{j}z_2$ can be written as

$$Z = (z_1 - z_2\mathbf{i})\,\mathbf{e} + (z_1 + z_2\mathbf{i})\,\mathbf{e}^\dagger = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \text{ where } \beta_1, \beta_2 \in \mathbb{C}(\mathbf{i}).$$

The following conjugations were introduced on bicomplex numbers [4, 6, 33, 34].

(i) $\overline{Z} := \overline{z_1} + \overline{z_2}\mathbf{j}$ (the bar-conjugation);

(ii) $Z^{\dagger} := z_1 - z_2\mathbf{j}$ (the †-conjugation);
(iii) $Z^{*} := (\overline{Z})^{\dagger} = \overline{z_1} - \overline{z_2}\mathbf{j}$ (the ∗-conjugation).
Besides, the above conjugations suggest to consider the three moduli for bicomplex numbers:
(i)$|Z|^2_{\mathbf{i}} := Z \cdot Z^{\dagger} = z_1^2 + z_2^2 \in \mathbb{C}(\mathbf{i})$;
(ii)$|Z|^2_{\mathbf{j}} := Z \cdot \bar{Z} = \eta_1^2 + \eta_2^2 \in \mathbb{C}(\mathbf{j})$, where $Z = \eta_1 + \eta_2\mathbf{i} = (x_1 + x_2\mathbf{j}) + (y_1 + y_2\mathbf{j})\,\mathbf{i}$;
(iii)$|Z|^2_{\mathbf{k}} := Z \cdot Z^{*} = |\beta_1|^2\,\mathbf{e} + |\beta_2|^2\,\mathbf{e}^{\dagger} \in \mathbb{D}^{+}$, where the set $\mathbb{D}^{+}$ can be described as $\mathbb{D}^{+} = \{v\mathbf{e} + \mu\mathbf{e}^{\dagger} \mid v, \mu \geq 0\}$.
In [6, 36, 38], the $\mathbf{k}$-modulus $|\cdot|_{\mathbf{k}}$ is defined by

$$|Z|_{\mathbf{k}} = |\beta_1|\,\mathbf{e} + |\beta_2|\,\mathbf{e}^{\dagger}.$$

The following partial order on hyperbolic numbers, introduced by Luna-Elizarrarás in[35, 36, 37].

Given hyperbolic numbers $\mathfrak{z}$ and $\mathfrak{w}$, we say that $\mathfrak{z} \preceq \mathfrak{w}$ if $\mathfrak{w} - \mathfrak{z} \in \mathbb{D}^{+}$. When $\mathfrak{w} - \mathfrak{z} \in \mathbb{D}^{+} \backslash \{0\}$, we write $\mathfrak{z} \prec \mathfrak{w}$.

The properties of $\mathbf{k}$-modulus are as follows:
(i) $|Z|_{\mathbf{k}} = 0$ if and only if $Z = 0$.
(ii) it satisfies the multiplicative property:

$$|ZW|_{\mathbf{k}} = |Z|_{\mathbf{k}} \cdot |W|_{\mathbf{k}}$$

(iii) the $\mathbf{k}$-modulus satisfies the triangle inequality:

$$|Z + W|_{\mathbf{k}} \preceq |Z|_{\mathbf{k}} + |W|_{\mathbf{k}}.$$

2.2. k-modulus and convergence.

We consider a convergence on bicomplex numbers in the following form.

Definition 2.1. [38, Definition 2.1] *Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of bicomplex numbers, we say that Z_n converges in the hyperbolic modulus to $Z_0 \in \mathbb{BC}$ if for any $\varepsilon \succ 0$ there exists $N \in \mathbb{N}$ such that*

$$\left|Z_n - Z_0\right|_{\mathbf{k}} \prec \varepsilon$$

for any $n \geq N$.

In this case we say that Z_0 is the limit of the sequence which we write as

$$\lim_{n\to\infty} Z_n = Z_0.$$

Definition 2.2. [38, Proposition 2.2] *Let $\{Z_n\}_{n=1}^{\infty}$ be a sequence of bicomplex numbers, then Z_n converges to Z_0 if and only if the complex sequence $\{\beta_{1,n}\}_{n=1}^{\infty}$ converges to $\beta_{1,0}$ and the complex sequence $\{\beta_{2,n}\}_{n=1}^{\infty}$ converges to $\beta_{2,0}$ where $Z_n = \beta_{1,n}\mathbf{e} + \beta_{2,n}\mathbf{e}^{\dagger}$ and $Z_0 = \beta_{1,0}\mathbf{e} + \beta_{2,0}\mathbf{e}^{\dagger}$. This type of convergence can be introduced for functions.*

Remark 2.3. *If the complex sequence $\{\beta_{1,n}\}_{n=1}^{\infty}$ converges to ∞ and the complex sequence $\{\beta_{2,n}\}_{n=1}^{\infty}$ converges to $\beta_{2,0}$, then $\lim_{n\to\infty} Z_n = \infty\mathbf{e} + \beta_{2,0}\mathbf{e}^{\dagger}$. If the complex* *sequence* $\{\beta_{1,n}\}_{n=1}^{\infty}$ *converges to* $\beta_{1,0}$ *and the complex sequence* $\{\beta_{2,n}\}_{n=1}^{\infty}$ *converges to* ∞*, then* $\lim_{n\to\infty} Z_n = \beta_{1,0}\mathbf{e} + \infty\mathbf{e}^{\dagger}$*. If the complex sequence* $\{\beta_{1,n}\}_{n=1}^{\infty}$ *converges to* ∞ *and the complex sequence* $\{\beta_{2,n}\}_{n=1}^{\infty}$ *converges to* ∞*, then* $\lim_{n\to\infty} Z_n = \infty\mathbf{e} + \infty\mathbf{e}^{\dagger}$

2.3. Sets of extended bicomplex numbers. The extended set of bicomplex numbers was introduced in [38]. The extended set of bicomplex numbers is

$$\overline{\mathbb{BC}} := \overline{\mathbb{C}(\mathbf{i})}\mathbf{e} + \overline{\mathbb{C}(\mathbf{i})}\mathbf{e}^{\dagger}$$

where $\overline{\mathbb{C}(\mathbf{i})}$ is the well-known compactification of the $\mathbb{C}(\mathbf{i})$-complex plane.

