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Δ-ENDOMORPHISM NEAR-RINGS

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The concept of a distributively generated near-rings arise if we define addition and multiplication of endomorphisms of the group $(G,+)$ in the usal manner. It is possibble to consider the set of the mappings of $(G,+)$ into itself which are similar to the endomorphisms of a group in such a way that their "linearity" is corrected by the elements from a normal subgroup Δ of the group $(G,+)$. These mappings are called Δ-endomorphisms of $(G,+)$. The set of Δ-endomorphisms of G generate (additively) a near-ring $\mathcal{E}_\Delta(G)$, whose defect depends on the shoice of the subgroup Δ. Also, Δ-endomorphisms for which is invariant every fully invariant subgroup of the group $(G,+)$, are investigated. In this case we obtain the subnearring $E_\Delta(G)$ of the near-ring $\mathcal{E}_\Delta(G)$. Some known properties of the endomorphism near-rings were transfered to the Δ-endomorphism near-rings.

Some elementary results relating to the E_Δ-invariant subgroups of $(G,+)$ are presented in Section 2. In Secton 3 we consider the structure of ideals of the near-ring $E_\Delta(G)$, generalizing the results which were obtained by H. Johnson in [8] and [9] for the near-ring of endomorphisms. The result in Section 4 refers to the problem embedding of near-rings into some near-ring of Δ-endomorphisms and generalizes the Theorem Heatherly and Malone in [7]. Also, a $\mathcal{D}$-direct sum of subnear-rings of the near-ring $E_\Delta(G)$ is considered, where $\mathcal{D}$ is a defect of $E_\Delta(G)$.

1. Preliminaries

Throughout this paper term "near-ring" shall mean "left near-ring" R satisfying $ox = o$ for all $x \in R$. The necessary definitions concerning near-rings with a defect of distributivity are now given.

A set of generators of the near-ring R is a multiplicative subsemigroup S of R whose elements generale $(R,+)$. Let S be a set of generators of the near-ring R and let

$$D_S = \{d\colon d = -(xs+ys)+(x+y)s, \quad x,y \in R, \quad s \in S\}.$$

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The normal subgroup D of the group $(R,+)$ which is generated by the set D_S is called the defect of distributivity of the near-ring R. Thus, for all $x,y \in R$ and $s \in S$ there exists $d \in D$ such that

$$(x+y)s = xs + ys + d.$$

The near-ring R with the defect D will be detoned by (R,S) when we wish to stress the set of generators S. A near-ring R is called D-distributive if $R = S$, i.e. for each $x,y,z \in R$ there exists $d \in D$ such that

$$(x+y)z = xz + yz + d.$$

Let (R,S) be a near-ring with the defect D and $A \subset R$. The normal subgroup $\bar{A}$ of $(R,+)$ generated by the set $A \cup AS$ has the elements of the form

$$\bar{a} = \sum_i (r_i \pm a_i s_i + m_i {a_i}' - r_i), \quad (r_i \in R,\ \ a_i, {a_i}' \in A,\ \ s_i \in S,\ \ m_i\text{– integers}).$$

For all r, $r_i \in R$, a_i, ${a_i}' \in A$ and s, $s_i \in S$ there exists d_1, $d_2 \in D$ such that

$$(r+\bar{a})s = rs + \bar{a}s + d_1 = rs + \left(\sum_i (r_i \pm a_i s_i + m_i {a_i}' - r_i)\right) s + d_1$$

$$(r+\bar{a})s = \sum_i (r_i s \pm a_i s_i s + m_i {a_i}' s - r_i s) + d_2 + d_1.$$

The normal subgroup D_r of the group $(R,+)$ generated by the elements $d_2 + d_1 = d \in D$ which have been obtained in the previous manner, is called a relative defect of the subset A with respect to R. It is obvious that $D_r \subseteq D$.

LEMMA 1.1. ([4]. *Lemma* 3.2*) Let (R,S) be a near-ring with defect. The normal subgroup B of the group $(R,+)$ is a right ideal of R if and only if B is an S-subgroup which contains the relative defect of the subset B with respect to R.*

PROPOSITION 1.2. ([5], *Coroll. of Lemma* 1.1*) Let (R,S) be a near-ring with defect and $A \subset R$. The normal subgroup $\bar{A}$ of $(R,+)$ generated by $A \cup RA \cup AS \cup RAS$ is an ideal of R if and only if $\bar{A}$ contains the relative defect of the subset $A \cup RA$ with respect to R.*

PROPOSITION 1.3. ([4], *Theorem* 2.3 *b) Every direct sum of the near-rings R_i with the defect D_i respectively, is a near-ring R whose defect is a direct sum of the defects D_i.*

2. Elementary properties of Δ-endomorphisms

Let $M_0(G)$ be a set of zero preserving mappings of the group $(G,+)$ into itself.

