PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 28 (42), 1980, pp. 25–32

TRUTH FILTERS IN DE MORGAN LATTICES

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Abstract. In this paper Stone-type theorems on existence of truth filters in De Morgan lattices are obtained. It is shown that $(^*)\cdots(a \wedge T) \cap T_0 = \emptyset$ is necessary and sufficient for existence of a truth filter containing **a**, where T_0 and T_1 are sets of zeroes and units. Also, it is proved that the filter F is contained in a truth filter iff for each member **a** of F holds $(^*)$. Some other related results are proved, too.

DEF 1. $(L, \leq, ^{-})$ is a *De Morgan lattice* iff

(DL) $(L, \leq)$ is a distributive lattice

(C_1) $(\forall x \in L)\bar{\bar{x}} = x$

(C_2) $(\forall x, y \in L)$ $(x \leq y \Rightarrow \bar{y} \leq \bar{x})$.

It is easy to prove that

LEMMA 1. (DeM) $\overline{x \vee y} = \bar{x} \wedge \bar{y}$, $\overline{x \wedge y} = \bar{x} \vee \bar{y}$ holds, where the least upper bound and the greatest lower bound in $(L, \leq)$ are denoted by $\vee$ and $\wedge$ respectively.

Separate study has been given to the structures defined above. These have been called *De Morgan lattices* in Monteiro [1960], *distributive ivolution lattices (i-lattices)* in Kalman [1958], and *quasi-Boolean algebras* in Bialnicki-Birula and Rasiowa [1957]. Most commonly used term in recent years in the first one. We shall use it in this paper. The term quasi-Boolean algebra is generally used for De Morgan lattices with least and greatest elements. It is obvious that the difference is immaterial.

De Morgan lattices were investigated in standard algebraical and topological manner (see, for example, Rasiowa [1974]). The rise of interest in them can be dated in the sixties when it was shown that various logical systems have appropriate interpretation in these lattices if they are equipped with some special kind of filters.

DEF 2. A filter F of a De Morgan lattice L is

(i) *consistent* iff $\neg(\exists x \in L)(x \in F \wedge \bar{x} \in F)$

(ii) *complete* iff $(\forall x \in L)(x \in F \vee \bar{x} \in F)$

(iii) *truth filter* iff it is consistent and complete.

It is obvious that a filter F is a fruth filter iff

$$(\forall x \in L)(x \in F \Leftrightarrow \bar{x} \notin F)$$

The definition above naturally leads to a question of a necessary and sufficient condition for a De Morgan lattice to have a truth filter, and it turns out that a quite simple condition works, namely that $^-$ has no fixed points, i.e. the following holds:

THEOREM *(Belnap and Spencer [1966])*

There exist a truth filter in a De Morgan lattice L iff

(C_3) $(\forall x \in L)\bar{x} \neq x$.

We shall call De Morgan lattice satisfying (C_3) *regular.*

It is easy to prove that each truth filter is prime but the converse does not hold. Concerning the maximal filters, they are neither always truth filters nor truth filters are always maximal.

Widely known Stone's theorem about distributive lattices states that for each nontrivial filter there exists a prime filter containing it. Also, as a consequence, for each a, b such that $\neg a \leq b$ there exists a prime filter containing a and not containing b.

Similar questions arise about truth filters in De Morgan lattices. It is our aim to give appropriate answers. To begin with, we are developing some necessary notions and appropriate algebraic apparatus.