Remark 2.4. *Similar definition as* $\overline{\mathbb{C}(\mathbf{i})} = \mathbb{C}(\mathbf{i}) \cup \{\infty\}$*, we have*

$$\overline{\mathbb{BC}} = \mathbb{BC} \cup \{\mathbb{C}(\mathbf{i})\mathbf{e} + \infty\mathbf{e}^{\dagger}\} \cup \{\infty\mathbf{e} + \mathbb{C}(\mathbf{i})\mathbf{e}^{\dagger}\} \cup \{\infty\mathbf{e} + \infty\mathbf{e}^{\dagger}\}.$$

3. Bicomplex Möbius transformation

It is well known that the Möbius transformation plays an important role in the study of complex analysis and there are many further discussions. Emanuello [21] studied the Möbius transformation in $\mathbb{R}^{1,1}$ and discussed its conformality, transitivity, and fixed points. Golberg [25] defined the $\mathbb{D}$−Möbius transformation in $\mathbb{D}^n$ and came to the conclusion that $\mathbb{D}$−Möbius transformation can be expressed as a product of the six elementary Möbius transformations.

The bicomplex Möbius transformation [15, 18, 22] $f(Z)$ is denoted by :

$$f(Z) = \frac{AZ + B}{CZ + D} \tag{4}$$

where $A, B, C, D \in \mathbb{BC}$ and $AD - BC \notin \mathfrak{S}_0$. Let $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^{\dagger}$ and

$$A = a_1\mathbf{e} + a_2\mathbf{e}^{\dagger} \quad B = b_1\mathbf{e} + b_2\mathbf{e}^{\dagger}$$
$$C = c_1\mathbf{e} + c_2\mathbf{e}^{\dagger} \quad D = d_1\mathbf{e} + d_2\mathbf{e}^{\dagger}.$$

We have

$$f(Z) = \frac{a_1\beta_1 + b_1}{c_1\beta_1 + d_1}\mathbf{e} + \frac{a_2\beta_2 + b_2}{c_2\beta_2 + d_2}\mathbf{e}^{\dagger} = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^{\dagger}, \tag{5}$$

where complex Möbius transformations f_1 and f_2 are called components of f. In [15, 22] , $\mathrm{GL}(2, \mathbb{BC})$ was defined by

$$\mathrm{GL}(2, \mathbb{BC}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathbb{BC}, AD - BC \notin \mathfrak{S}_0 \right\}.$$

The collection of all bicomplex Möbius transformation for which $AD - BC$ takes the value 1 forms a group which can be identified with $\mathrm{PSL}(2, \mathbb{BC})$.

We can represent $f(Z)$ via a matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2, \mathbb{BC}). \tag{6}$$

Definition 3.1. [22, Corollary 6.2.6] *The group of bicomplex Möbius transformations acting in* $\overline{\mathbb{BC}}$ *is called the General Möbius group and is denoted by* $\mathrm{Aut}(\overline{\mathbb{BC}})$.

Remark 3.2. [22, Corollary 6.2.6] *The group of* $\mathrm{Aut}(\overline{\mathbb{BC}})$ *is the collection of bicomplex Möbius transformations*

$$f(Z) = \frac{AZ+B}{CZ+D}.$$

Lemma 3.3. [32, Theorem 3.1, Theorem 3.3] *Let* $\mathcal{A} = \mathcal{A}_1\mathbf{e} + \mathcal{A}_2\mathbf{e}^\dagger$ *be any bicomplex matrix in* $\mathrm{GL}(2,\mathbb{BC})$. *Then*

$$\det\mathcal{A} = \det\mathcal{A}_1\mathbf{e} + \det\mathcal{A}_2\mathbf{e}^\dagger \quad \det\mathcal{A}^{-1} = \det\mathcal{A}_1^{-1}\mathbf{e} + \det\mathcal{A}_2^{-1}\mathbf{e}^\dagger \tag{7}$$

$\det\mathcal{A} = AD - BC$ *is called the determinant of* $\mathcal{A}$.

By Lemma 3.3, for any $\mathcal{A} \in \mathrm{PSL}(2,\mathbb{BC})$, we have

$$\det\mathcal{A}_1 = \det\mathcal{A}_2 = 1.$$

In order to introduce the following important properties, we need to generalize the trace of the matrix to $\mathrm{GL}(2,\mathbb{BC})$.

Definition 3.4. [15, 21] *Let* $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}(2,\mathbb{BC})$*, the trace of* $\mathcal{A}$ *is defined by*

$$\mathrm{tr}\mathcal{A} = A + D. \tag{8}$$

Naturally, $\mathrm{tr}^2(f)$ is defined by

$$\mathrm{tr}^2(f) = \frac{\mathrm{tr}^2(\mathcal{A})}{\det(\mathcal{A})}, \quad \mathcal{A} \in \mathrm{GL}(2,\mathbb{BC}), \tag{9}$$

where $\mathcal{A}$ is any matrix which projects to f. According to (3) and (5), for any $\mathcal{A} \in \mathrm{PSL}(2,\mathbb{BC})$, we have

$$\begin{aligned} \mathrm{tr}^2(f) &= \left[(a_1+d_1)\mathbf{e} + (a_2+d_2)\mathbf{e}^\dagger\right]^2 \\ &= (a_1+d_1)^2\mathbf{e} + (a_2+d_2)^2\mathbf{e}^\dagger \\ &= \mathrm{tr}^2(f_1)\mathbf{e} + \mathrm{tr}^2(f_2)\mathbf{e}^\dagger. \end{aligned}$$

The transitivity of the bicomplex Möbius transformations of $\mathrm{Aut}(\overline{\mathbb{BC}})$ are only one transitive. It is not two transitive in general.

Lemma 3.5. [21, 22] *The bicomplex Möbius transformations are transitive, but not 2-transitive.*

To see that bicomplex Möbius group is not two-transitive, we give a proof by counterexample([15, Chapter 4]). We choose four different points

$$Z_1 = \mathbf{ie}, Z_2 = 2\mathbf{ie}, Z_3 = \mathbf{ie}^\dagger \text{ and } Z_4 = 3\mathbf{i}$$

in $\mathbb{BC}$ to form two sets $\{Z_1, Z_2\}$ and $\{Z_3, Z_4\}$. We choose f that satisfies

$$f(Z_1) = Z_3 \quad f(Z_2) = Z_4,$$

the component f_2 of f has to satisfy both $f_2(0) = \mathbf{i}$ and $f_2(0) = \mathbf{j}$. It is a contradiction.

Based on [15, Definition 2] and [22, Proposition 6.2.5], we have the following definition.

