PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 28 (42), 1980, pp. 51–59

ON THE RADICALS OF NEAR-RINGS WITH DEFECT OF DISTRIBUTIVITY

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In the near-right with defect of distributivity as well as in distributively generated (d.g.) near-rings, beside the radicals J_0, J_1 and J_2, other Jacobson-type radicals are defined. The quasi-radical $Q(R)$ of the near-ring R is the intersection of all maximal right ideals of R. The radical subgroup $R_s(R)$ of R is the intersection of all maximal R-subgroups of R.

In this paper we exted some known results for d.g. near-right to near-rings with defect of distributivity. The connection between these radicals and some hereditary properties can be investigated in the case of near-rings with defect. Preliminary definitions are contained in Section 1. Also, we prove one lemma which determines the form of the elements of the right ideal which is generated by some subset of R.

In Section 2 some results refer to the connection between the radicalt $J_2(R)$, $Q(R)$ and $R_s(R)$ for certain types of near-rings with defect. We first generalize the results of Biedleman ([3]), Lemmas 4 and 5) and ([2], Th. 2.2). The Theorem 2.4 refers to the conditions under which the near-ring R with defect of distributivity is Noertherian and gereralizes the Theorem 2.4 of [2]. We know that the radical $J_2(R)$ of a near-ring R contains all nilpotent right R-subgroups. However, the Theorem 2.5 of [9] gives the conditions under which the radical $J_2(R)$ of a d.g. near-ring R contains all nilpotent left R-subgroups. Our Theorem 2.7 extends this result to near-rings with defect. Also, we generalize the Theorem 28 of of [4], which refers to a connection between the radical subgroup $R_s(R)$.

In Section 3 we consider some hereditary properties of the radicals J_0 and J_2 of a near-ring R with defect. The Theorems 3.2, 3.4 and 3.5 generalize the results ([8], Thms 5,6) and ([10], Th. 6.34). Also, the results ([8], Lemma 13, Prop. 7) are extended to a near-ring with defect of distributivity.

This paper forms part of the author's doctoral dissertation to be submitted to the University of Sarajevo in 1979. I wish to express my gratitude to Prof. V. Perić for much helpful discussion and advice.

1. Definitions and preliminary results

In this paper by "near-ring" is meant a left near-ring R satisfying $0x = 0$ for all $x \in R$. The necessary definitions concerning near-rings with defect of distributivity are now given. A set of generators of the near-ring R is a multiplicative subsemigroup $(S, \cdot)$ of the semigroup $(R, \cdot)$, whose elements generate $(R, +)$. Let S be a set of generators of the near-ring R and let

$$D_s = \{d : d = -(xs + ys) + (x + y)s, \ \ x, y \in R, \ \ s \in S\}.$$

The normal subgroup D of the group $(R, +)$ which is generated by the set D_s is called the defect of distributivity of the near-ring R. We say that R is the near-ring with the defect D. The near-ring R will be detoned by (R, S) when we wish to stress the set of generators S.

Let (R, S) be a near-ring with the defect D and let $A \subset R$. The normal subgroup $\bar{A}$ of $(R, +)$ generated by the set $A \cup AS$, has the elements of the form

$$\bar{a} = \sum_i (r_i \pm a_i s_i + m_i a_i{}' - r_i), \ \ (r_i \in R, \ \ a_i, \ \ a_i{}' \in A, \ \ s \in S, \ \ m_i\text{-integers}).$$

For all r, $r_i \in R$, a_i, $a_i{}' \in A$ and s, $s_i \in S$, there exist d_1, $d_2 \in D$ such that

$$\begin{aligned}(r + \bar{a})s &= rs + \bar{a}s + d_1 \\ &= rs + \left(\sum_i (r_i \pm a_i s_i + m_i a_i{}' - r_i)\right) s + d_1 = \\ &= rs + \sum_i (r_i s \pm a_i s_i s + m_i a_i{}' s - r_i s) + d_2 + d_1\end{aligned}$$

The normal subgroup D_r of the group $(R, +)$ generated by the elements $d_2 + d_1 \in D$ which are obtained in the previous manner, is called a relative defect of the subset A with respect to R. It is clear that $D_r \subseteq D$. For basic properties of near-rings with defect see [5].