DEF 3

$$\begin{array}{ll}
I(A) = \{x \in L \mid (\exists a_1, \dots, a_n \in A) x \leq a_1 \vee \cdots \vee a_n\} & \\
F(A) = \{x \in L \mid (\exists a_1, \dots, a_n \in A) x \geq a_1 \wedge \cdots \wedge a_n\} & \\
T_0 = \{x \in L \mid x = x \wedge \bar{x}\}, & I_0 = I(T_0) \\
T_1 = \{x \in L \mid x = x \vee \bar{x}\}, & F_1 = F(T_1) \\
\bar{A} = \{\bar{x} \in L \mid x \in A\}, & a \wedge A = \{a \wedge x \mid x \in A\} \\
A \wedge B = \{x \wedge y \mid x \in A, y \in B\}, & [a,b] = \{x \in L \mid a \leq x \leq b\}.
\end{array}$$

$I(A)$ and $F(A)$ will be called *ideal* and *filter generated by the set* A. T_0 and T_1 are sets of *zeroes* and *units* respectively.

The following lemma states some basic properties of notions introduced above.

LEMMA 2. Let $x, y \in L$ then:

(i) $x \in T_0 \Leftrightarrow x \leq \bar{x}$; $x \in T_0$ and $y \leq x \Rightarrow y \in T_0$

(ii) $x \in T_1 \Leftrightarrow x \geq \bar{x}$; $x \in T_1$ and $y \geq x \Rightarrow y \in T_1$

(iii) $x \wedge \bar{x} \in T_0$, $x \vee \bar{x} \in T_1$

(iv) $\bar{T}_0 = T_1$, $\bar{T}_1 = T_0$

(v) F is a filter iff $\bar{F}$ is an ideal.

(vi) $\overline{I(A)} = F(\overline{A})$. $\overline{F(A)} = I(\overline{A})$

(vii) $\bar{I}_0 = F_1$, $\bar{F}_1 = I_0$

(viii) L is regular iff $T_0 \cap T_1 = \emptyset$

(ix) Filter F is consistent iff $F \cap T_0 = \emptyset$

(x) If F is a truth filter, then F is prime.

(xi) If F is a truth filter, then $F \supseteq F_1$ and $F \cap I_0 = \emptyset$.

PROOF:

(i), (ii), (iii) and (iv) are simple applications of definitions and axioms of De Morgan lattice.

(v) follows from the fact that $^-$ is an isomorphism from De Morgan lattice $(L, \leq, ^-)$ to De Morgan lattice $(L, \geq, ^-)$.

(vi) Because of the equivalence $x \leq a_1 \vee \cdots \vee a_n \Leftrightarrow \bar{x} \geq \bar{a}_1 \wedge \cdots \bar{a}_n x$ belongs to $I(A)$ iff $\bar{x}$ belongs to $F(\bar{A})$. The second equality is dual to the first.

(vii) is a consequence of (vi), if we take $A = T_0$ using (iv).

(viii) $T_0 \cap T_1 = \emptyset \Leftrightarrow \neg(\exists x \in L)\ (x \in T_0 \text{ and } x \in T_1)$

because of (i), (ii) $\Leftrightarrow \neg(\exists x \in L)\ (x \leq \bar{x} \text{ and } x \geq \bar{x})$

$\Leftrightarrow \neg(\exists x \in L)\bar{x} = x$

$\Leftrightarrow$ L is regular.

(ix) $F \cap T_0 = \emptyset \Leftrightarrow \neg(\exists x \in L)\ (x \in T_0 \text{ and } x \in F)$

because of (i): $\Leftrightarrow \neg(\exists x \in L)\ (x \leq \bar{x} \text{ and } x \in F)$

$\Leftrightarrow \neg(\exists x \in L)\ (x \leq \bar{x} \text{ and } \bar{x} \in F \text{ and } x \in F)$

$\Leftrightarrow F$ is consistent.

(x) Suppose not. Then there exists a truth filter F and $x, y \in L$ such that $x \wedge y \in F$ and $x \notin F$ and $y \notin F$. As a truth filter, F is also complete, so $\bar{x} \in F$ and $\bar{y} \in F$. Because F is a filter, we have $\bar{x} \wedge \bar{y} \in F$; but $\bar{x} \wedge \bar{y} = \overline{x \vee y}$, so $\overline{x \vee y} \in F$ which contradicts the consistency of F.