Definition. Let Δ be a normal subgroup of the group $(G,+)$. The mapping $f \in M_0(G)$ with $(\Delta)f \subseteq \Delta$ is called Δ-endomorphism of the group $(G,+)$ if for all $x, y \in G$ there exists $\delta \in \Delta$ such that

$$(x+y)f = (x)f + (y)f + \delta.$$

It is easy to prove by induction that for each $x_1, \dots, x_n \in G$ and some Δ-endomorphism f there exists $\delta \in \Delta$ such that

$$(x_1 + \ldots + x_n)f = (x_1)f + \cdots + (x_n)f + \delta.$$

In the case $\Delta = (0)$ we obtain the endomorphisms of the group $(G,+)$. The set of all Δ-endomorphisms of the group $(G,+)$ will be denoted by $\mathcal{E}nd_\Delta(G)$. This set is a semigroup with respect to composition.

Let us denote by $(G,\Delta)_0$ the set of all mappings $h\colon\ G \to \Delta$ with $(0)h = 0$. It is clear that $(G,\Delta)_0 \subseteq \mathcal{E}nd_\Delta(G)$. Thus, for $\Delta \neq (0)$ it follows that $\mathcal{E}nd_\Delta(G) \neq End(G)$.

If $(G,+)$ is non-commutative, then the set of all Δ-endomorphisms of G will not be closed under pointwise addition. However, the set of all (finite) sums and differences of Δ-endomorphisms of G forms a near-ring, which will be designated by $\mathcal{E}_\Delta(G)$. Namely, if $f = \sum_i (\pm t_i)$ and $h = \sum_j (\pm t_j{}')$, $(t_i, t_j{}' \in \mathcal{E}nd_\Delta(G))$, then for all $x \in G$ we have

$$\begin{aligned}
(x)fh &= \sum_j \pm \left(\sum_i ((\pm x)t_i) \right) t_j{}' \\
&= \sum_j \pm \left(\sum_i (\pm x)t_i t_j{}' + \delta_{ij} \right) \\
&= \sum_j \pm \left(\sum_i (\pm x)t_i t_j{}' \right) + \delta, \quad (\delta_{ij}, \delta \in \Delta).
\end{aligned}$$

But, the element $\delta \in \Delta$ depends on x. If we put $\delta = (x)\alpha$, then $\alpha \in (G,\Delta)_0$ i.e. $\alpha \in \mathcal{E}nd_\Delta(G)$. Hence,

$$(x)fh = (x)\left[\left(\sum_j \left(\pm \sum_i t_i t_j{}' \right) \right) + \alpha \right], \text{ i.e.}$$

$$fh = \sum_j \left(\sum_i (\pm t_{ij}) \right) + \alpha,$$

where $t_i t_j{}' = t_{ij} \in \mathcal{E}nd_\Delta(G)$ and $\alpha \in \mathcal{E}nd_\Delta(G)$.

The normal subgroup $\mathcal{D}$ of the group $(\mathcal{E}_\Delta(G),\ +)$ generated by

$$\{\delta\colon \delta = -(ht+ft)+(h+f)t, \quad h,f \in \mathcal{E}_\Delta(G),\ \ t \in \mathcal{E}nd_\Delta(G)\}$$

is a defect of distributivity of the near-ring $\mathcal{E}_\Delta(G)$. It is clear that $\mathcal{D} \subseteq (G,\Delta)_0$. For example, the near-ring $\mathcal{E}_\Delta(Z_4) = \{f_0, f_1, \dots, f_{15}\}$, where $\Delta = \{0,2\}$, has the defect $\mathcal{D} = \{f_0, f_3, f_{12}, f_{13}\}$ (table 1).