LEMMA 1.1. *Let (R, S) be a near-ring with defect and let $A \subset R$. The normal subgroup $\bar{A}$ of $(R, +)$ generated by $A \cup AS$ is the right ideal of R if and only if $\bar{A}$ contains a relative defect of the subset A with respect to R.*

PROOF. 1° It is known (see [5]) that the elements of $\bar{A}$ have the form

$$\bar{a} = \sum_i (r_1 \pm a_i s_i + m_i a_i{}' - r_i), \ \ (r_i \in R, \ \ a_i, \ \ a_i{}' \in A, \ \ s_i \in S, \ \ m_i\text{-integers}).$$

Suppose that $\bar{A}$ contains the relative defect D_r of the subset A with respect to R. By definition of the relative defect for all $r_i \in R$, a_i,$a_i{}' \in A$ and s, $s_i \in S$ there exists $d \in D_r$ such that

$$\bar{a}s = \sum_i (r_i s \pm a_i s_i s + m_i a_i{}' s - r_i s) + d, \quad (m_i\text{-integers}).$$

Since $D_r \subseteq \bar{A}$, it follows that $\bar{a}s \in \bar{A}$ for all $\bar{a} \in \bar{A}$ and $s \in S$. Thus, $\bar{A}$ is a S-subgroup. By Lemma 3.2 of [5], $\bar{A}$ is a right ideal of R.

$2°$ Conversely, let $\bar{A}$ be a right ideal of R. By definition of the relative defect D_r, if $d \in D_r$ then there exist $r \in R$, $a \in A$ and $s \in S$ such that

$$(r+\bar{a})s = rs + \sum_i (r_i s \pm a_i s_i s + m_i a_i' s - r_i s) + d,$$

where

$$\bar{a} = \sum_i (r_i \pm a_i s_i + m_i a_i' - r_i) \in \bar{A}, \ \ (r_i \in R, \ \ a_i, \ \ a_i' \in A, \ \ s_i \in S, \ \ m_i\text{-integers}).$$

Therefore,

$$-rs + (r+\bar{a})s = \sum_i (r_i s \pm a_i s_i s + m_i a_i' s - r_i s) + d.$$

Since $-rs + (r+a)s \in \bar{A}$ and $\sum_i (r_i s \pm a_i s_i s + m_i a_i' s - r_i s) \in \bar{A}$, it follows that $d \in \bar{A}$. Hence, $D_r \subseteq \bar{A}$.

Corollary 1. *Let (R,S) be a near-ring with defect and let $A \subset R$. The normal subgroup $\bar{A}$ of $(R,+)$ generated by $A \cup RA \cup AS \cup RAS$ is an ideal of R if and only if $\bar{A}$ contains relative defect of the subset $A \cup RA$ with respect to R.*

Proof. The normal subgroup $\bar{A}$ of $(R,+)$ generated by $A \cup RA \cup AS \cup RAS$ has the elements of the form

$$\bar{a} = \sum_i (r_i \pm x_i a_i s_i \pm a_i' s_i' \pm y_i a_i'' + m_i a_i''' - r_i),$$
$$(r_i, x_i, y_i \in R, a_i, a_i'.a_i'', a_i''' \in A, \ \ s_i, s_i', \in S, \ \ m_i\text{-integers}).$$

If in place of the set A in the Lemma 1.1 we take consideration the set RA, then $\bar{A}$ is a right ideal of R. Also $\bar{A}$ is a left ideal of R.

Corollary 2. *Let (R,S) be a near-ring with a defect and with identity and let $A \subset R$. The set*

$$\bar{A} = \left\{ \sum_i (r_i \pm a_i s_i - r_i) \colon r_i \in R, \ \ a_i \in A, \ \ s_i \in S \right\}$$

is a right ideal of R (generated by A) is and only if $\bar{A}$ contains the relative defect of the subset A with respect to R.

2. Some properties of radicals of the near-rings with defect of distributivity

An ideal B of near-ring R is called a small ideal if and only if $R = C$ for each other ideal C such that $R = B + C$. The following theorem generalizes Lemma 4 of [3].