(xi) Let $p \in T_1$. By the definition of T_1, $p = p \vee \bar{p}$. Because of the completeness of the truth filter F, we have $p \in F$ or $\bar{p} \in F$. As F is a filter we have $p \vee \bar{p} \in F$, so $p \in F$. Consequently $T_1 \subseteq F$. Because F is a filter, it contains a filter generated by T_1, so $F_1 \subseteq F$. Also, F is consistent, so $\bar{F}_1 \cap F = \emptyset$. From (vii) we have $\bar{F}_1 = I_0$; hence $I_0 \cap F = \emptyset$.

DEF 4 Let $p \in T_1$, define $f_p : L \to L$ as

$$f_p(x) = p \wedge (\bar{p} \vee x), \text{ for } x \in L.$$

By the above definition, a class of mapping is introduced in L. Next lemma describes some usefull properties of these mappings.

LEMMA 3. Let $p, q \in T_1$ and $x \in L$, then:
(i) $f_p(x) = \bar{p} \vee (p \wedge x)$
(ii) f_p is a homomorphism from L to L.
(iii) $f_p(L) = [\bar{p}, p]$
(iv) $f_p \circ f_p = f_p$
(v) $f_p \circ f_q = f_{f_{p(q)}}$
(vi) $f_p = f_q \Rightarrow p = q$
(vii) $x \in T_0 \Rightarrow f_p(x) \in T_0$; $x \in T_1 \Rightarrow f_p(x) \in T_1$
(viii) $x \in I_0 \Rightarrow f_p(x) \in I_0$; $x \in F_1 \Rightarrow f_p(x) \in F_1$
(ix) $f_p(p \wedge x) = f_p(x)$, $f_p(p \vee x) = p$
$f_p(\bar{p} \vee x) = f_p(x)$, $f_p(\bar{p} \wedge x) = \bar{p}$.

PROOF:
(i) Because of $p \in T_1 \Leftrightarrow p = p \vee \bar{p} \Leftrightarrow \bar{p} = \bar{p} \wedge p$ we have:

$$f_p(x) = p \wedge (\bar{p} \vee x) = (p \wedge \bar{p}) \vee (p \wedge x) = \bar{p} \vee (p \wedge x).$$

(ii) $f_p(x \vee y) = p \wedge (\bar{p} \vee x \vee y) = p \wedge (\bar{p} \vee x \vee \bar{p} \vee y)$
$= (p \wedge (\bar{p} \vee x)) \vee (p \wedge (\bar{p} \vee y)) = f_p(x) \vee f_p(y)$
$f_p(\bar{x}) = p \wedge (\bar{p} \vee \bar{x}) = p \wedge (\overline{p \wedge x}) = \overline{\bar{p} \vee (p \wedge x)} = \overline{f_p(x)}$
$f_p(x \wedge y) = f_p\overline{(\bar{x} \vee \bar{y})} = \overline{f_p(x)} \vee \overline{f_p(y)} = f_p(x) \wedge f_p(y)$
(iii) $\bar{p} \leq \bar{p} \vee (p \wedge x) = p \wedge (\bar{p} \vee x) \leq p \Rightarrow \bar{p} \leq f_p(x) \leq p \cdots (1)$
$x \in [\bar{p}, p] \Rightarrow \bar{p} \leq x \leq p \Rightarrow \bar{p} \vee x = x \Rightarrow p \wedge (\bar{p} \vee x) = p \wedge x = x.$
So, for $x \in (\bar{p}, p]$ we have $f_p(x) = x$; together with (1) it implies (iii)
(iv) $f_p(f_p(x)) = f_p(p \wedge (\bar{p} \vee x)) = \bar{p} \vee (p \wedge (p \wedge (\bar{p} \vee x)))$
$= \bar{p} \vee (p \wedge (p \wedge (\bar{p} \vee x)) = (\bar{p} \vee p) \wedge (\bar{p} \vee \bar{p} \vee x)$
$= p \wedge (\bar{p} \vee x) = f_p(x)$
(v) $F_{f_{p(q)}}(x) = \overline{f_p(q)} \vee (f_p(q) \wedge x) = f_p(\bar{q}) \vee (f_p(q) \wedge x)$
$= (p \wedge (p \wedge (\bar{p} \vee \bar{q})) \vee ((\bar{p} \vee (p \wedge q)) \wedge x)$
$= (p \wedge \bar{p}) \vee (p \wedge \bar{q}) \vee (\bar{p} \wedge x) \vee (p \wedge q \wedge x)$
$= (\bar{p} \wedge (\bar{p} \wedge x)) \vee (p \wedge \bar{q}) \vee (p \wedge q \wedge x)$
$= \bar{p} \vee (p \wedge \bar{q}) \vee (p \wedge q \wedge x)$
$= \bar{p} \vee (p \wedge (\bar{q} \vee (q \wedge x)))$
$= f_p(\bar{q} \vee (q \wedge x)) = f_p(f_q(x))$
$= f_p \circ f_q(x)$

