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# ON OSCILLATION FUNCTION OF ONE CLASS OF STOCHASTIC PROCESSES

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0. Let $X = \{X(t),\ 0 \leq t \leq 1\}$ be a stochastic process of second order, i.e. a process for which the inequality $\|X(t)\| < \infty$ holds for any $t$, where the norm of arbitrary random variable $z$ is defined by $\|z\| = (z, z)^{1/2} = (E|z|^2)^{1/2}$. By *the convergence* of a sequence of random variables we mean the convergence in the norm, i.e. the convergence in quadratic mean. We say that the *left (right) limit of* $X$ *at* $t$ exists if there exists a random variable $X(t-0)$ $(X(t+0))$, such that $X(t-0) = \underset{u\to t-0}{\text{l.i.m.}} X(u)$ $(X(t+0) = \underset{u\to t+0}{\text{l.i.m.}} X(u))$. If at least one of the equalities $X(t-0) = X(t) = X(t+0)$ do not hold, we say that $X$ has *the discontinuity of the kind at* $t$. If at least one of limits $X(t-0)$, $x(t+0)$ do not exist, we say that $X$ has *the discontinuity of the second kind at* $t$; if only $X(t-0)$ $(X(t+0))$ does not exist, then we say that $X$ has *the left (right) discontinuity of the second kind at* $t$.

In the following we shall suppose, without loss of generality, that, if for some $t$ there exists only one of limits $X(t-0)$, $X(t+0)$, then it is equal to $X(t)$, and if there exist the both limits $X(t-0)$ and $X(t+0)$, then the eqality $X(t-0) = X(t)$ is satisfied. We shall say that $X$ is *mean square continuous from the left (right) at* $t$ if the eqality $X(t-0) = X(t)$ $(X(t) = X(t+0))$ holds. The process $X$ is *mean square cotninuos from the left (right)* if it is mean square continuous from the left (right) at any $t$.

Let us define the function $\omega = \omega(t)$ by

$$\omega(t) = \sup_{(t_n),(t_n{}')\in\Gamma_t} \overline{\lim_{n\to\infty}} \|X(t_n) - X(t_n{}')\|, \qquad t \in [0;1], \tag{1}$$

where $\Gamma_t$ denotes the set of all sequences which converge to $t$ and whose members are from [0; 1]; the function $\omega$ we shall call *the oscillation function of the process* $X$. If the set $[0;\ 1] \cap [t-h; t+h]$ we denote by $i_{t,h}$, then it is easy to show that the following equality holds:

$$\omega(t) = \inf_{h>0} \sup_{u,v\in i_{t,h}} \|X(u) - X(v)\|, \qquad t \in [0;1]. \tag{2}$$

In this paper we shall prove some properties of the function $\omega$, and some statements about stochastic processes we shall prove by means of the function $\omega$; also, we shall that, in one special case, for any function $\omega$ of fixed properties there exists a stochastic process (not unique) whose oscillation function is just equal to given function $\omega$.

1. It is evident that the equality $\omega(t) = 0$ holds if and only if $X$ is mean square continuous at $t$. The following lemma contains the proposition which is well known for real function, [3].

LEMMA 1. *The function $\omega$ is an upper semi-continuous function.*

PROOF. Let $t$ be arbitrary point from $[0;1]$ and $\omega(t) = s > 0$. In order the function $\omega$ to be upper semi-continuous at $t$ it is necessary and sufficient that for any $\varepsilon > 0$ there $\delta > 0$, such that the inequality $\omega(u) \le s + \varepsilon$ holds for each $u \in i_{t,\delta}$. Let us suppose that this is not the case, that is that there is $\varepsilon_0 > 0$, such that for each $\delta > 0$ there is at least one $u \in i_{t,\delta}$ for which the inequality $\omega(u) > s + \varepsilon_0$ is satisfied. This means that, on at least one side of $t$, there is a sequence $(u_n)$, converging to $t$, whose members have the property

$$\omega(u_n) > s + \varepsilon_0, \qquad n = 1, 2, \ldots;$$

that implies, by reason of the definition (1), that for each $n$ there are $u_n{}'$, $u_n{}''$ $(u_n{}', u_n{}'' \in i_{t,3|t-u_n|/2})$, such that

$$\|X(u_n{}') - X(u_n{}'')\| > s + \varepsilon_0/2,$$

which gives as a consequence

$$\overline{\lim_{n\to\infty}} \|X(u_n{}') - X(u_n{}'')\| \ge s + \varepsilon_0/2,$$

which contradicts the assumption $\omega(t) = s$.