THEOREM 2.1. *Let R be a near-ring with the defect D which is a small ideal of R. If R contains identity and $(R,+)$ is a solvable group, then the quasi-radical $Q(R)$ coincides with the radical $J_2(R)$. Moreover, the commutator subgroup R' of $(R,+)$ and defect ace contained in every maximal right ideal of R.*

PROOF. Let A be a maximal right ideal of R. By Corollary 2.3 od [10] $D + A$ is a right ideal of R. Since A is the maximal right ideal we have either $D + A = A$ i.e. $D \subseteq A$ for $D + A = R$. However, this last is impossible, because D is a small ideal. For the group $(R,+)$ there is a solvable series of R-subgroups. Also, $R \supset A \supset (0)$ is a normal series of R-subgroups. By Lemma 1.3 of [2] there is a refinment of this last series which is a solvable series of R-subgroups, too. Thus, if $(R/A,+)$ weren't abelian, then there would exists a solvable series of R-subgroups $(0) \subseteq \cdots \subseteq A \subseteq \cdots \subseteq C \subseteq R$, where $D \subseteq A$. Therefore, by Lemma 3.2 of [5] C is a right ideal of R. But, it is contradictory to the fact that A is a maximal right ideal of R. Consequently, $(R/A,+)$ is an abelian group. Thus, $R' \subset A$. Hence $B = R' + D$ is an ideal of R. One the other hand, by the Corollary of Theorem 2.6 of [5], R/B is a d.g. near-ring. Hence, by Theorem 4.4.3 of [6] it follows that R/B is a ring. If there is the R-subgroup B_1 such that $B_1 \supset A$, then the ring R/B has a subgroup $B_1/B \supset A/B$. However, every R-subgroup in R/B is a right ideal of R/B. But A/B is a maximal right ideal, thus $R/B = B_1/B$, i.e. $B_1 = R$. Consequently, every maximal right ideal of R is strictly maximal and $Q(R) = J_2(R)$.

The following two theorems generalize the Theorem 2.2 of [2] and the Lemma 5 of [3].

THEOREM 2.2. *Let R be a near-ring with the defect D which is a small ideal of R and R contains left idetity. If $(R,+)$ is a solvable group and R satisfies the descending chain condition (d.c.c.) on R-subgroups, then the radical $J_2(R)$ is nilpotent and quasiregular. Moreover, the near-ring $R/J_2(R)$ is a ring.*

PROOF. By theorem 2.1 we have $Q(R) = J_2(R)$. By using the Theorem 5.48 (b) (c) (d) of [10] and Remark 5.67 (j) of [10] it follows that the radical $J_2(R)$ is nilpotent and quasiregular. It needs to be proved that $R/J_2(R)$ is a ring, Since $D \subseteq Q(R) = J_2(R)$ then, by Corollary of Theorem 2.6 of [5], $R/J_2(R)$ is a d.g. near-ring. Also, by the Theorem 2.1, $R' \subseteq J_2(R)$ is a ring.

THEOREM 2.3. *Let R be a near-ring with the defect D which is a small ideal of R. If R contains identity and $(R,+)$ is a solvable group, then the radical $J_2(R)$ is a small right ideal of R.*

PROOF. By Theorem 2.1 we have $Q(R) = J_2(R)$. Thus, by Theorem 9 of [3] it follows that $J_2(R)$ is a small right ideal of R.

Beidleman([2], Th. 2.4) proved the following theorem: If R is a d.g. near-ring with identity whose additive group $(R,+)$ is solvable and R satisfies the d.c.c. on R-subgroups, then R in Noetherian. We proove a similar theorem for near-rings with defect of distributivity.

THEOREM 2.4. *Let R be a near-ring with the defect D which is a small ideal of R and let $(R,+)$ be a solvable group. If R contains identity and R satisfies the d.c.c. on R-subgroups, then R is Noetherian.*