(vi) $f_p = f_q \Rightarrow f_p(p \vee q) = f_q(p \vee q)$

$\Rightarrow p \wedge (\bar{p} \vee p \vee q) = q \wedge (\bar{q} \vee p \vee q)$

$\Rightarrow p \wedge (p \vee q) = q \wedge (p \vee q)$

$\Rightarrow p = q$

(vii) $x \in T_0 \Rightarrow x = x \wedge \bar{x}$

$\Rightarrow f_p(x) = f_p(x) \wedge \overline{f_p(x)}$ (f_p is an homomorphism)

$\Rightarrow f_p(x) \in T_0$.

Similarily, $x \in T_1 \Rightarrow f_p(x) \in T_1$

(viii) $x \in I_0 \Rightarrow x \leq q_1 \vee \cdots \vee q_n$, for some $q_1, \ldots, q_n \in T_0$

$\Rightarrow f_p(x) \leq f_p(q_1) \vee \cdots \vee f_p(q_n)$.

Last inequality holds because f_p is an homomorphism. Using (vii) we have $f_p(q_i) \in T_0$, hence $f_p(x) \in I_0$. Second implication in (viii) is dual to the first.

(ix) is a simple consequence of the definition of f_p and (i).

Now, we are ready to prove:

THEOREM 1. *Let $F \subset L$ be a filter in L. F is contained in some truth filter T in L iff $(F \wedge T_1) \subset T_0 = \emptyset$.*

PROOF:

$(\Rightarrow)$

Suppose not. Then $(F \wedge T_1) \cap T_0 \neq \emptyset$ and there exists a truth filter $T (\supseteq F)$ in L. Because the intersection of sets above is non-empty we have:

$$(\exists a \in F)(\exists p \in T_1) a \wedge p \in T_0. \tag{1}$$

But, because of (xi) Lemma 2, T as a truth filter contains T_1, so $p \in T$. Hence, $p \wedge a \in T$ which implies $T \cap T_0 \neq \emptyset$ which contradicts (xi) Lemma 2.

$(\Leftarrow)$

Suppose $(F \wedge T_1) \cap T_0 = \emptyset$. We will prove the existence of a truth filter containing F in a several steps.