If the commutator subgroup G' of $(G,+)$ is a subset of Δ, then $\mathcal{E}_\Delta(G)$ is a $\mathcal{D}$-distributive near-ring, where $\mathcal{D}$ is the defect of $\mathcal{E}_\Delta(G)$. Let G be a nilpotent group and Δ its maximal subgroup. Then by Corollary 10.3.2 of [6] it follows that the near-ring $\mathcal{E}_\Delta(G)$ is $\mathcal{D}$-distributive, where $\mathcal{D}$ is the defect of $\mathcal{E}_\Delta(G)$.

Let (R,S) be a near-ring with the defect D. For all $s \in S$ and $x \in R$ there is a map $f_s\colon x \to xs$ from R into R. These maps are D-endomorphisms. Let us denote by $\mathcal{E}_D(R)$ the near-ring of "right multiplications" of the near-ring R with the defect D. The defect of distributivity of $\mathcal{E}_D(R)$ is the set

$$\{f_d\colon (x)f_d = xd,\ \ x \in R,\ \ d \in D\}.$$

Proposition 2.1. *If Δ is a proper normal subgroup of the group $(G,+)$, then $\mathcal{E}_\Delta(G) \subset M_0(G)$.*

Proof. Anyhow $\mathcal{E}_\Delta(G) \subseteq M_0(G)$. If $(0) \neq \Delta \neq G$ and $y \in G\backslash\Delta$, then the map $h \in M_0(G)$ can be defined as follows

$$x(h) = \begin{cases} y, & x \in \Delta,\, x \neq 0 \\ 0 & x = 0 \\ x, & x \notin \Delta \end{cases}$$

Since $(\Delta)\ \mathcal{E}_\Delta(G) \subseteq \Delta$, we have $h \notin \mathcal{E}\Delta(G)$.

If B is a fully invariant subgroup of the group $(G,+)$, then B must not be invariant with respect to all Δ-endomorphisms of $(G,+)$. For example, the subgroup $B = \{0,2,4\}$ of $(Z_6,+)$ is not invariant with respect to the Δ-endomorphism $f = \binom{012345}{003003}$, where $\Delta = \{0,3\}$.

Let Δ be a proper normal subgroup of the group $(G,+)$. There exist nontrivial Δ-endomorphisms for which are invariant all subgroups of $(G,+)$. For instance, the mapping $f \in M_0(G)$ with $(x)f = x$ for all $x = \Delta$, and $(x)f = 0$ for all $x \in G\backslash\Delta$ is such a Δ-endomorphism. Let us denote by $End_\Delta(G)$ the biggest subsemigroup of the semigroup $\mathcal{E}nd_\Delta(G)$ for which are invariant all fully invariant subgroups of the group $(G,+)$. If we denote by $E_\Delta(G)$ the additive group generated by $\mathcal{E}nd_\Delta(G)$, then $E_\Delta(G)$ is a near-ring whose set of generators $End_\Delta(G)$ is contained in a set of generators $\mathcal{E}nd_\Delta(G)$ of the near-ring $\mathcal{E}_\Delta(G)$. Every fully invariant subgroup of $(G,+)$ which is invariant with respect to $End_\Delta(G)$, is invariant with respect to

$E_\Delta(G)$ as well. For this reason we say that the subgroups of this kind are E_Δ-invariant.

EXAMPLE 1. The group $(Z_6,+)$ has 96 Δ-endomorphisms for which only the subgroup $\Delta=\{0,3\}$ is invariant. However, the set $End_\Delta(Z_6)=\{f_0,f_1,\dots,f_{23}\}$ contains all Δ-endomorphisms of $(Z_6,+)$ for which both subgroups Δ and $B=\{0,2,4\}$ are invariant (table 2). If we take for Δ the subgroup B, then there exist 486 Δ-endomorphisms. But by claiming that both subgroups of $(Z_6,+)$ are invariant this number will be reduced to 54.

If Δ is a fully invariant subgroup of $(G,+)$, then a near-ring $E_\Delta(G)$ contains the endomorphism near-ring $E(G)$. A several following propositions are related to the elementary proposities of E_Δ-invariant subgroup and they generalize the corresponding results of M. Jonson in [8].

PROPOSITION 2.2. *Let Δ be a fully invariant subgroup of $(G,+)$ and let $y\in G$, $(y\neq 0)$. If $\mathcal{H}$ is a right $E_\Delta(G)$-subgroup, then $(y)\mathcal{H}$ is E_Δ-invariant subgroup of $(G,+)$.*

The proof is quite analagous with that in ([8], Lemma 3.1).