COROLLARY 1.1. *The set $D_s = \{t : \omega(t) \ge s\}$ is closed for any $s \ge 0$,* [3].

COROLLARY 1.2. *The function $\omega$ is continuous at all points at which it is equal to zero.*

LEMMA 2. *If $X(t_0 - 0)$ exists, then $\omega(t_0 - 0)$ exists and $\omega(t_0 - 0) = 0$.*

PROOF. Let $(t_n)$ be an arbitrary increasing sequence converging to $t_0$; we are going to show that $\omega(t_n) \to 0$ when $n \to \infty$. For each $\varepsilon > 0$ there is $h_\varepsilon > 0$, such that the inequality

$$\|X(u) - X(v)\| < \varepsilon \tag{3}$$

ia true for all $u$, $v \in (t_0 - h_\varepsilon; t_0)$; let us denote by $k_\varepsilon$ the smallest natural number such that $t_{k_\varepsilon} \in (t_0 - h_\varepsilon; t_0)$. From (3) it follows that for arbitraty sequences $(t'_{k,n})$, $(t''_{k,n})$ from $\Gamma_{t_k}$ it will be

$$\overline{\lim_{n\to\infty}} \|X(t'_{k,n}) - X(t''_{k,n})\| \le \varepsilon \text{ for each } k \ge k_\varepsilon,$$

which is equivalent to the fact that $\omega(t_k) \to 0$ when $k \to \infty$. As the same conclusion holds for each sequence increasingly converging to $t_0$, our lemma is proved.

LEMMA 3. *If the process $X$ is mean square continuous from the left on everywhere dense set $E$,* Leb $(E) = 1$, *then for each $\varepsilon > 0$ there exists a set $C \subset$* $[0;1]$, Leb $(C) \geq 1 - \varepsilon$, *such that $X$ is mean square continuous on $C$.*

PROOF. From the fact that the function $\omega$ is measurable [2], it follows that for any $\varepsilon > 0$ there is a continuous function $\omega_c$, such that [2]

$$\mathrm{Leb}\,(\{t\colon \omega(t) = \omega_c(t)\}) \geq 1 - \varepsilon;$$

put $C = \{t : \omega(t) = \omega_c(t)\}$. Since $\omega_c(t-0) = 0$ for all $t \in C \cap E$, and the function $\omega_c$ is continuous, it follows that $\omega_c(t) = 0$ for all $t \in C \cap E$. But, as the set $C \cap E$ is dence in $C$, this implies that the equality $\omega_c(t) = 0$ holds for each $t \in C$, which means that $X$ is mean square continuous on $C$, as we wanted to prove.

Let us denote by $\Gamma_t^+$ the set of all sequences which decreasingly converge to $t$, and by $\omega^+ = \omega^+(t)$ the function defined by

$$\omega^+(t) = \sup_{(t_n),(t_n{}')\in\Gamma_t^+} \overline{\lim_{n\to\infty}} \|X(t_n) - X(t_n{}')\|, \quad t \in [0;1). \tag{4}$$

It is easy to see that the equality $\omega^+(t) = 0$ holds if and only if $X(t+0)$ exists, which immediately implies the inequality

$$\omega^+(t) \leq \omega(t) \ \text{ for each } \quad t \in [0;1). \tag{5}$$

The function $\omega^+$ we shall call the right oscillation function of $X$.

THEOREM 1. *Suppose that the process $X$ is mean square continuous from the left everywhere except at some set $D^-$, which is at most countable. Then the following statements are true:*

I. *The process $X$ has at most countably many right discontinuities of the second kind.*

II. *The set $D_s^+ = \{t\colon \omega^+(t) \geq s\}$ is nowhere dense for any $s > 0$.*

PROOF. I. This statement is equivalent to the statement that the set $D^+ = \{t\colon \omega^+(t) > 0\}$ is at most contable. Let us suppose that this is not true, i.e. that

$$\mathrm{card}\,(D^+) = \aleph_1. \tag{6}$$

This implies that there is $s > 0$, such that

$$\mathrm{card}\,(D_s^+) = \aleph_1; \tag{7}$$

for, if the contrary is the case, i.e. if $\mathrm{card}\,(D_s^+) \leq \aleph_0$ for any $s > 0$, then the set $D^+ = \cup_{n=1}^{\infty} D_{1/n}^+$ is also at most countable, contary to the hypothesis (6). Let