1° *The filter $F_1 = F(F \cup T_1)$ is consistent*

Suppose the opposite – i.e. F_1 is inconsistent. That means that there exists $b, \bar{b} \in F_1$, so, for some $b, b \wedge \bar{b} \in F_1$. As F_1 is a filter generated by $F \cup T_1$ we have, for some $a_1, \ldots, a_k \in F$ and $p_1, \ldots, p_m \in T_1$:

$$a_1 \wedge \cdots \wedge a_k \wedge p_1 \wedge \cdots \wedge p_m \leq b \wedge \bar{b}. \tag{2}$$

F is a filter, so $a = a_1 \wedge \cdots \wedge a_k \in F$ and $b \wedge \bar{b} \in T_0$, therefore

$$a \wedge p_1 \wedge \cdots \wedge p_m \in T_0. \tag{3}$$

Applying the f_{p_m} onto (3) and using the fact that f_{p_m} is a homomorphism which preserves T_0, we obtain:

$$f_{p_m}(a) \wedge f_{p_m}(p_1 \wedge \cdots \wedge p_{m-1} \wedge p_m) \in T_0. \tag{4}$$

Because of $f_p(p \wedge x) = f_p(x)$ ((ix) Lemma 3) (4) becomes:

$$f_{p_m}(a) \wedge f_{p_m}(p_1) \wedge \cdots \wedge f_{p_m}(p_{m-1}) \in T_0. \tag{5}$$

As f_p preserves T_1 ((vii) Lemma 3) (5) becomes

$$f_{p_m}(a) \wedge p_1^1 \wedge \cdots \wedge p_{m-1}^1 \in T_0, \ \ p_i^1 \in T_1. \tag{6}$$

Finally, we have that the number of units in the (3) is decreased by one in (6). Repeating the procedure above $m-1$ times we will obtain:

$$f_{p_1^m}(f_{p_2^{m-1}} \cdots (f_{p_m^1}(a)) \cdots) \in T_0. \tag{7}$$

Because of (v) and (vii) from Lemma 3, a composition of f_p mappings is one f_p mapping so, for some $p \in T_1$,

$$f_p(a) \in T_0. \tag{8}$$

Because of $p \wedge a \leq \bar{p} \vee (p \wedge a) = f_p(a)$, (8) implies

$$p \wedge a \in T_0$$

wich condraticts $(F \wedge T_1) \cap T_0 = \emptyset$.

$2°$ *The set of all consistent filters containing F_1 has a maximal element.*

Let $\mathcal{G}_{F_1} = \{G \supseteq F_1 \mid G \text{ is a consistent filter in } L\}$.

Because of $1°$, F_1 is consistent, so $F_1 \in \mathcal{G}_{F_1}$, hence $\mathcal{G}_{F_1} \neq \emptyset$.

Let $\mathcal{L}$ be chain in $\mathcal{G}_{F_1}$, and let $C = \cup\mathcal{L}$. As a union of filter, C is a filter. C is also consistent, because $x, \bar{x} \in C$ implies $x \wedge \bar{x} \in C$, so for some $G \in \mathcal{L}, x \wedge \bar{x} \in G$ which contradicts a consistency of G.

As $\mathcal{G}_{F_1}$ is non-empty and closed over the unions of chains it fulfills the conditions for the application of Zorn's Lemma. Hence, $\mathcal{G}_{F_1}$ has a maximal elements. Let us denote one of them by T.

$3°$ *T is a truth filter and contains F*

As T belongs to $\mathcal{G}_{F_1}$, it is consistent and contains F. To complete the proof it suffices to prove that T is complete.

Suppose not. Then, there exists $a, \bar{a} \in L$ such that $a \notin T$, $\bar{a} \notin T$. Because of that, the filters $F(T \cup \{a\})$ and $F(T \cup \{\bar{a}\})$ are different from T and greater than T. Hence, as T is a maximal consistent filter containing F_1, these filters must be inconsistent - i.e. because of (ix) Lemma 2, some zeroes belongs to them. So, we have:

$$(9) \qquad (\exists q_1, q_2 \in T_0)(q_1 \in F(T \cup \{a\}) \text{ and } q_2 \in F(T \cup \{\bar{a}\}))$$

Using the definition of the notion "filter generated by..." (9) becomes:

$$(10) \qquad (\exists q_1, q_2 \in T_0)(\exists t_1, t_2 \in T)(t_1 \wedge a \leq q_1 \text{ and } t_2 \wedge \bar{a} \leq q_2).$$