COROLLARY. Let B be E_Δ-invariant subgroup of $(G,+)$ and let $y\in B$, $(y\neq 0)$. If $\mathcal{H}$ is a right $E_\Delta(G)$-subgroup, then $(y)\mathcal{H}$ is E_Δ-invariant subgroup of $(G,+)$.

DEFINITION. Let B be a subgroup of the group $(G,+)$ and $\mathcal{H}\subseteq M_0(G)$. If B is an invariant subgroup with respect to $\mathcal{H}$, then we say that $\mathcal{H}$ acts transitively on B if for all $x\in \mathbf{B}$, $(x\neq 0)$ we have $(x)\mathcal{H}=B$.

DEFINITION. The group $(G,+)$ is called E_Δ-simple if and only if $(G,+)$ has not proper E_Δ-invariant subgroups.

Using Corollary of Proposition 2.2 we obtain the following.

PROPOSITION 2.3. *Let B be an E_Δ-invariant subgroup of the group $(G,+)$. Then B is a minimal E_Δ-invariant subgroup of $(G,+)$ if and only if $E_\Delta(G)$ acts transitively on B.*

COROLLARY. Let Δ be a fully invariant subgroup of $(G,+)$. $E_\Delta(G)$ acts transitively on G if and only if G is E_Δ-simple.

Let G be a group and $B\subset G$. Denote by $\mathcal{A}(B)$ a right annihilator of B in $E_\Delta(G)$, that is, $\mathcal{A}(B)=\{f\in E_\Delta(G)\colon\ (b)f=0 \text{ for all } b\in B\}$.

PROPOSITION 2.4. *Let B_i $(i\in I)$ be a collection of minimal E_Δ-invariant subgroups of the group $(G,+)$ and let $\mathcal{N}$ be a right $E_\Delta(G)$-subgroup of $E_\Delta(G)$ containing only nilpotent elements. Then $\mathcal{N}\subseteq\cap_i\mathcal{A}(B_i)$.*

PROOF. Let $h\in\mathcal{N}$ and suppose that for some $b\in B_p$ $(p\in I)$, $(b)h\neq 0$. By Proposition 2.2 $(b)hE_\Delta(G)$ is E_Δ-invariant subgroup. Since B_p is a minimal E_Δ-invariant subgroup of $(G,+)$, there exists $f\in E_\Delta(G)$ such that $(b)hf=b$. Hence

hf is not nilpotent. On the other hand, $hf \in \mathcal{N}$ and this contradiction establishes the proposition.

The next proposition is easily verified.

Proposition 2.5. *Let B_i $(i \in I)$ be a collection of E_Δ-invariant subgroups of the group $(G,+)$. If $\Delta \subseteq \sum_i B_i$ then $\sum_i B_i$ is E_Δ-invariant subgroup.*

3. The ideal structurs of $E_\Delta(G)$

The results in this section refer to the ideal structures of the near-ring $E_\Delta(G)$. The results of M. Johnson ([8], Lemmas 6.1, 8.5, Thms 6.2, 6.11, 6.12, Propositions 8.9, 8.15) and ([9], Lemma 11, Thms 8 and 16) become a special case of these, when we take an endomorphism near-ring $E(G)$ instead $E_\Delta(G)$.

If $\mathcal{H}$ is a subset of $E_\Delta(G)$, we define

$$\Im(\mathcal{H}) = \{(x)h : x \in G, \quad h \in \mathcal{H}\}.$$

Obvious, $\Im(\mathcal{D}) \subseteq \Delta$, where $\mathcal{D}$ is the defect of the near-ring $E_\Delta(G)$.

Proposition 3.1. *Let B be an E_Δ-invariant subgroup of the group $(G,+)$. If $\Im(\mathcal{D}_r) \subseteq B$, where $\mathcal{D}_r$ is the relative defect of the subset $\mathcal{B} = \{f \in E_\Delta(G) : \Im(f) \subseteq B\}$ with respect to $E_\Delta(G)$, then $\mathcal{B}$ is an ideal of $E_\Delta(G)$.*

Proof. It is easy to show that $\mathcal{B}$ is a normal subgroup of $(E_\Delta(G),+)$ and $E_\Delta(G)$-subgroup of $E_\Delta(G)$. If $\delta \in \mathcal{D}_r$ then $\delta \in \mathcal{B}$ because $\Im(\mathcal{D}_r) \subseteq B$. Hence $\mathcal{B}$ contains the relative defect of the subset $\mathcal{B}$ with respect to $E_\Delta(G)$. Therefore, by Lemma 1.1 it follows that $\mathcal{B}$ is a right ideal of $E_\Delta(G)$. Also, $\mathcal{B}$ is a lift $E_\Delta(G)$-subgroup. Thus $\mathcal{B}$ is an ideal of $E_\Delta(G)$.