PROOF. By Theorem 2.2 it follows that the radical $J_2(R)$ is nilpotent. Let $A_0 = R$ and, for each positive integer k, let A_k, denote the right ideal of R that is generated by finite products of k elements from $J_2(R)$. Since the radical $J_2(R)$ is nilpotent then there exists the smallest positive integer n such that $A_n = (0)$. Therefore, $R = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_n = (0)$ is a normal series of R-subgroups. Then a solvable series $R = B_0 \supseteq B_1 \supseteq \cdots \supseteq B_p = (0)$ is a refinment of the upper normal series ([2], Lemmas 1,3). Thus, B_k/B_{k+1} are abelian groups for every k. If $x \in J_2(R)$ then $B_k x \subseteq B_{k+1}$, i.e. $J_2 \subseteq (B_{k+1}\colon B_k)$. Consequently, for all $b_k \in B_k$, $r \in R$ we obtain $(b_k + B_{k+1})(r + J_2) = b_k r + B_{k+1}$. By Theorem 2.2, $R/J_2(R)$ is a ring and B_k/B_{k+1} can be considered as an R/J_2 module. The remaining part of the proof is similar to the proof of Theorem 2.4 in [2].

Beidleman ([4], Th. 28) proved that in d.g. near-ring with identity, whose additive group is nilpotent, the radical subgroup coincides with the radical $J_2(R)$. A similar result gives the following theorem.

THEOREM 2.5. *Let R be a near-ring whose defect D is a subset of the commutator subgroup of $(R,+)$ and let every maximal subgroup of $(R,+)$ be an R-subgroup. If $(R,+)$ is a nilpotent group, and R has identity, then the radical $J_2(R)$ is the radical subgroup of R.*

PROOF. Let $(B,+)$ be a maximal subgroup of $(R,+)$. It is a maximal R-subgroup, too. By Corollary 10.3.2 of [7] every maximal subgroup of $(R,+)$ is a normal subgroup and contains the commutator subgroup of $(R,+)$. Since, the defect D of the near-ring R is a subset of the commutator subgroup of $(R,+)$ it follows that D is cotained in B. Therefore by Lemma 3.2 of [5] B is a right ideal of R. Consequently, the radical $J_2(R)$ is the radical subgroup.

COROLLARY. *Let R be a near-ring whose defect D is a subset of the commutator subgroup of $(R,+)$ and let every maximal subgroup of $(R,+)$ be an R-subgroup. If $(R,+)$ is a nilpotent group and R has identity, then the radical $J_2(R)$ is quasiregular.*

PROOF. By Theorem 2.2 of [1] the radical subgroup $R_s(R)$ is a quasiregular R-subgroup.

THEOREM 2.6. *Let R be a near-ring whose defect D is a subsat of the commutator subgroup of $(R,+)$ and let every maximal subgroup of $(R,+)$ be an R-subgroup. If $(R,+)$ is a nilpotent group and R has identity, then $R/J_2(R)$ is a ring.*

Proof. From Theorem 2.5 the radical $J_2(R)$ is equal to the intersection of all maximal R-subgroups. By Corollary 10.3.2 of [7] the commutator subgroup R' of $(R,+)$ is contained in every maximal R-subgroup of R. Hence $R' \subseteq J_2(R)$. Using Theorem 4.4.4 of [6] we obtain that the defect D of R is contained in every maximal R-subgroup. Thus $D \subseteq J_2(R)$. Because of the Corollary of Theorem 2.6 in [5], $R/J_2(R)$ is d.g. near-ring. Hence, $R/J_2(R)$ is a ring ([6], Th. 4.4.3).

If R is a near-ring, then by Corollary 5.45 of [10] the radical $J_2(R)$ contains all nilpotent right R-subgroups. Laxton ([9], Th. 2.5) has proved that the radical $J_2(R)$ of d.g. near-ring identity and satisfying the d.c.c. for R-subgroups, contains all nilpotent left R-subgroups. In this sence we have the following theorem.

Theorem 2.7. *Let for any nilpotent left R-subgroup A of the near-ring (R,S) with defect, the normal subgroup of $(R,+)$ generated by $A \cup AS$ contains the relative defect of the subset A with respect to R. If R has identity and R satisfies the d.c.c. on R-subgroups, then the radical $J_2(R)$ contains all nilpotent left R-subgroups.*