Definition 3.6. *Suppose that f and g are two different bicomplex Möbius transformations in $\mathrm{Aut}(\overline{\mathbb{BC}})$, f and g are conjugate, denoted as*

$$f \sim g, \tag{10}$$

if and only if there exists bicomplex Möbius transformations h in $\mathrm{Aut}(\overline{\mathbb{BC}})$ such that

$$f = hgh^{-1}.$$

Remark 3.7. *A bicomplex Möbius transformation $f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger$ is conjugate to a bicomplex Möbius transformation $g(Z) = g_1(\beta_1)\mathbf{e}+g_2(\beta_2)\mathbf{e}^\dagger$ if and only if there exists bicomplex Möbius transformations $h(Z) = h_1(\beta_1)\mathbf{e} + h_2(\beta_2)\mathbf{e}^\dagger$ such that $g(Z) = hfh^{-1}(Z)$. For any bicomplex Möbius transformation $h(Z) = h_1(\beta_1)\mathbf{e} + h_2(\beta_2)\mathbf{e}^\dagger$, by (5), we have*

$$hfh^{-1}(Z) = h_1f_1h_1^{-1}(\beta_1)\mathbf{e} + h_2f_2h_2^{-1}(\beta_2)\mathbf{e}^\dagger. \tag{11}$$

The following lemmas, though rather trivial, are useful.

Lemma 3.8. [15, Theorem 2] *Let $f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger$ and $g(Z) = g_1(\beta_1)\mathbf{e} + g_2(\beta_2)\mathbf{e}^\dagger$ in $\mathrm{Aut}(\overline{\mathbb{BC}})$, $f \sim g$ if and only if both $f_1 \sim g_1$ and $f_2 \sim g_2$.*

Lemma 3.9. [15, Theorem 3] *Suppose that f and g are two different bicomplex Möbius transformations in $\mathrm{Aut}(\overline{\mathbb{BC}})$ where neither component is the identity. Then $f \sim g$ if and only if $\mathrm{tr}^2(f) = \mathrm{tr}^2(g)$.*

Remark 3.10. *It is not hard to conclude that the trace of bicomplex Möbius transformations is invariant under conjugate transformations.*

3.1. Fixed Points.

Definition 3.11. [18, Definition 14] *Let f be any bicomplex Möbius transformations in $\mathrm{Aut}(\overline{\mathbb{BC}})$, then $Z \in \overline{\mathbb{BC}}$ is called a fixed point of f if and only if $f(Z) = Z$.*

Example 3.12. *(i) the mapping $Z + 1$ has one fixed point $\infty\mathbf{e} + \infty\mathbf{e}^\dagger$.*
(ii)the mapping $\frac{1}{\beta_1}\mathbf{e} + (\beta_2 + 1)\mathbf{e}^\dagger$ has two fixed points, $\mathbf{e} + \infty\mathbf{e}^\dagger$ and $-\mathbf{e} + \infty\mathbf{e}^\dagger$.
(iii)the mapping $1/Z$ has four fixed points,$\pm 1, \pm \boldsymbol{k}$.

Remark 3.13. *If a point $Z = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger \in \overline{\mathbb{BC}}$ satisfies $f(Z) = Z, Z$ is called a fixed point of f. Further,*

$$f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger = \beta_1\mathbf{e} + \beta_2\mathbf{e}^\dagger.$$

So Z is a fixed point of f if and only if

$$f_1(\beta_1) = \beta_1, f_2(\beta_2) = \beta_2.$$

Pogorui [41] studied the roots of polynomials with bicomplex coefficients, by Pogorui's bicomplex polynomial theory, Chinmay [18] obtained the fixed points of bicomplex Möbius transformations.

Lemma 3.14. [18, Theorem 4] *Let f be a bicomplex Möbius transformation where neither component is the identity. Then f has either one fixed point, two fixed points or four fixed points.*

The following lemma is crucial for us.

Lemma 3.15. [22, Corollary, 6.3.2] *Let $f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger$ be a bicomplex Möbius transformation where neither component is identity. Then*

(i) *$f(Z)$ has precisely one fixed point if and only if $f(Z)$ is conjugate to* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(ii) *$f(Z)$ has precisely two fixed points if and only if $f(Z)$ is conjugate to*

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{e} + \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\mathbf{e}^\dagger \quad \text{or} \quad \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\mathbf{e} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{e}^\dagger,$$

where $\lambda \neq 1$.

(iii) *$f(Z)$ has precisely four fixed points if and only if $f(Z)$ is conjugate to*

$$\begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}\mathbf{e} + \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix}\mathbf{e}^\dagger,$$

where $u \neq 1, \lambda \neq 1$.

3.2. The elementary group.

Definition 3.16. [25, Definition,2] *The hyperbolic-valued function*

$$\|\cdot\|_{\mathbb{D}} : \mathrm{GL}(2, \mathbb{BC}) \to \mathbb{D}^+,$$

defined as follows:

$$\|\mathcal{A}\|_{\mathbb{D}} := \|\mathcal{A}_1\|\mathbf{e} + \|\mathcal{A}_2\|\mathbf{e}^\dagger, \tag{12}$$

is called a hyperbolic-norm in $\mathrm{GL}(2, \mathbb{BC})$.

The norm $\|\mathcal{A}_i\|(i = 1, 2)$ is given by

$$\|\mathcal{A}_i\| = (|a_i|^2 + |b_i|^2 + |c_i|^2 + |d_i|^2)^{1/2}(i = 1, 2).$$

A subgroup G of $\mathrm{GL}(2, \mathbb{C})$ is discrete if one point of G is isolated:

$$\inf\{\|X - I\| : X \in \mathrm{G}, X \neq I\} > 0.$$

Similar definition as [8, Chapter 2.3] , a subgroup G of $\mathrm{GL}(2, \mathbb{BC})$ is discrete if one point of G is isolated:

Definition 3.17. *A subgroup G of* $\mathrm{GL}(2, \mathbb{BC})$ *is discrete if one point of G is isolated.*

$$\inf\left\{\|X - I\|_{\mathbb{D}} : X \in G, X \neq I\right\} > 0, \tag{13}$$

In terms of sequences, G is discrete if and only if $\mathcal{A}_n \to I$ and $\mathcal{A}_n \in$ G implies that $\mathcal{A}_n = I$ for almost all n.
Let $X = X_1\mathbf{e} + X_2\mathbf{e}^{\dagger}$, by (12), we have

$$\|X - I\|_{\mathbb{D}} = \|X_1 - I\|\,\mathbf{e} + \|X_2 - I\|\,\mathbf{e}^{\dagger}. \tag{14}$$

So

$$\begin{aligned} &\inf\left\{\|X - I\|_{\mathbb{D}} : X \in \mathrm{G}, X \neq I\right\} \\ =&\inf\{\|X_1 - I\| : X_1 \in \mathrm{GL}(2,\mathbb{C}), X_1 \neq I\}\,\mathbf{e} \\ +&\inf\{\|X_2 - I\| : X_2 \in \mathrm{GL}(2,\mathbb{C}), X_2 \neq I\}\,\mathbf{e}^{\dagger}. \end{aligned}$$

It is well known that elementary groups have been defined in [8, Definition 5.1.1]. We shall generalize Beardon's definition to bicomplex numbers and obtain the following definition.