Proposition 3.2. *Let $\Delta \neq G$ be a nonzero fully invariant subgroup of the group $(G,+)$. Then $E_\Delta(G)$ is not a simple near-ring.*

Proof. Let $\mathcal{D}_r$ be a relative defect of the subset

$$\mathcal{B} = \{f \in E_\Delta(G) : \Im(f) \subseteq \Delta\}$$

with respect to $E_\Delta(G)$. Because $\mathcal{D}_r \subseteq \mathcal{D} \subseteq (G,\Delta)_0$, we have $\Im(\mathcal{D}_r) \subseteq \Delta$. By Proposition 3.1, $\mathcal{D}$ is an ideal of $E_\Delta(G)$. Since $\Delta \neq G$ it follows that the identity map is not in $\mathcal{B}$, i.e. $\mathcal{B} \neq E_\Delta(G)$. Let us define the map $h \in (G,\Delta)_0$ as follows

$$(x)h = \begin{cases} x, & x \in \Delta \\ 0, & x \notin \Delta \end{cases}$$

This map is a nonzero Δ-endomorphism and $\Im(h) \subseteq \Delta$, i.e. $h \in \mathcal{B}$. Hence, $\mathcal{B}$ is a proper ideal of $E_\Delta(G)$.

Proposition 3.3. *Let Δ be a fully invariant subgroup of the group $(G,+)\cdot$ $E_\Delta(G)$ is simple if and only if G is E_Δ-simple.*

PROOF. If G is a nonzero E_Δ-simple group it must be either $\Delta = (0)$ or $\Delta = G$. For $\Delta = (0)$ the results follows from ([8], Th. 6.12) and for $\Delta = G$ it follows from ([2], Lemma 4).

Conversely, let now $E_\Delta(G)$ be a simple near-ring. If $\Delta = (0)$ the result follows from ([8], Th. 6.12). If $\Delta \neq (0)$ then it is not a proper subgroup of G. Namely, if $\Delta \neq G$ then by Proposition 3.2 $E_\Delta(G)$ is not a simple near-ring. Thus, let $\Delta = G$, i.e. $E_\Delta(G) = H_0(G)$. If B is a proper subgroup of $(G,+)$, then there always exists $f \in M_0(G)$ for that B is not invariant. Therefore, G is an E_Δ-simple group.

THEOREM 3.4. *If B is a sum of all minimal nozero E_Δ-invariant subgroups of a finite group $(G,+)$ and $\Delta \subseteq B$ is fully invariant subgroup of $(G,+)$, then $\mathcal{B} = \{h \in E_\Delta(G) \colon \Im(h) \subseteq B\}$ is a proper nonzero ideal of $E_\Delta(G)$.*

PROOF. By Proposition 2.5 it follows that B is E_Δ-invariant subgroup. If $\mathcal{D}_r$ is a relative defect of the subset $\mathcal{B}$ with respect to $E_\Delta(G)$, then $\mathcal{D}_r \subseteq \mathcal{D} \subseteq (G,\Delta)_0$. Since, $\Delta \subseteq B$ we have $\Im(\mathcal{D}_r) \subseteq B$. Thus, by Proposition 3.1 $\mathcal{B}$ is an ideal of $E_\Delta(G)$. Clearly, $\mathcal{B} \neq E_\Delta(G)$. Let $\{x_1, \dots, x_n\} = G$. By Proposition 2.2 $(x_p)E_\Delta(G)(p = 1, \dots, n)$ is E_Δ-invariant subgroup of $(G,+)$. Thus, $(x_p)E_\Delta(G) \cap B \neq (0)$ for all $p = 1, \dots, n$. Now the proof is similar to the proof of the Theorem 6.2 in [8].