$s = s_0$ be one of values for which (7) is true. Since, by reason of Corollary 1.1, the set $D^+_{s0}$ is closed (namely, we can show, by the procedure which is similar to that from Lemma 1, that the function $\omega^+$ is upper semi-continuous), it has to contain one perfect subset $P_{s_0}$, such that $\mathrm{card}\,(P_{s_0}) = \aleph_1$, [3]. From the assumption $\mathrm{card}\,(D^-) \le \aleph_0$ it follows $\mathrm{card}\,(D^- \cap P_{s_0}) \le \aleph_0$, which means that there are at most countably many values $t$ for which the inequalities $\overline{\omega^+(t-0)} \ge s_0$ hold; this implies, for the set $P_{s_0}$ is perfect and $\mathrm{card}\,(P_{s_0}) = \aleph_1$, that $\mathrm{card}\,(\{t \colon \overline{\omega^+(t+0)} \ge s_0\}) = \aleph_1$. But, that means that there are continuously many values $t$ for which the inequalities $\overline{\omega^+(t-0)} \ne \overline{\omega^+(t+0)}$ hold, which is impossible, [3]. Hence, it must be $\mathrm{card}\,(D^+_s) \le \aleph_0$ for any $s > 0$, that is $\mathrm{card}\,(D^+) \le \aleph_0$.

II. Let us suppose that the statement does not hold, i.e. that, for some $s > 0$, there are $t_0 \in D^+_s$ and $h > 0$, such that in the neighbourhood $i_{t_0,h}$ to $t_0$ there is no interval whose all points are from the complement $\bar{D}^+_s$ of the set $D^+_s$; hence, the set $D^+_s \cap i_{t_0,h}$ is dense in $i_{t_0,h}$. From that, and from the fact that the set $D^+_s$ is closed, it follows that $i_{t_0,h} \subset D^+_s$, which contradicts the statement from. I. Thus the proof is completed.

It is clear that the result from I is stronger than the statement (i) from [1].

Note that in proofs of statement, in which the mean square continuity from the left of the process $X$ is presupposed, only the assumption about the existence of left limits of $X$ is used.

2. We showed that any stochastic process, mean square continuous from the left, uniquely determines a non-negative function $\omega^+$ with the following properties:

(a) $\omega^+$ is upper semi-continuous function;

(b) $\omega^+(t-0) = 0$ for any $t \in (0;1]$;

(c) $\mathrm{card}\,(D^+) \le \aleph_0$;

(d) the set $D^+_s$ is nowhere dense for any $s > 0$.

The natural question is: if $\omega_0$ is arbitrary non-negative function with the above properties (a)–(d), does there always exist a process $X$, whose function $\omega^+$, defined by (4), satisfies the equality

$$\omega^+(t) = \omega_0(t) \text{ for each } t.$$

If we were to answer that question, we need some preliminary results.

Lemma 4. *Suppose that a non-negative upper semi-continuous function $\omega_0$, defined on $[0;\,1]$, satisfies the condition $\omega_0(t-0) = 0$ for all $t \in (0;1]$. If the set $D = \{t \colon \omega_0(t) > 0\}$ is at most countable and nowhere dense, then there exists a process $X$, whose right oscillation function satisfied the equality*

$$\omega^+(t) = \omega_0(t) \text{ for each } t. \tag{8}$$

Proof. First of all we shall show that for each $u \in [0;1)$ and any $s > 0$ there exists a process $X_{u,s}$, whose right oscillation function $\omega^+_{u,s}$ is defined by

$$\omega^+_{u,s}(t) = \begin{cases} s & \text{for } t = u, \\ 0 & \text{for } t \ne u. \end{cases} \tag{9}$$

Really, if $W = \{W(t), 0 \leq t \leq 1\}$ is Brownian motion process (i.e. process such that $P\{W(0)=0\}=1$, and for all $t,s \in [0;1]$ the random variable $W(t)-W(s)$ has the probability distribution $\mathcal{N}(0, \mid t-s \mid))$, and if the process $X_{u,s}$ is defined by

$$X_{u,s}(t) = \begin{cases} 0, & t \leq u, \\ s \cdot W\left(\frac{1}{2}\left(\sin\frac{1}{t-u}+1\right)\right), & t > u, \end{cases} \tag{10}$$

then the oscillation function $\omega^+_{u,s}$ of $X_{u,s}$ has the form (9).