Definition 3.18. *A subgroup G of* GL$(2,\mathbb{BC})$ *is called elementary if there exists a finite orbit in* $\overline{\mathbb{BC}}$.

Remark 3.19. *A bicomplex Möbius transformation $f(Z)$ is said to be finite order if there exists $n \in \mathbb{N}$ such that $f(Z)^n = I$.*

Let G be an elementary group and suppose that the finite orbit is $\{Z_1, Z_2, \cdots, Z_n\}$. If f is in G then there exist an integer m such that f^m fixes Z_j, $j = 1, 2, \cdots, n$. With this available we can now classify the elementary groups into four types.

Type 1 : suppose that $n = 1$ and Z_1 is in $\overline{\mathbb{BC}}$.

In this case, G is conjugate to a group, every element of which fixes $\infty\mathbf{e}+\infty\mathbf{e}^{\dagger}$. According to Lemma 3.5, the bicomplex transformations are transitive. So the conclusion is obvious.

Type 2 : suppose that $n = 2$ and Z_1, Z_2 are in $\overline{\mathbb{BC}}$.

In this case, G is conjugate to a group, every element of which leaves $\{0, \infty\mathbf{e}\}$ or $\{0, \infty\mathbf{e}^{\dagger}\}$ invariant .

There exist $f \in$ G such that $f(Z_1) = Z_1$ and $f(Z_2) = Z_2$. By Lemma 3.15, we have f_1 has two fixed points or f_2 has two fixed points. Without loss of generality, suppose that f_1 has one fixed point and f_2 has two fixed points. Let $Z_1 = \beta_{1,1}\mathbf{e} + \beta_{2,1}\mathbf{e}^{\dagger}, Z_2 = \beta_{1,1}\mathbf{e} + \beta_{2,2}\mathbf{e}^{\dagger}$, for any g $\in$ G.

(i) $\mathrm{g}(Z_1) = Z_1$ and $\mathrm{g}(Z_2) = Z_2$.

$$\mathrm{g}(Z_1) = \mathrm{g}(\beta_{1,1}\mathbf{e} + \beta_{2,1}\mathbf{e}^{\dagger}) = \mathrm{g}_1(\beta_{1,1})\mathbf{e} + \mathrm{g}_2(\beta_{2,1})\mathbf{e}^{\dagger} = \beta_{1,1}\mathbf{e} + \beta_{2,1}\mathbf{e}^{\dagger} = Z_1$$
$$\mathrm{g}(Z_2) = \mathrm{g}(\beta_{1,1}\mathbf{e} + \beta_{2,2}\mathbf{e}^{\dagger}) = \mathrm{g}_1(\beta_{1,1})\mathbf{e} + \mathrm{g}_2(\beta_{2,2})\mathbf{e}^{\dagger} = \beta_{1,1}\mathbf{e} + \beta_{2,2}\mathbf{e}^{\dagger} = Z_2.$$

By [8, Chapter 4.3], there exists complex Möbius transformations

$$h_1(\beta_{1,1}) = 0, h_2(\beta_{2,1}) = 0, h_2(\beta_{2,2}) = \infty,$$

such that

$$h_1 g_1 h_1^{-1}(0) = 0, h_2 g_2 h_2^{-1}(0) = 0, h_2 g_2 h_2^{-1}(\infty) = \infty.$$

Let $h(Z) = h_1(\beta_1)\mathbf{e} + h_2(\beta_2)\mathbf{e}^{\dagger}$, we have

$$hgh^{-1}(0) = h_1g_1h_1^{-1}(0)\mathbf{e} + h_2g_2h_2^{-1}(0)\mathbf{e}^{\dagger} = 0$$
$$hgh^{-1}(\infty\mathbf{e}^{\dagger}) = h_1g_1h_1^{-1}(0)\mathbf{e} + h_2g_2h_2^{-1}(\infty)\mathbf{e}^{\dagger} = \infty\mathbf{e}^{\dagger}.$$

(ii) $g(Z_1) = Z_2$ and $g(Z_2) = Z_1$.

$$g(Z_1) = f(\beta_{1,1}\mathbf{e} + \beta_{2,1}\mathbf{e}^{\dagger}) = g_1(\beta_{1,1})\mathbf{e} + g_2(\beta_{2,1})\mathbf{e}^{\dagger} = \beta_{1,1}\mathbf{e} + \beta_{2,2}\mathbf{e}^{\dagger} = Z_2$$
$$g(Z_2) = g(\beta_{1,1}\mathbf{e} + \beta_{2,2}\mathbf{e}^{\dagger}) = g_1(\beta_{1,1})\mathbf{e} + g_2(\beta_{2,2})\mathbf{e}^{\dagger} = \beta_{1,1}\mathbf{e} + \beta_{2,1}\mathbf{e}^{\dagger} = Z_1.$$

By [8, Chapter 4.3], there exists complex Möbius transformations

$$h_1(\beta_{1,1}) = 0, h_2(\beta_{2,1}) = 0, h_2(\beta_{2,2}) = \infty,$$

such that

$$h_1g_1h_1^{-1}(0) = 0, h_2g_2h_2^{-1}(0) = \infty, h_2g_2h_2^{-1}(\infty) = 0.$$

Let $h(Z) = h_1(\beta_1)\mathbf{e} + h_2(\beta_2)\mathbf{e}^{\dagger}$, we have

$$hgh^{-1}(0) = h_1g_1h_1^{-1}(0)\mathbf{e} + h_2g_2h_2^{-1}(0)\mathbf{e}^{\dagger} = \infty\mathbf{e}^{\dagger}$$
$$hgh^{-1}(\infty\mathbf{e}) = h_1g_1h_1^{-1}(\infty)\mathbf{e} + h_2g_2h_2^{-1}(0)\mathbf{e}^{\dagger} = 0.$$

This shows that G is conjugate to a group, every element of which leaves $\{0, \infty\mathbf{e}^{\dagger}\}$ invariant .

