PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 28 (42), 1980, pp. 19–23

A NOTE ON STRONGLY REVERSIBLE SEMIPRIMARY SEMIGROUPS

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(Communicated March 15, 1979.)

The strongly reversible semigroups are defined by G. Thierrin, [8]. These semigroups are studied in details by A. Spoletini-Cherubini and A. Varisco, [6], [7]. The commutative primary semigroups are considered by M. Satyanarayana, [5]. Similarly, we consider here strongly reversible semiprimary semigroups.

The semiprimary semigroups are caracterised by Theorem 1. By the Theorem 2. the semiprimary ideal in the strongly reversible semigroups is caracterised. Then by the Theorem 3. strongly reversible semiprimary semigroups are caracterised by totally ordered completely prime ideals. Strongly reversible semigroups in which all the ideals are completely prime are caracterised by Theorem 4. It consequently follows that the regular strongly reversible semigroup is a primary if and only if it is simiprimary.

Notions which we use further on can be found in [1] or [4].

The ideal I of the semigroup S is completely prime if

$$(\forall a, b \in S)(ab \in I \Rightarrow a \in I \vee b \in I).$$

The semigroup S is a completely prime ideal.

The ideal I of the semigroup S is semiprime if

$$(\forall a \in S)(a^2 \in I \Rightarrow a \in I).$$

It follows that every completely prime ideal is semiprime.

The ideal I of the semigroup S is primary if

$$(\forall a, b \in S)(ab \in I \Rightarrow a \in I \vee (\exists n \in N)(b^n \in I)).$$

The semigroup S is primary if all its ideals are primary, [5].

Obviously, every completely prime ideal is a primary.

Definition 1. *The ideal I of semigroup S is semiprimary if*

$$(\forall a, b \in S)(ab \in I \Rightarrow (\exists m \in N)(a^m \in I) \vee (\exists n \in N)(b^n \in I)).$$

The semigroup S is semiprimary if all its ideals are semiprimary.

Obviously, every primary semigroup is semiprimary.

Theorem 1. *Let S be a semigroup. Then, the following conditions are equivalent:*

(i) *S is simiprimary.*

(ii) $(\forall a, b \in S)$ $((\exists m \in N)$ $(a^m \in SabS) \vee (\exists n \in N)$ $(b^n \in SabS))$.

(iii) *Every principal ideal of S is semiprimary.*

Proof. (i) $\Rightarrow$ (ii). Let S be a semiprimary semigroup. Then for any $a, b \in S$ the ideal $SabS$ is semiprimary and since

$$a^2b^2 = a(ab)b \in SabS$$

then

$$(\exists m \in N)((a^2)^m \in SabS) \vee (\exists n \in N)((b^2)^n \in SabS).$$

(ii) $\Rightarrow$ (iii). Let (ii) is valid and let $[x]$ ae a principal ideal of S generated by $x \in S$. Then we have

$$\begin{aligned} ab \in [x] &\Rightarrow SabS \subset [x] \\ &\Rightarrow (\exists m \in N)(a^m \in [x]) \vee (\exists n \in N)(b^n \in [x]). \end{aligned}$$

(iii) $\Rightarrow$ (i). This is obvious.

Definition 2. [8]. The semigroup S is strongly reversible if for every $a, b \in S$ there exists $h, k, l \in N$ such that

$$(ab)^h = a^k b^l = b^l a^k.$$

Denote with $\mathcal{I}$ a class of all strongly reversible semigroups.

Lemma 1. *Let $S \in \mathcal{I}$. Then, every semiprime ideal of S is two-sided.*

Proof. Let $S \in \mathcal{I}$ and let R be its right semiprime ideal. For any $b \in R$ $a \in S$ there exitst $h, k, l \in N$ such that (1) holds. From $b^l a^k \in R$, it follows $(ab)^h \in R$. So we have that $ab \in R$. Similarly, it holds for the left ideal of the semigroup S.

Furter on we are going to use the notion of a redical on an ideal.

Defition 3. [3]. A redikal of the ideal I, denoted by rad (I), of the semigroup S is the set of all $x \in S$ such that some power of x is in I, i.e.

$$\operatorname{rad}(I) = \{x \in S \mid (\exists n \in N)(x^n \in I)\}.$$

Lemma 2. *Let I be the ideal of the semigroup $S \in \mathcal{I}$. Then rad (I) is semiprime ideal of the semigroup S containing I.*

Proof. It is clear that $I \subset \operatorname{rad}(I)$. Let $a \in \operatorname{rad}(I)$. Then, there exists $n \in N$ such that $a^n \in I$. Let b be any element of S. From $S \in \mathcal{I}$ it follows that exists $h, k, l \in N$ such that (1) holds. From (1) we have $(ab)^{nh} = a^{nh}b^{nl} \in I$, so it follows $ab \in \operatorname{rad}(I)$. Similarly, $ba \in \operatorname{rad}(I)$. So, rad (I) is an ideal of semigroup S. It is clear that rad (I) is a semiprime.

From the Theorem II 3.7. [4] and Lemma 2. we have.

Lemma 3. *Let I be an ideal of the semigroup $S \in \mathcal{I}$. Then,* rad (I) *is an intersection of completely prime ideals.*

Lemma 4. *Let I be an ideal of the semigroup S. Then, I is semiprime if and only if $I = \operatorname{rad}(I)$.*

Proof. Let I be a semiprime ideal of the semigroup S. For $x \in \operatorname{rad}(I)$ there exists $n \in N$ such that $x^n \in I$. It follows that $x \in I$. So, $\operatorname{rad}(I) \subset I$. As the opposite inclusion is valid we have $I = \operatorname{rad}(I)$.