Put $D = \{t_1, t_2, \dots\}$. For any $t_i \in D$, because the set $D$ is nowhere dense, it can be contructed a sequence of intervals $(a_{i,k}; b_{i,k}]$ $k = 1, 2, \dots$, with the following properties (compare with [4]):

1. $(a_{i,k}; b_{i,k}]$ does not contain points from $D$, $k = 1, 2, \dots$;
2. $a_{i,k} > t_i$ for all $k = 1, 2, \dots$;
3. $(a_{i,k}; b_{i,k}] \cap (a_{i,j}; b_{i,j}] = \emptyset$ for all $j, k = 1, 2, \dots$ and $j \neq k$;
4. $b_{i,k} \to t_i$ when $k \to \infty$;

for the sequence of intervals with the above properties we say that conveges to $t_i$ (it is clear that it converges descreasingly). These convergent sequences can be contructed so that

$$\bigcap_{i(t_i \in D)} \bigcup_{k=1}^{\infty} (a_{i,k}; b_{i,k}] = \emptyset.$$

Let $Z$ be a process, defined on $[0;1]$, continuous on $(0;1]$, and such that its right oscillation at $t=0$ is $\omega^+_Z(0) = 1$ (we can, for example, put $Z(t) = X_{0,1}(t)$, $0 \leq t \leq 1$, where the process $X_{0,1}$ is defined by (10) for $u=0$ and $s=1$). Put $T_i = \cup_{k=1}^{\infty}(a_{i,k}; b_{i,k}]$, $i = 1, 2, \dots$, and the process $X_i$, $i = 1, 2, \dots$, define by

$$X_i(t) = \begin{cases} 0, & t \in \bar{T}_i, \\ \omega_0(t_i) Z\left(\frac{t-t_i}{1-t_i}\right), & t \in T_i. \end{cases}$$

Finally, if the process $X$ is defined by

$$X(t) = \begin{cases} 0, & t \in \overline{\cup_i T_i}, \\ X_i(t), & t \in T_i, \end{cases} \tag{11}$$

then it is easy to see that the right oscillation function $\omega^+$ of $X$ satisfies (8). The proof is completed.

It can happen that $X$ has discontinuities of the first kind on the ends of intervals $(a_{i,k}; b_{i,k}]$ for some or all values of indices $i, k$. Let us show it is possible to contruct a process $X$, which has no discontinuities of the first kind, and whose right oscillation function $\omega^+$ satisfies (8). Suppose that on $[0;1]$ a mean square continuous process $Z$ is defined, such that $P\{Z(0)=0\}=P\{Z(1)=0\}=1$ and $\max_{0\le t\le 1}\|Z(t)\|=1$. By using denotations from Lemma 4, we can define the process $X_i^*$, $i=1,2,\dots$, by

$$X_i^*(t)=\begin{cases} 0, & t\in \bar{T}_i \\ \omega_0(t_i)Z\left(\dfrac{t-a_{i,k}}{b_{i,k}-a_{i,k}}\right), & t\in (a_{i,k};b_{i,k}], \quad k=1,2,\dots \end{cases}$$

If in (11) we exchange $X_i$ by $X_i^*$ for $i=1,2,\dots$, we shall see that so obtained process $X$ has no discontinuities of the first kind and that its right oscillation function $\omega^+$ satisfies (8).

COROLLARY 4.1. *Let $\omega_0$ be a non-negative function, defined on $[0;1]$, and satisfying the conditions* (a) – (d). *If the indicator function of the set $\{t\colon 0<\omega_0(t)\le \varepsilon\}$ we denote by $I_\varepsilon=I_\varepsilon(t)$, then for any $\varepsilon>0$ there exists a process $X_\varepsilon$, whose right oscillation function $\omega_\varepsilon^+$ satisfies the equality*

$$\omega_\varepsilon^+(t)=(1-I_\varepsilon(t))\omega_0(t), \quad t\in[0;1).$$

LEMMA 5. *Suppose that $X_1$ and $X_2$ are arbitrary stochastic processes of second order, and that the process $X_0$ is defined by $X_0(t)=X_1(t)+X_2(t)$, $0\le t\le 1$. If $\omega_i$ is the oscillation function of $X_i$, $i=0,1,2$, then the inequality*

$$\omega_0(t)\le \omega_1(t)+\omega_2(t), \quad 0\le t\le 1, \tag{12}$$

*holds. This inequality becomes equality if the following conditions are satisfied:*

(i) *processes $X_1$ and $X_2$ are mutually orthogonal;*

(ii) $D_1\cap D_2=\emptyset$ where $D_i=\{t\colon \omega_i(t)>0\}$, $i=1,2$.