Similarly, if f_1 has two fixed points and f_2 has one fixed point, we have G is conjugate to a group, every element of which leaves $\{0, \infty\mathbf{e}\}$ invariant .

Type 3 : suppose that $n = 4$ and Z_1, Z_2, Z_3, Z_4 are in $\overline{\mathbb{BC}}$.

In this case, G is conjugate to a group, every element of which leaves

$$\{0, \quad \infty\mathbf{e}, \quad \infty\mathbf{e}^{\dagger}, \quad \infty\mathbf{e} + \infty\mathbf{e}^{\dagger}\}$$

invariant.

There exist $f \in$ G such that $f(Z_i) = Z_i, i = 1, 2, 3, 4$. By Lemma 3.15, we have f_1 has two fixed points and f_2 has two fixed points. Let $Z_1 = \beta_{1,1}\mathbf{e}+\beta_{2,1}\mathbf{e}^{\dagger}, Z_2 = \beta_{1,1}\mathbf{e} + \beta_{2,2}\mathbf{e}^{\dagger}, Z_3 = \beta_{1,2}\mathbf{e} + \beta_{2,1}\mathbf{e}^{\dagger}, Z_4 = \beta_{1,2}\mathbf{e} + \beta_{2,2}\mathbf{e}^{\dagger}$.
By [8, Chapter 4.3], there exists complex Möbius transformations

$$h_1(\beta_{1,1}) = 0, \quad h_1(\beta_{1,2}) = \infty$$
$$h_2(\beta_{2,1}) = 0, \quad h_2(\beta_{2,2}) = \infty,$$

such that

$$h_1f_1h_1^{-1}(0) = 0, \quad h_1f_1h_1^{-1}(\infty) = \infty$$
$$h_2f_2h_2^{-1}(0) = 0, \quad h_2f_2h_2^{-1}(\infty) = \infty.$$

Let $h(Z) = h_1(\beta_1)\mathbf{e} + h_2(\beta_2)\mathbf{e}^\dagger$, we have

$$hfh^{-1}(0) = h_1 f_1 h_1^{-1}(0)\mathbf{e} + h_2 f_2 h_2^{-1}(0)\mathbf{e}^\dagger = 0$$
$$hfh^{-1}(\infty\mathbf{e}) = h_1 f_1 h_1^{-1}(\infty)\mathbf{e} + h_2 f_2 h_2^{-1}(0)\mathbf{e}^\dagger = \infty\mathbf{e}$$
$$hfh^{-1}(\infty\mathbf{e}^\dagger) = h_1 f_1 h_1^{-1}(0)\mathbf{e} + h_2 f_2 h_2^{-1}(\infty)\mathbf{e}^\dagger = \infty\mathbf{e}^\dagger$$
$$hfh^{-1}(\infty\mathbf{e} + \infty\mathbf{e}^\dagger) = h_1 f_1 h_1^{-1}(\infty)\mathbf{e} + h_2 f_2 h_2^{-1}(\infty)\mathbf{e}^\dagger = \infty\mathbf{e} + \infty\mathbf{e}^\dagger$$

Similar discussion as type 2, for any $f \in$ G, there exists h such that hfh^{-1} leaves $\{0, \infty\mathbf{e}, \infty\mathbf{e}^\dagger, \infty\mathbf{e} + \infty\mathbf{e}^\dagger\}$ invariant.

Type 4 : suppose that $n = 3$ or $n \geq 5$.

If $n = 3$, by Lemma 3.14, f^m has three fixed points and so is identity: thus each non-trivial element of G is finite order.

Similarly, for $n \geq 5$, each non-trivial element of G is finite order.

Finally, we shall study the elementary bicomplex Möbius groups and give necessary condition for a group to be non-elementary. Let fix(f) be the set of fixed points of f, we have

Theorem 3.20. *Let $f(Z) = f_1(\beta_1)\mathbf{e} + f_2(\beta_2)\mathbf{e}^\dagger$ be a bicomplex Möbius transformation and f_1, f_2 are not of order two. Define the map $\Theta(g) = gfg^{-1}$. If for some n, we have $\Theta^n(g) = f$, then $\langle f, g\rangle$ is elementary.*

Proof. Defined $g_0 = g = g_{0,1}\mathbf{e} + g_{0,2}\mathbf{e}^\dagger$ and $g_n = \Theta^n(g)$ so for $m \geq 0$,

$$g_{m+1} = g_m f(g_m)^{-1} = g_{m,1} f_1 (g_{m,1})^{-1}\mathbf{e} + g_{m,2} f_2 (g_{m,2})^{-1}\mathbf{e}^\dagger. \tag{15}$$

Case 1. f has exactly one fixed point. According to Lemma 3.15 , let

$$f(Z) = (\beta_1 + 1)\mathbf{e} + (\beta_2 + 1)\mathbf{e}^\dagger.$$

Since $g_1, g_2, ..., g_n$ are conjugate to f, then g_i has a unique fixed point ($i = 1, 2, ..., n$). Since

$$g_{n+1}(\infty\mathbf{e} + \infty\mathbf{e}^\dagger) = \infty\mathbf{e} + \infty\mathbf{e}^\dagger \tag{16}$$

and

$$g_{n+1}(g_n(\infty\mathbf{e} + \infty\mathbf{e}^\dagger)) = \infty\mathbf{e} + \infty\mathbf{e}^\dagger, \tag{17}$$

we have

$$g_n(\infty\mathbf{e} + \infty\mathbf{e}^\dagger) = \infty\mathbf{e} + \infty\mathbf{e}^\dagger. \tag{18}$$

It follows that $g_1, g_2, ..., g_n$ each fix $\infty\mathbf{e} + \infty\mathbf{e}^\dagger$. Thus $\langle f, g\rangle$ is elementary.