Proof. Let A be any nilpotent left R-subgroup. Then for some integer p we have $A^p = (0)$. Using Theorem 5.42 of [10] we need to prove that A is contained in every maximal ideal of R. Let us suppose the opposite: that A is not contained in the maximal ideal B. Let us denote by $\bar{A}$ the minimal ideal of R which containes A. By Corollary 2 of Lemma 1.1, $\bar{A}$ has the elements of the form $\sum_i (z_i \pm a_i' s_i - z_i)$, $(z_i \in R,\ a_i' \in A,\ s_i \in S)$, because A is a left R-subgroup of R. Since we supposed that A is not contained in B, so $\bar{A}$ is not contained in the maximal ideal B. By Corollary 2.3 of [10], $\bar{A} + B$ is an ideal of R. But $\bar{A} + B$ contains the maximal ideal B and thus must be $R = \bar{A} + B$. Therefore $e = y + b$, where $y \in A$, $b \in B$ and e is identity. Since $A^p = (0)$ it follows that $a_1 \dots a_{p-1} e = a_1 \dots a_{p-1}(y+b) = a_1 \dots a_{p-1} y + a_1 \dots a_{p-1} b$, $(a_1, \dots, a_{p-1} \in A)$. For $y = \sum_i (x_i \pm a_i' s_i - x_i)(x_i \in R,\ a_i' \in A,\ s_i \in S)$ we obtain

$$a_1 \dots a_{p-1} y = \sum_i (a_1 \dots a_{p-1} x_i \pm a_1 \dots a_{p-1}\ a_i' s_i - a_i \dots a_{p-1} x_i) = 0,$$

because $a_1 \dots a_{p-1} a_i' \in A^p$ and therefore $a_1 \dots a_{p-1} a_i' = 0$. Thus, $a_1 \dots a_{p-1} = a_1 \dots a_{p-1} y + a_1 \dots a_{p-1}\ b = a_1 \dots a_{p-1} b \in B$. Consequently $A^{p-1} \subseteq B$. Repeating the above process, we obtain $A \subseteq B$. This contradicts that A is not contained in B. Thus, every nilpotent left R-subgroup is contained in some maximal ideal of R, i.e. $J_2(R)$ contains every left nilpotent R-subgroup ([10], Th. 5.42b).

3. Some hereditary properties of radicals of near-rings with defect

Kaarli ([8], Thms 5,6) and Pilz ([10], Th. 6.34) consider some heriditary properties of the radicals $J_0(R)$ and $J_2(R)$ for d.g. near-rings. Similar properties we consider in near-rings with defect of distributivity.

Proposition 3.1. *Let (R,S) be a near-ring with defect and let q be a quasiregular element of R which is contained in some R-subgroup B of R. If the* *normal subgroup of $(R,+)$ generated by the set $\{q^2-q\}\cup(q^2-q)S$ contains the relative defect of the element q^2-q with respect to R, and if the right ideal B_q of B generated by $\{qb-b\colon b\in B)$ is a right R-subgroup, then the quasiregular element q of R is quasiregular in the R-subgroup B.*

PROOF. Let B be an R-subgroup of R and $q\in B$. Because of Lemma 10 of [8], q is a quasiregular element in R if and only if the right ideal of R generated by element q is equal to the right ideal of R generated by element q^2-q. By Lemma 1.1 in this case q has the form

$$q=\sum_i(r_i\pm(q^2-q)s_i+m_i(q^2-q)-r_i),\ \ (r_i\in R,\ \ q\in B,\ \ s_i\in S,\ \ m_i\text{-integers}).$$

Multiplying from the left side by q we obtain

$$q^2=\sum_i(qr_i\pm(qq^2-q^2)s_i+m_i(qq^2-q^2)-qr_i).$$

Since B is an R-subgroup, we have

$$q^2=\sum_i(qr_i\pm(qb-b)s_i+m_i(qb-b)-qr_i),$$

where $q^2=b\in B$. Let us denote by B_q the right ideal of B generated by the set $\{qb-b\colon b\in B\}$. Then $q^2\in B_q$ since $(qb-b)s_i\in B_q$ and $(q^2-q)\in B_q$, i.e. $q\in B_q$. If $b\in B$, then $qb\in B_q$. Hence $b\in B_q$, i.e. $B\subseteq B_q$. Thus $B_q=B$ and q is a quasiregular element of B.