PROOF. The inequality (12) follows immediately from the properties of norm and function $\overline{\lim}$ and sup. If the condition (i) is satisfied, then for each $t$ and arbitrary sequences $(t_n),(t_n{}')$ from $\Gamma_t$ the equality

$$\|X_0(t_n)-X_0(t_n{}')\|=\|X_1(t_n)-X_1(t_n{}')\|+\|X_2(t_n)-X_2(t_n{}')\|, \quad n=1,2,\dots,$$

holds. We shall show that, from the assumption that the condition (ii) is also satisfied, it follows

$$\overline{\lim_{n\to\infty}}\|X_0(t_n)-X_0(t_n{}')\|=\sum_{i=1}^{2}\overline{\lim_{n\to\infty}}\|X_i(t_n)-X_i(t_n{}')\|. \tag{13}$$

The condition (ii) implies that $t$ can belong te at most one of the sets $D_1,D_2$; if $t\in\overline{D_1\cup D_2}$, then the both sides in (13) are obviously equal to zero. If $t$ belongs to one of the sets $D_1,D_2$, for example $t\in D_1$, then it holds

$$\begin{aligned}&\left|\|X_0(t_n)-X_0(t_n{}')\|-\sum_{i=1}^{2}\overline{\lim_{n\to\infty}}\|X_i(t_n)-X_i(t_n{}')\|\right|\le \\ &\le\left|\|X_1(t_n)-X_1(t_n{}')\|-\overline{\lim_{n\to\infty}}\|X_1(t_n)-X_1(t_n{}')\|\right|+\|X_2(t_n)-X_2(t_n{}')\|.\end{aligned}\tag{14}$$

From the definition of $\overline{\lim}$ and the fact that $t \in \bar{D}_2$ it follows that the right side in (14) will be smaller than arbitrary $\varepsilon > 0$ for infinitely many values of $n$. Thus we proved that (13) is true. This implies, by reason of (ii), that the equality

$$\omega_0(t) = \omega_1(t) + \omega_2(t), \quad 0 \leq t \leq 1,$$

holds, as we wanted to prove.

COROLLARY 5.1. *If $X_1$ and $X_2$ are arbitrary processes of second order and if a process ${X_0}'$ is defined as in Lemma 5, then it holds $D_0 \subseteq D_1 \cup D_2$. That inclusion becomes equality if at least one of the conditions* (i) *and* (ii) *is satisfied.*

It is clear that, analogously, it can be shown that Lemma 5 and Corollary 5.1 remain valid also for right oscillation functions $\omega_i^+$, i.e. for corresponding sets $D_i^+$, $i = 0, 1, 2$.

LEMMA 6. *If the sequence $X_1, X_2, \ldots$ of stochastic processes converges uniformly to some process $X$, then the sequence of corresponding oscillation functions $\omega_1, \omega_2, \ldots$ converges uniformly to oscillation function $\omega$ of $X$.*

PROOF. From the uniform convergence of the sequence $(X_n)$, i.e. from

$$\sup_{0 \leq t \leq 1} \|X(t) - X_k(t)\| \to 0, \qquad n \to \infty,$$

it follows that for any $\varepsilon > 0$ there is $k_\varepsilon$ such that

$$|\|X(u) - X(v)\| - \|X_k(u) - X_k(v)\|| < \varepsilon \text{ for all } u, v \in [0; 1] \text{ and } k > k_\varepsilon;$$

that implies the following inequalities

$$\sup_{u,v \in i_{t,h}} \|X_k(u) - X_k(v)\| - \varepsilon \leq \sup_{u,v \in i_{t,h}} \|X(u) - X(v)\| \leq$$
$$\leq \sup_{u,v \in i_{t,h}} \|X_k(u) - X_k(v)\| + \varepsilon \text{ for any } t \text{ and all } k > k_\varepsilon,$$

which hold for each $h > 0$. This, by reason of (2), means that it will be

$$| \omega(t) - \omega_k(t) | \leq \varepsilon \text{ for any } t \text{ and all } k > k_\varepsilon,$$

which is equivalent to the statement that $\omega_k$ converges uniformly to $\omega$ when $k \to \infty$, as we wanted to show.

It is easy to see that the statement from Lemma 6 remains valid if we exchange the oscillation functions by the right oscillations functions.