Case 2. f has exactly two fixed points. We may assume that

$$f(Z) = (\beta_1 + 1)\mathbf{e} + k\beta_2\mathbf{e}^\dagger (k \neq 0, 1).$$

So $g_1, g_2, ..., g_n$ each have exactly two fixed points. Since

$$g_{n+1}(\infty\mathbf{e} + \infty\mathbf{e}^\dagger) = \infty\mathbf{e} + \infty\mathbf{e}^\dagger \text{ and } g_{n+1}(\infty\mathbf{e}) = \infty\mathbf{e}.$$

Then

$$\text{fix}(g_{n+1}) = \{\infty\mathbf{e} + \infty\mathbf{e}^\dagger, \infty\mathbf{e}\}.$$

Thus

$$\{\infty\mathbf{e} + \infty\mathbf{e}^\dagger, \infty\mathbf{e}\} = \{g_{n,1}(\infty)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^\dagger, g_{n,1}(\infty)\mathbf{e} + g_{n,2}(0)\mathbf{e}^\dagger\}.$$

If

$$g_{n,1}(\infty)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^{\dagger} = \infty\mathbf{e}$$
$$g_{n,1}(\infty)\mathbf{e} + g_{n,2}(0)\mathbf{e}^{\dagger} = \infty\mathbf{e} + \infty\mathbf{e}^{\dagger},$$

we have

$$g_{n,1}(\infty) = \infty,\ g_{n,2}(\infty) = 0,\ g_{n,2}(0) = \infty.$$

Since

$$\text{fix}(g_{n,2}) = \{g_{n-1,2}(0), g_{n-1,2}(\infty)\}$$
$$g_{n-1,2}(0) \neq 0, \infty \text{ and } g_{n-1,2}(\infty) \neq 0, \infty.$$

We have $g_{n,2}$, and hence f_2, is of order two. It is a contradiction.

Thus

$$g_{n,1}(\infty)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^{\dagger} = \infty\mathbf{e} + \infty\mathbf{e}^{\dagger}$$
$$g_{n,1}(\infty)\mathbf{e} + g_{n,2}(0)\mathbf{e}^{\dagger} = \infty\mathbf{e}.$$

It follows that $g_1, g_2, ..., g_n$ each fix $\infty\mathbf{e}$ and $\infty\mathbf{e} + \infty\mathbf{e}^{\dagger}$. So g leave $\{\infty\mathbf{e} + \infty\mathbf{e}^{\dagger}, \infty\mathbf{e}\}$ invariant, $\langle f, g\rangle$ is elementary.

Case 3. f has exactly four fixed points. We may assume that

$$f(Z) = \lambda\beta_1\mathbf{e} + k\beta_2\mathbf{e}^{\dagger}(\lambda, k \neq 0, 1).$$

So $g_1, g_2, ..., g_n$ each have four fixed points. Now suppose that

$$g_{n+1} = f.$$

Then

$$\begin{aligned}\text{fix}(g_{n+1}) &= \{0, \infty\mathbf{e}, \infty\mathbf{e}^{\dagger}, \infty\mathbf{e} + \infty\mathbf{e}^{\dagger}\} \\ &= \{g_{n,1}(0)\mathbf{e} + g_{n,2}(0)\mathbf{e}^{\dagger}, g_{n,1}(0)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^{\dagger}, \\ &\quad g_{n,1}(\infty)\mathbf{e} + g_{n,2}(0)\mathbf{e}^{\dagger}, g_{n,1}(\infty)\mathbf{e} + g_{n,2}(\infty)\mathbf{e}^{\dagger}\}.\end{aligned}$$

Similar discussions as case 2, we have $\langle f, g\rangle$ is elementary. □

3.3. Geometric representations of bicomplex Möbius transformations. The hyperbolic Möbius transformations were introduced in [25]. In the case dimension two, the six types of maps defined by Luna-Elizarrarás et al. are as follows:

1. The reflection in $\Gamma\left(\hat{h}, v\right)$

$$F_{\Gamma(\hat{h},v)}(Z) = F_{\Gamma_1(\hat{h}_1,v_1)}(\beta_1)\mathbf{e} + F_{\Gamma_2(\hat{h}_2,v_2)}(\beta_2)\mathbf{e}^{\dagger},\ F_{\Gamma(\hat{h},v)}(\infty_{\mathbb{BC}}) = \infty_{\mathbb{BC}},$$

where hyperbolic straight line $\Gamma\left(\hat{h}, v\right) = \Gamma_1\left(\hat{h}_1, v_1\right)\mathbf{e} + \Gamma_2\left(\hat{h}_2, v_2\right)\mathbf{e}^{\dagger}$ and $F_{\Gamma_i(\hat{h}_i,v_i)}(\beta_i)$ is a reflection in complex straight line $\Gamma_i\left(\hat{h}_i, v_i\right)(i = 1, 2)$.

2. The reflection in $\mathbb{S}(Z_0, R)$

$$F_{\mathbb{S}(Z_0,R)}(Z) = F_{\mathbb{S}_1(\beta_{1,0},r_1)}(\beta_1)\mathbf{e} + F_{\mathbb{S}_2(\beta_{2,0},r_2)}(\beta_2)\mathbf{e}^{\dagger},$$

where bicomplex circumference $\mathbb{S}(Z_0, R) = \mathbb{S}_1\left(\beta_{1,0}, r_1\right)\mathbf{e} + \mathbb{S}_2\left(\beta_{2,0}, r_2\right)\mathbf{e}^{\dagger}$ and $F_{\mathbb{S}_i(\beta_{i,0},r_i)}(\beta_i)$ is a reflection in complex circumference $\mathbb{S}_2\left(\beta_{i,0}, r_i\right)(i = 1, 2)$.

3. Translations

$$F_W(Z) = Z + W = (\beta_1 + w_1)\mathbf{e} + (\beta_2 + w_2)\mathbf{e}^\dagger,\ F_W(\infty_{\mathbb{BC}}) = \infty_{\mathbb{BC}},$$

where $W = w_1\mathbf{e} + w_2\mathbf{e}^\dagger,\ \infty_{\mathbb{BC}} = \infty\mathbf{e} + \infty\mathbf{e}^\dagger$.

4. Stretching

$$F_\vartheta(Z) = \vartheta Z = (\vartheta_1\beta_1)\mathbf{e} + (\vartheta_2\beta_2)\mathbf{e}^\dagger,\ F_\vartheta(\infty_{\mathbb{BC}}) = \infty_{\mathbb{BC}},$$

where $\vartheta = \vartheta_1\mathbf{e} + \vartheta_2\mathbf{e}^\dagger \succ 0,\ \infty_{\mathbb{BC}} = \infty\mathbf{e} + \infty\mathbf{e}^\dagger$.