From (10) we infer that for some $q_1, q_2 \in T_0$ and $t_1, t_2 \in T$:

$$(11) \qquad t_1 \wedge t_2 \wedge a \leq q_1 \text{ and } t_1 \wedge t_2 \wedge \bar{a} \leq q_2.$$

From (11) follows:

$$(12) \qquad t_1 \wedge t_2 \wedge (a \vee \bar{a}) \leq q_1 \vee q_2.$$

$a \vee \bar{a} \in T_1 \subseteq F_1 \subseteq T$, so $a \vee \bar{a} \in T$. Hence, the left side of inequality (12) belongs to T and, consequently, right side belongs to T. That is, $q_1 \vee q_2 \in T$. T is consistent filter; so, $\overline{q_1 \vee q_2} = \bar{q}_1 \wedge \bar{q}_2 \notin T$. But, $\bar{q}_1, \bar{q}_2 \in T_1 \subseteq F_1 \subseteq T$ and $\bar{q}_1 \wedge \bar{q}_2 \in T$. Contradiction.

Some other related statements on truth filters follows easilly from Theorem 1.

Corollary 1. 1. *Let $a \in L$. There there exists a truth filter T such that $a \in T$ iff $(a \wedge T_1) \cap T_0 = \emptyset$.*

Proof: There exists a truth filter containing a if and only if there exists a truth filter containing a filter generated by $\{a\}$. But $F_a = F(\{a\}) = \{x \in L \mid x \geq a\}$. So, $(a \wedge T_1) \cap T_0 = \emptyset$ is equivalent to $(F_a \wedge T_1) \cap T_0 = \emptyset$. The last is, according to Theorem 1, equivalent to the existence of fruth filter containing F_a and a.

Corollary 1.2. *Let $a, b \in L$ such that $\neg a \leq b$. There exists a truth filter containing a and not containing b iff $(a \wedge \bar{b} \wedge T_1) \cap T_0 = \emptyset$.*

Proof: Because of completeness of any truth filter, the filter with desired property fulfills $a \in T$ and $\bar{b} \in T$. As T is a filter $a \wedge \bar{b} \in T$ holds. So, there exists a truth filter containing a and not containing b iff there exists a truth filter containing $a \wedge \bar{b}$. The last is, according to Corollary 1.1 equivalent to

$$(a \wedge \bar{b} \wedge T_1 \cap T_0 = \emptyset.$$

Lemma 4. *Let $p \in T_1$ and $x \in L$, then:*

$$p \wedge x \in T \text{ iff } f_p(x) \in T_0.$$

Proof: $p \wedge x \in T_0 \Rightarrow f_p(p \wedge x) = f_p(x) \in T_0$ (Lemma 3 (vvii), (ix))

$$f_p(x) \in T_0 \Rightarrow \bar{p} \vee (p \wedge x) \in T_0 \Rightarrow p \wedge x \in T_0.$$

Corollary 1.3. *There exists a truth filter T in a De Morgan lattice L iff L is regular.*

Proof:

($\Rightarrow$) If there exists a truth filter in a De Morgan lattice L, then $(\forall x \in L)(x \in T \Leftrightarrow \bar{x} \notin T)$ so x and $\bar{x}$ must be defferent, i.e. L is regular.

($\Leftarrow$) Let L be a regular De Morgan lattice and let $p \in T_1$. If $(p \wedge T_1) \cap T_0 \neq \emptyset$ then $p \wedge q \in T_0$ for some $q \in T_1$. According to Lemma 4, the last is equivalent to $f_p(q) \in T_0$. But, because of Lemma 3 (vii) $f_p(q) \in T_1$ also holds. So, $T_0 \cap T_1 \neq \emptyset$ which, because of Lemma 2 (viii) contradicts the regularity of L. Hence, $(p \wedge T_1) \cap T_0 \neq \emptyset$ – i.e. there exists a truth filter containing p.