THEOREM 2. *Suppose that $\omega_0$ is a non-negative function, defined on* $[0; 1]$ *and satisfying conditions* (a)–(d). *Then there exists a process $X$, whose right oscillation function $\omega^+$ satisfied the equality*

$$\omega^+(t) = \omega_0(t) \text{ for any } t \in [0; 1).$$

PROOF. Denote by $I_n = I_n(t)$ the indicator function of the set $\{t\colon 0 < \omega_0(t) \le 1/2^n\}$. From Corollary 4.1 it follows that for each $n = 1, 2, \dots$ there is a process $X_n$, whose right oscillation function $\omega_n^+$ satisfied the equality $\omega_n^+(t) = (1 - I_n(t))\omega_0(t)$, $t \in [0; 1)$. It is easy to see that the sequence $(\omega_n^+)$ converges uniformly to $\omega_0$. If we show that processes $X_n$, $n = 1, 2, \dots$, can be constructed in such a way that the sequence $(X_n)$ converges uniformly to some process $X$ (i.e. that $(X_n)$ is a Cauchy sequence in the sense of the uniform convergence), then, by reason of Lemma 6, it will imply that our statement is true.

Let us construct processes $X_n$, $n = 1, 2, \dots$. Put $D_1 = \left\{t\colon \omega_0(t) > \frac{1}{2}\right\}$ and define the function $\omega_1 = \omega_1(t)$ by

$$\omega_1(t) = \begin{cases} 0, & t \in \bar{D}_1, \\ \omega_0(t), & t \in D_1. \end{cases}$$

As the function $\omega_1$ satisfies all conditions from Lemma 4, it must exist a process $\bar{X}_1$, whose right oscillation function $\overline{\omega_1^+}$ satisfied the equality

$$\overline{\omega_1^+}(t) = \omega_1(t), \quad t \in [0; 1).$$

Put $D_2 = \left\{t\colon \frac{1}{4} < \omega_0(t) \le \frac{1}{2}\right\}$ and define the function $\omega_2 = \omega_2(t)$ by

$$\omega_2(t) = \begin{cases} 0, & t \in \bar{D}_2, \\ \omega_0(t), & t \in D_2. \end{cases}$$

According to Lemma 4 there is a process $\bar{X}_2$, whose right oscillation function $\overline{\omega_2^+}$ satisfies the equality

$$\overline{\omega_2^+}(t) = \omega_2(t), \quad t \in [0; 1).$$

It is clear that a process $\bar{X}_2$ can be constructed in such a way that it is orthogonal to $\bar{X}_1$, and that its norm satisfies the inequality

$$\sup_{0 \le t \le 1} \|\bar{X}_2(t)\| < 1.$$

By the described procedure we obtain the sequence of sets $D_n = \{t\colon 1/2^n < \omega_0(t) \le 1/2^{n-1}\}$, $n = 1, 2, \dots$, and corresponding sequence $(\bar{X}_n)$ of mutually orthogonal processes, whose norms satisfy the inequalities

$$\sup_{0 \le t \le 1} \|\bar{X}_n(t)\| < \frac{1}{2^{n-2}}, \quad n = 2, 3, \dots. \tag{15}$$

The new processes $X_n$, $n = 1, 2, \dots$, we shall define by

$$X_n(t) = \sum_{k=1}^{n} \bar{X}_k(t), \quad t \in [0; 1], \ n = 1, 2, \dots.$$

Since the process $X_n$, for any $n = 1, 2, \ldots$, satisfies the conditions (i) and (ii) from Lemma 5, it follows that for the right oscillation function $\omega_n^+$ of $X_n$ the equality

$$\omega_n^+(t) = \sum_{k=1}^{n} \overline{\omega_k^+}(t), \quad t \in [0; 1),$$

will be satisfied. From the definition of $\omega_k^+$, i.e. of $\omega_k^+$, $k = 1, 2, \ldots$, it follows that

$$\omega_n^+(t) = (1 - I_n(t))\omega_0(t), \;\; t \in [0; 1), \quad n = 1, 2, \ldots .$$

For arbitrary natural numbers $n$ and $m$ (we can suppose that, for example, $n > m$) it will be, by reason of mutual orthogonality of processes $\bar{X}_k$, $k = 1, 2, \ldots$, and by reason of (15),

$$\|X_n(t) - X_m(t)\| \leq \sum_{k=m+1}^{n} \frac{1}{2^{k-2}} \to 0, \quad n, m \to \infty,$$

which means that the sequence $(X_n)$ converges uniform $y$ to some process $X$. The proof is completed.

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