THEOREM 3.2. *Let (R,S) be a near-ring with defect and let B be some R-subgroup of R. If for any quasiregular element q of R which is contained in B, the normal subgroup of $(R,+)$ generated by the set $\{q^2-q\}\cup(q^2-q)S$ contains the relative defect of the element q^2-q with respect to R, and if the right ideal B_q of B generated by $\{qb-b\colon b\in B\}$, is a right R-subgroup, then $J_0(B)\supseteq J_0(R)\cap B$ and $J_{1/2}(B)\supseteq J_{1/2}(R)\cap B$.*

PROOF. By Theorem 5.37c of [10], $J_0(R)$ is the greatest quasiregular ideal of R. Therefore, the elements for $J_0(R)\cap B$ are quasiregular in R and belong to B. By Proposition 3.1 it follows that the elements from $J_0(R)\cap B$ are quasiregular in R-subgroup B. Applying the Theorem 5.37c of [10] to near-ring B, we obtain $J_0(R)\cap B\subseteq J_0(B)$. The inklusion $J_{1/2}(B)\supseteq J_{1/2}(R)\cap B$ can be proved in a similar way. For this, the word "ideal" ought to be exchanged with the word "right ideal". After that we make use of Theorem 5.37a of [10] and the Theorem 3.1.

COROLLARY. *Let (R,S) be a near-ring with defect and let q be quasiregular element of R. If for any R-subgroup $B\subseteq J_0(R)$ of R with $q\in B$, the normal subgroup of $(R,+)$ generated by the set $\{q^2-q\}\cup(q^2-q)S$ contains the relative defect of the element q^2-q with respect to R, and if the right ideal B_q of B generated by $\{qb-b\colon b\in B\}$ is a R-subgroup, then $J_0(R)=J_0(J_0(R))$ and $J_0(B)=B$.*

Proof. Applying the Theorem 3.2 to the R-subgroup $J_0(R)$ we shall obtain $J_0(R) = J_0(J_0(R))$. Let now $B \subseteq J_{1/2}(R)$, where B is an R-subgroup of R. By Theorem 3.2 we have $J_{1/2}(B) \supseteq B$, i.e. $J_{1/2}(B) = B$. By using Theorem 5.37a, c of [10] we obtain $B = J_{1/2}(B) \supseteq J_0(B)$, i.e. $B = J_0(B)$.

Theorem 3.3. *Let (R, S) be a near-ring with defect and let C be an ideal of R with the set of generators $S' \subseteq S$. Let every maximal ideal A of C contains the relative defect of subset A with respect to R. If G is an R-group of type 0 and $GC \neq \{0\}$, then G is a C-group of type 0.*

Proof. Let G be a monogenic by $q \in G$ and let $X = \{c \in C\colon gc = 0\}$. By Lemma 4 of [8] we have $gC = G$, i.e. G is R-isomorphic to the factor near-ring C/X. Thus, there exists $a \in C$ and $a \notin X$ such that $ga = g$. For all $b \in C$ we have $gb = g(ab)$, i.e. $b - ab \in X$. By using Zorn's lemma, there exists the maximal right ideal Y of the near-ring C which contains the set X but does not contain the element $a \in C$. Let us assume that $Y \neq X$. This assumption will lead us tos a contradiction. G is an R-subgroup of type 0, i.e. G does not have nontrivial right ideals. Since G is R-isomorphis to the factor near-ring C/X, a right ideal of R generated by Y is equal to C, i.e. $(Y)_R = C$. Thus, by Lemma 1.1 the element $a \in (Y)_R$ has the form

$$a = \sum_i (r_i \pm y_i s_i + m_i y_i{}' - r_i), \quad (r_i \in R,\ y_i,\ y_i{}' \in Y,\ s_i \in S,\ m_i\text{-integers}).$$

On the othe hand, $a = \sum_j (\pm s_j{}')$, where $s_j{}' \in S$ and $S' \subseteq S$. Let us denote by D_r the relative defect of subset Y with respect to R. Thus,

$$\begin{aligned}
a^2 &= \sum_i (r_i \pm y_i s_i + m_i y_i{}' - r_i) \sum_j (\pm s_j{}') \\
&= \sum_j \left(\pm \sum_i (r_i \pm y_i s_i + m_i y_i{}' - r_i) s_j{}' \right) \\
&= \sum_j \left(\pm \sum_i (r_i s_j{}' \pm y_i s_i s_j{}' + m_i y_i{}' s_j{}' - r_i s_j{}' + d_j) \right), \quad (d_j \in D_r) \\
&= \sum_j \left(\pm \sum_i (r_i s_j{}' \pm y_i s_i s_j{}' + m_i y_i{}' s_j{}' - r_i s_j{}') \right) + d, \quad (d \in D_r).
\end{aligned}$$