5. Orthogonal

$$\|F_{ort}(Z)\|_{\mathbb{D}^2} = \|Z\|_{\mathbb{D}^2}\ F_{ort}(\infty_{\mathbb{BC}}) = \infty_{\mathbb{BC}},$$

where $F_{ort}(Z) = T_1(\beta_1)\mathbf{e} + T_2(\beta_2)\mathbf{e}^\dagger$. T_1 and T_2 are $\mathbb{R}$-linear orthogonal mappings

6. Any mix of 1 and 2. For example,

$$F_{mix}(Z) = F_{\mathbb{S}_1(\beta_{1,0},r_1)}(\beta_1)\mathbf{e} + F_{\Gamma_2(\hat{h}_2,v_2)}(\beta_2)\mathbf{e}^\dagger.$$

Based on [15, Definition 3.1] and [25, Definition 11], we have the following definition.

Definition 3.21. *A map $F : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ is called a hyperbolic Möbius transformation if it is a superposition of a finite number of the above six types of maps.*

Chen and Dai[15] described the relation between the hyperbolic Möbius transformation and bicomplex Möbius transformation. They obtained the following crucial results.

Lemma 3.22. [15, Lemma 3.2] *The translation $F_W(Z)$ is a superposition of two reflections in bicomplex circumference, and the stretching $F_\vartheta(Z)$ is a superposition of two reflections in hyperbolic straight line.*

Theorem 3.23. [15, Theorem 3.3] *The map $F(Z)$ is a superposition of an even number of reflections of the form $F_{\Gamma(\hat{h},v)}(Z)$, $F_{\mathbb{S}(Z_0,R)}(Z)$ and $F_{mix}(Z)$ if and only if $F(Z)$ is a bicomplex Möbius transformation.*

4. The proof of bicomplex Jørgensen' inequality.

Proof of Theorem 1.2. Suppose that $\langle f, g\rangle$ is discrete nonelementary group. Select matrices $\mathcal{A}$ and $\mathcal{B}$ representing f and g respectively in $\mathrm{PSL}(2,\mathbb{BC})$ and define

$$\mathcal{B}_0 = \mathcal{B},\quad \mathcal{B}_{n+1} = \mathcal{B}_n\mathcal{A}\mathcal{B}_n^{-1}. \tag{19}$$

Case 1. f has one fixed point in $\overline{\mathbb{BC}}$.

According to Lemma 3.9 and Lemma 3.15, we may assume that

$$\mathcal{A} = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\mathbf{e} + \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\mathbf{e}^\dagger$$

$$\mathcal{B} = \begin{pmatrix}a_{0,1} & b_{0,1}\\ c_{0,1} & d_{0,1}\end{pmatrix}\mathbf{e} + \begin{pmatrix}a_{0,2} & b_{0,2}\\ c_{0,2} & d_{0,2}\end{pmatrix}\mathbf{e}^\dagger,$$

where $c_{0,1} \neq 0$ or $c_{0,2} \neq 0$ (else $\langle f, g\rangle$ is elementary). Without loss of generality, suppose that $c_{0,1} \neq 0$ and $c_{0,2} = 0$. Since

$$|\mathrm{tr}^2(f) - 4|_{\mathbf{k}} + |\mathrm{tr}(fgf^{-1}g^{-1}) - 2|_{\mathbf{k}} \prec 1 \tag{20}$$

we have

$$\Big(|\mathrm{tr}^2(f_1) - 4| + |\mathrm{tr}(f_1g_1f_1^{-1}g_1^{-1}) - 2|\Big)\mathbf{e} + \Big(|\mathrm{tr}^2(f_2) - 4| + |\mathrm{tr}(f_2g_2f_2^{-1}g_2^{-1}) - 2|\Big)\mathbf{e}^{\dagger} \prec 1.$$

So

$$\begin{cases} |\mathrm{tr}^2(f_1) - 4| + |\mathrm{tr}(f_1g_1f_1^{-1}g_1^{-1}) - 2| < 1, \\ |\mathrm{tr}^2(f_2) - 4| + |\mathrm{tr}(f_2g_2f_2^{-1}g_2^{-1}) - 2| < 1. \end{cases} \tag{21}$$

It is easy to check that $|c_{0,1}| < 1$.

The relation (19) yields

$$\begin{aligned} a_{n+1,1} &= 1 - a_{n,1}c_{n,1}, & a_{n+1,2} &= 1 \\ b_{n+1,1} &= a_{n,1}^2, & b_{n+1,2} &= a_{n,2}^2 \\ c_{n+1,1} &= -(-c_{0,1})^{2^n}, & c_{n+1,2} &= 0 \\ d_{n+1,1} &= 1 + a_{n,1}c_{n,1}, & d_{n+1,2} &= 1, \end{aligned}$$

where

$$a_{n,2} = \begin{cases} 1 & \text{if } n \geq 1 \\ a_{0,2} & \text{if } n = 0. \end{cases}$$

Since $|c_{0,1}| < 1$, we have $|c_{n,1}| \to 0$, and $|a_{n,1}| \leq n + |a_{0,1}|$. So $a_{n,1}c_{n,1} \to 0$, $|a_{n,1}| \to 1$, and $|d_{n,1}| \to 1$.

If $\langle f, g\rangle$ is discrete, $\mathcal{B}_{n+1} = \mathcal{A}$, for sufficiently large n. According to Theorem 3.20 we have $\langle f, g\rangle$ is elementary.

Similarly, if $c_{0,1} = 0, c_{0,2} \neq 0$ or $c_{0,1} \neq 0, c_{0,2} \neq 0$, we have $\langle f, g\rangle$ is elementary.

Case 2. f has two fixed point in $\overline{\mathbb{BC}}$.

According to Lemma 3.9 and Lemma 3.15, we may assume that

$$\mathcal{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{e} + \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}\mathbf{e}^{\dagger} \ (u \neq 0, 1.)$$

$$\mathcal{B} = \begin{pmatrix} a_{0,1} & b_{0,1} \\ c_{0,1} & d_{0,1} \end{pmatrix}\mathbf{e} + \begin{pmatrix} a_{0,2} & b_{0,2} \\ c_{0,2} & d_{0,2} \end{pmatrix}\mathbf{e}^{\dagger},$$

where $c_{0,1} \neq 0$ or $b_{0,2}c_{0,2} \neq 0$ (else $\langle f, g\rangle$ is elementary). Without loss of generality, suppose that $c_{0,1} \neq 0$ and $b_{0,2}c_{0,2} \neq 0$.