Conversely, let $I = \operatorname{rad}(I)$. For $x^2 \in I$ we have that $x \in \operatorname{rad}(I) = I$. Hence, I is a semiprime ideal.

The following theorem gives a caracterisation of the semiprimary ideal in the class of semigroups $\mathcal{I}$.

Theorem 2. *Let $S \in \mathcal{I}$. Then, the ideal I of the semigroup S is semiprimary if and only if* rad (I) *is completely prime ideal of S.*

Proof. Let I be the semiprimary ideal of the semigroup $S \in \mathcal{I}$. For $ab \in \operatorname{rad}(I)$ there exists $t \in N$ such that $(ab)^t \in I$ and there exists $s, p, q \in N$ such that

$$(ab)^s = a^p b^q = b^q a^p. \tag{2}$$

From (2) we have

$$a^{pt} b^{qt} = b^{qt} a^{pt} = (ab)^{st} \in I. \tag{3}$$

From the supposition of the ideal I from (3) we have

$$(\exists m \in N)((a^{pt})^m \in I) \vee (\exists n \in N)((b^{qt}) \in I)$$

from which we have

$$a \in \mathrm{rad}\,(I) \vee b \in \mathrm{rad}\,(I).$$

Hence, rad (I) is completely prime ideal.

Conversely, let rad (I), (this is an ideal, Lemma 2.) be completely prime ideal. Then, we have

$$\begin{aligned} ab \in I \Rightarrow ab \in \mathrm{rad}\,(I) &\Rightarrow a \in \mathrm{rad}\,(I) \vee b \in \mathrm{rad}\,(I) \\ &\Rightarrow (\exists m \in N)(a^m \in I) \vee (\exists n \in N)(b^n \in I). \end{aligned}$$

COROLLARY 1. *Let $S \in \mathcal{I}$. Then a radical of the primary ideal is completely prime.*

We proove the next theorem in which strogly reversible semiprimary semigroups are caracterised by totally ordered completely prime ideals.

THEOREM 3. *Let $S \in \mathcal{I}$. Then S is semiprimary if and only if completely prime ideals of the semigroup S are totally ordered.*

PROOF. Let I_1, I_2 be completely prime ideals of the semiprimary semigroup $S \in \mathcal{I}$. Suppose the contrary, i.e. that $I_1 \not\subset I_2$ and $I_2 \not\subset I_1$. Then there exists $a \in I_1 \backslash I_2$ and $b \in I_2 \backslash I_1$, so $ab \in I_1 \cap I_2$, $a \notin I_1 \cap I_2$, $b \notin I_1 \cap I_2$. As $I_1 \cap I_2$ is a semiprime ideal, (Theorem II 3.7. [4]), from Lemma 4. we have that $I_1 \cap I_2 = \mathrm{rad}\,(I_1 \cap I_2)$. Since, S is a semiprimary semigroup, rad $(I_1 \cap I_2)$ is completely prime ideal, (Theorem 2.). Then $I_1 \cap I_2$ is completely prime, i.e. from $ab \in I_1 \cap I_2$ it follows that $a \in I_1 \cap I_2$ or $b \in I_1 \cap I_2$, which is impossible.

Conversely, let completely prime ideals of the semigroup S be totally ordered and let I be any ideal of S. Then from Lemma 3. it follows that $\mathrm{rad}\,(I) = \cap I_i$, ($I \subset I_i$ and I_i are completely prime ideals). Let $a \notin \cap I_i$, $b \notin \cap I_i$. Then there exists completely erime ideals I_j, I_k such that $a \notin I_j$, $b \notin I_k$. From the supposition of the theorem it follows $I_j \subset I_k$ or $I_k \subset I_j$. Suppose that $I_j \subset I_k$. Then $a \notin I_j$. Since I_j is a completely prime ideal, it follows $ab \notin I_j$. So, we have that $ab \notin \cap I_i$. By contrapositive we conclude that $\cap I_i = \mathrm{rad}\,(I)$ is a completely prime ideal. From this and from Theorem 2. we have that I is a semiprimary ideal. Therefore, S is a semiprimary semigroup.

By the theorem that follows strongly reversible semigroups in which all ideals are completely prime are caracterised.

THEOREM 4. *Let $S \in \mathcal{I}$. Then, the following conditions are equivalent:*

(i) *Every ideal of S is completely prime.*

(ii) *S is primary and regular.*

(iii) *S is semiprimary and regular.*

PROOF. (i) $\Rightarrow$ (ii). Let every ideal of the semigroup S be completely prime. Then, every ideal of S is semiprime, so S is intra-regular semigroup, (Propriété)

6., [2]). It follows from the above that S is a regular semigroup, (Proposition 10., [7]). That S is primary follows immediately.

(ii) $\Rightarrow$ (iii). Trivial.

(iii) $\Rightarrow$ (i). Let S be a semiprimary and regular semigroup.

Since, $S \in \mathcal{I}$, it follows that S is an intra-regular semigroup, (Proposition 10., [7]), and every ideal of S is semiprime, (Propriétê 6., [2]). Hence, every ideal of the semigroup S is semiprimary and semiprime. Therefore, every ideal of S is completely prime.

The proof for the next two consequences follows immediately.

Corollary 2. *Let $S \in \mathcal{I}$ be regular semigroup. Then, the following conditions are equivalent:*

(i) *Every ideal of S is completely prime.*

(ii) *S is primary.*

(iii) *S is semiprimary.*

(iv) *Completely prime ideals of S are totally ordered.*

Corollary 3. *Let $S \in \mathcal{I}$ be semiprimary. Then, every ideal of S is completely prime if and only if S is regular semigroup.*

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