So, the Belnap and Spenser theorem mentioned at the begining of this paper is proved as a corollary to the Theorem 1.

Theorem 2. $a \in I_0 \Rightarrow (a \wedge T_1) \cap T_0 \neq \emptyset$.

Proof: Because of Lemma 2 (xi) $T \cap I_0 \neq \emptyset$ for any truth filter T. So, there is not a truth filter containing $a \in I_0$. According to Corallary 1.1 the last is equivalent to $(a \wedge T_1) \cap T_0 \neq \emptyset$.

Therefore, we have proved Theorem 2 as a corollary to theorem 1. But, the proof of theorem 1 uses a Zorn's Lemma which is not necessary for the proof of Theorem 2. We are giving a straightforvard proof:

Let $a \in I_0$ – i.e. there are $q_1, \dots, q_n \in T_0$ such that

$$a \leq q_1 \vee \cdots \vee q_n. \tag{1}$$

Applying $f_{\bar{q}_n}$ onto (1) we will (as in the step 1° of the proof of Theorem 1 – which does not uses a Zorn's Lemma) obtain:

$$f_{\bar{q}_n}(a) \leq f_{\bar{q}_n}(q_1) \vee \cdots \vee f_{\bar{q}_n}(q_{n-1}). \tag{2}$$

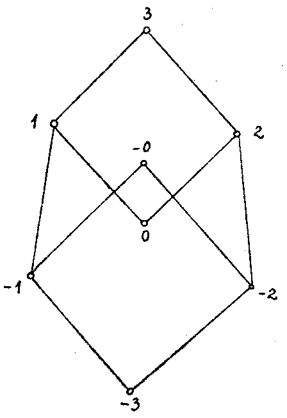
$$f_{p1}(\cdots(f_{p_n}(a))\cdots) \leq f_p(q), \tag{3}$$

where $p, p_1, \dots, p_n \in T_1$ and $q \in T_0$. The composition of the f_p mappings is one f_p mappings and $f_p(q)$ belongs to T_0 so (3) becomes:

$$f_r(a) \in T_0, \text{ for some } r \in T_1. \tag{4}$$

The last is equivalent to

$$(a \wedge T_1) \cap T_0 \neq \emptyset.$$



$\bar{a} = -a$

The theorem above leads to a natural question: Does the converse holds? – i.e. Is $a \in I_0$ equivalent to $(a \wedge T_1) \cap T_0 \neq \emptyset$? It is obvious that this equivalence will simplify the criteria for existence of a truth filter containing a. But, the answer is negative – i.e. the implication in Theorem 2 is “proper”. We will prove this with a following example:

Let M_0 be a following regular De Morgan lattice:

M_0 has some important properties (see, for example, Anderson and Belnap [1975]), but, anyway M_0^ω is a regular De Morgan lattice as a direct product of ω regular De Morgan lattices. Let L be a complete sublattice of M_0 generated by $e_1 = (2, 1, 1, \dots)$, $e_2 = (2, 2, 1, \dots)$, $e_3 = (1, 1, 2, \dots), \dots$ i.e. e_k is a sequence of 1 exept that 2 is on the k-th place.

$(0, 0, 0, \dots) \in L$ because of $(0, 0, 0, \dots) = \wedge_{k \in \omega} e_k$. So, $(-0, -0, -0, \dots) \in L$. $(-0, -0, -0, \dots) \wedge e_k = -e_k$ which means that $(-0, -0, -0, \dots)$ fulfills condition $((-0, -0, -0, \dots) \wedge T_1) \cap T_0 \neq \emptyset$; but $(-0, -0, -0, \dots) \notin I_0$.

Let us mention that $a \in I_0$ and $(a \wedge T_1) \cap T_0 \neq \emptyset$ are equivalent in any finitely generated De Morgan lattice.

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