By (20), we have

$$|c_{0,1}| < 1 \ \text{ and } \ \nu = (1 + |b_{0,2}c_{0,2}|)\left|u - \frac{1}{u}\right|^2 < 1.$$

The relation (19) yields

$$\begin{aligned} a_{n+1,1} &= 1 - a_{n,1}c_{n,1} \quad a_{n+1,2} = a_{n,2}d_{n,2}u - \frac{b_{n,2}c_{n,2}}{u} \\ b_{n+1,1} &= a_{n,1}^2 \qquad\quad b_{n+1,2} = a_{n,2}b_{n,2}\left(\frac{1}{u} - u\right) \\ c_{n+1,1} &= -(-c_{0,1})^{2^n} \quad c_{n+1,2} = c_{n,2}d_{n,2}\left(u - \frac{1}{u}\right) \\ d_{n+1,1} &= 1 + a_{n,1}c_{n,1} \quad d_{n+1,2} = \frac{a_{n,2}d_{n,2}}{u} - b_{n,2}c_{n,2}u. \end{aligned}$$

So

$$\begin{aligned} |b_{n+1,2}c_{n+1,2}| \leq& \left| b_{n,2}c_{n,2}(1 + b_{n,2}c_{n,2})\left(u - \frac{1}{u}\right)^2 \right| \\ \leq& \nu^n |b_{0,2}c_{0,2}|. \end{aligned}$$

Thus

$$b_{n,2}c_{n,2} \to 0 \text{ and } a_{n,2}d_{n,2} \to 1.$$

Also, we obtain

$$a_{n+1,2} \to u, \quad d_{n+1,2} \to \frac{1}{u}.$$

Now

$$\left|\frac{b_{n+1,2}}{b_{n,2}}\right| = \left|a_{n,2}\left(\frac{1}{u} - u\right)\right| \leq \nu^{\frac{1}{2}}|u|.$$

Thus

$$\frac{b_{n,2}}{u^n} \to 0.$$

Similarly, $c_{n,2}u^n \to 0$. So

$$\begin{aligned} &\mathcal{A}^{-n}\mathcal{B}_{2n}\mathcal{A}^n \\ &= \begin{pmatrix} a_{2n,1} - nc_{2n,1} & n(a_{2n,1} - nc_{2n,1}) + b_{2n,1} - nd_{2n,1} \\ c_{2n,1} & d_{2n,1} \end{pmatrix} \mathbf{e} + \begin{pmatrix} a_{2n,2} & \frac{b_{2n,2}}{u^{2n}} \\ c_{2n,2}u^{2n} & d_{2n,2} \end{pmatrix} \mathbf{e}^\dagger \\ &= \begin{pmatrix} a_{2n,1} - nc_{2n,1} & n(a_{2n,1} - d_{2n,1}) - n^2c_{2n,1} + b_{2n,1} \\ c_{2n,1} & d_{2n,1} \end{pmatrix} \mathbf{e} + \begin{pmatrix} a_{2n,2} & \frac{b_{2n,2}}{u^{2n}} \\ c_{2n,2}u^{2n} & d_{2n,2} \end{pmatrix} \mathbf{e}^\dagger. \end{aligned}$$

$$\begin{aligned} &a_{2n,1} - nc_{2n,1} = a_{2n,1} + n(-c_{0,1})^{2^{2n-1}} \\ &n(a_{2n,1} - d_{2n,1}) - n^2c_{2n,1} + b_{2n,1} = -2na_{2n-1,1}c_{2n-1,1} - n^2(-c_{0,1})^{2^{2n-1}} + b_{2n,1}. \end{aligned}$$

Since $|c_{0,1}| < 1$, we have

$$|-2na_{2n-1,1}c_{2n-1,1}| \leq 2n(2n - 1 + |a_{0,1}|)|c_{0,1}|^{2n-1}.$$

Thus

$$a_{2n,1} - nc_{2n,1} \to 1 \text{ and } n(a_{2n,1} - d_{2n,1}) - n^2c_{2n,1} + b_{2n,1} \to 1.$$

As $\langle f, g\rangle$ is discrete, we must have

$$\mathcal{A}^{-n}\mathcal{B}_{2n}\mathcal{A}^{n} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{e} + \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}\mathbf{e}^{\dagger} \ (u \neq 0, 1.)$$

for all large n. So $\mathcal{B}_{2n} = \mathcal{A}$ for infinitely many n. According to Theorem 3.20, we have $\langle f, g\rangle$ is elementary.

Similarly, if $c_{0,1} = 0, b_{0,2}c_{0,2} \neq 0$ or $c_{0,1} \neq 0, b_{0,2}c_{0,2} = 0$, we have $\langle f, g\rangle$ is elementary.

Case 3. f has four fixed points . Let

$$\mathcal{A} = \begin{pmatrix} v & 0 \\ 0 & \frac{1}{v} \end{pmatrix}\mathbf{e} + \begin{pmatrix} u & 0 \\ 0 & \frac{1}{u} \end{pmatrix}\mathbf{e}^{\dagger} \ (u, v \neq 0, 1.)$$

$$\mathcal{B} = \begin{pmatrix} a_{0,1} & b_{0,1} \\ c_{0,1} & d_{0,1} \end{pmatrix}\mathbf{e} + \begin{pmatrix} a_{0,2} & b_{0,2} \\ c_{0,2} & d_{0,2} \end{pmatrix}\mathbf{e}^{\dagger},$$

where $b_{0,1}c_{0,1} \neq 0$ or $b_{0,2}c_{0,2} \neq 0$ (else $\langle f, g\rangle$ is elementary).

The relation (19) yields

$$\begin{aligned} a_{n+1,1} &= a_{n,1}d_{n,1}v - \frac{b_{n,1}c_{n,1}}{u} & a_{n+1,2} &= a_{n,2}d_{n,2}u - \frac{b_{n,2}c_{n,2}}{u} \\ b_{n+1,1} &= a_{n,1}b_{n,1}\left(\frac{1}{v} - v\right) & b_{n+1,2} &= a_{n,2}b_{n,2}\left(\frac{1}{u} - u\right) \\ c_{n+1,1} &= c_{n,1}d_{n,1}\left(v - \frac{1}{v}\right) & c_{n+1,2} &= c_{n,2}d_{n,2}\left(u - \frac{1}{u}\right) \\ d_{n+1,1} &= \frac{a_{n,1}d_{n,1}}{v} - b_{n,1}c_{n,1}v & d_{n+1,2} &= \frac{a_{n,2}d_{n,2}}{u} - b_{n,2}c_{n,2}u. \end{aligned}$$

Similar discussions as case 2, if $\langle f, g\rangle$ is discrete, we have

$$\mathcal{B}_{2n} = A$$

for all sufficiently large n. This shows that $\langle f, g\rangle$ is elementary. □

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