Hence,

$$a^2 - d = \sum_j \left(\pm \sum_i (a_{ij} \pm y_i{}'' s_j{}' + m_i y_{ij}{}' - a_{ij}) \right),$$

where

$$r_i s_j{}' = a_{ij} \in C, \ \ y_i s_i = y_i{}'' \in Y, \ \ y_i{}' s_j{}' = y_{ij}{}' \in X.$$

The ideal C of R contains the relative defect of subset Y with respect to C, since $S' \subseteq S$. By Lemma 1.1, the elements of the form

$$\sum_j \left(\pm \sum_i (a_{ij} \pm y_i{''} s_j{'} + m_i y_{ij}{'} - a_{ij}) \right), \qquad (a_{ij} \in C,\ y_i{''},\ y_{ij}{'} \in Y,\ s_j{'} \in S',\ m_i\text{-integers})$$

belong to the right ideal of C generated by the subset Y, i.e. $a^2 - d \in (Y)_C = Y$. Since $d \in Y$ we have $a^2 \in Y$. At the beginning of this proof we established that there exists $a \in C$ such that $b - ab \in X$ for each $b \in C$. If $b = a$ we have $a^2 - a \in X$, i.e. $a^2 - a \in Y$. From $a^2 \in Y$ and $a^2 - a \in Y$ we have $a \in Y$, i.e. $ab \in Y$, because Y is a right ideal of C. Also, from $b - ab \in X$ it follows that $b - ab \in Y$. Therefore, $b \in Y$ i.e. $C \subseteq Y$. This inclusion and $Y \subseteq C$ give $Y = C$. It is contradictory to the fact that Y is a maximal right ideal of C. Thus, our supposition that $X \neq Y$ is impossible, i.e. $X = Y$. Therefore, the C-group C/X is of type 0. Above we established that the R-group G is isomorphic to the factor near-ring C/X. Hence, G is a C-group of type 0.

THEOREM 3.4. *Let (R, S) be a near-ring with defect and let C be a direct sumand of R with a set of generators $S' \subseteq S$. If every maximal right ideal A of C contains the relative defect of the subset A with respect to R, then $J_0(C) = J_0(R) \cap C$.*

PROOF. By Theorem 5.18 of [10] we have $J_0(C) \supseteq J_0(R) \cap C$. On the other hand, by using the Theorem 3.3 it follows that, if G is an R-group of type 0 and $GC \neq (0)$, then G is a C-group of type 0. Because of Lemma 9 of [8] we obtain $J_0(C) \subseteq J_0(R) \cap C$. Thus, $J_0(C) = J_0(R) \cap C$.

THEOREM 3.5. *Let C be an ideal of a near-ring (R, S) with defect. If every 2-modular ideal A of C contains the relative defect of subset A with respect to R, then $J_2(C) = J_2(R) \cap C$.*

PROOF. The proof that A is an R-subgroup of R is analogous to the proof of Theorem 6.34 of [10]. Let D_r be the relative defect of subset A with respect to R. Foe all $s \in S$, $c \in C$ and $a \in A$ there exists $d \in D_r$ such that

$$(c + a)(\pm s) - c(\pm s) = \pm(cs + as + d) - c(\pm s).$$

Since $D_r \subseteq A$ and A is an R-subgroup, we have

$$(c + a)(\pm s) - c(\pm s) = \pm(cs + a') \pm cs \in A,$$

where $as + d = a' \in A$, $cs \in C$. It is easy to show, by induction on n, that for all $r \in R\,(r = \sum_i (\pm s_i))$, $c \in C$ and $a \in A$, $(c + a)r - cr \in A$. Hence, $(c + A) \overset{df}{=} cr + A$ is well-defined. From this fact and by using Proposition 3.30 of [10] and Theorem 5.18 of [10] we obtain $J_2(C) = J_2(R) \cap C$.

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