PROPOSITION 3.5. *Let B be a sum of all minimal nonzero E_Δ-invariant subgroups of a finite group $(G,+)$ and let $\Delta \subseteq B$ be a fully invariant subgroup of $(G,+)$. If $\mathcal{H}$ is a minimal right $E_\Delta(G)$-subgroup of $E_\Delta(G)$ then $\Im(\mathcal{H}) \subseteq B$.*

The proof is the same as that in ([9]*, Proposition* 6.*)*

THEOREM 3.6. *Let B a minimal nonzero E_Δ-invariant subgroup of the group $(G,+)$. If $b \in B (b \neq 0)$, then $\mathcal{A}(b)$ is a maximal right ideal of $E_\Delta(G)$.*

PROOF. If $\Delta = G$ then $E_\Delta(G) = M_0(G)$. In this case the result follows from ([10], Th. 3). If $\Delta = (0)$ then result follows by Lemma 8.5 of [8]. Let now $\Delta \neq (0)$ and $\Delta \neq G$. Since $e \notin \mathcal{A}(b)$ (e is the identity map), we have that $\mathcal{A}(b) \neq E_\Delta(G)$. Let us suppose that there is a right ideal $\mathcal{P}$ of $E_\Delta(G)$ such that $\mathcal{A}(b)$ is a proper subset of $\mathcal{P}$. By Corollary of Proposition 2.2 it follows that $(b)\mathcal{P}$ is an E_Δ-invariant subgroup of $(G,+)$. Thus, either $(b)\mathcal{P} = B$ or $(b)\mathcal{P} = (0)$, because B is a minimal E_Δ-invariant subgroup. Since $\mathcal{A}(b) \subset \mathcal{P}$ we have $(b)\mathcal{P} = B$. Consequently, there exists $f \in \mathcal{P}$ such that $(b)f = b$. Let $h = -f + e$, where e is the identity map of G itself. Clarly $h \in \mathcal{A}(b)$. Thus, $e = h + f \in \mathcal{P}$ and $\mathcal{P} = E_\Delta(G)$. Therefore, $\mathcal{A}(b)$ is a maximal ideal of $E_\Delta(G)$.

THEOREM 3.7. *Let B be a minimal nonzero E_Δ-invariant subgroup of the group $(G,+)$. Then $\mathcal{A}(B)$ is a maximal ideal of $E_\Delta(G)$.*

The proof is similar to the proof of the Proposition 8.15 in [8].

EXAMPLE 2. Let $E_\Delta(Z_6)$ be a near-ring of Δ-endomorphisms of the group $(Z_6,+)$ (table 2). The subgroups $B_1 = \Delta = \{0,3\}$ and $B_2 = \{0,2,4\}$ of $(Z_6,+)$

are minimal E_Δ-invariant subgroups. The annihilator ideals

$$\mathcal{A}(B_1) = \{f_0, f_2, f_4, f_6, f_7, f_9, f_{12}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}\}$$

and

$$\mathcal{A}(B_2) = \{f_0, f_3, f_9, f_{11}, f_{12}, f_{13}, f_{14}, f_{21}\}$$

are maximal ideals of $E_\Delta(Z_6)$.

The following theorem gives another type of a maximal right ideal of $E_\Delta(G)$ and generalizes the Proposition 8.9 in [8].

Theorem 3.8. *Let B be a maximal E_Δ-invariant subgroup of a finite group $(G,+)$ and let $\Delta \subseteq B$ be a fully invariant subgroup of $(G,+)$. If $x \in G \backslash B$ then $\mathcal{B} = \{\beta \in E_\Delta(G) \colon (x)\beta \in B\}$ is a maximal right ideal of $E_\Delta(G)$.*

Proof. It is easy to show that $\mathcal{B}$ is a normal $E_\Delta(G)$-subgroup. Let $\mathcal{D}_r$ be a relative defect of the subset $\mathcal{B}$ with respect to $E_\Delta(G)$. Since $\mathcal{D}_r \subseteq \mathcal{D} \subseteq (G,\Delta)_0$ we have $\mathcal{D}_r \subseteq \mathcal{B}$. Thus, by Lemma 1.1 it follows that $\mathcal{B}$ is a right ideal of $E_\Delta(G)$. Morover, $\mathcal{B} \neq E_\Delta(G)$, because $\mathcal{B}$ contains no the identity map e of G into itself.

We will prove that $\mathcal{B}$ is a maximal right ideal of $E_\Delta(G)$. Let $\mathcal{P}$ be a right ideal of $E_\Delta(G)$ such that $\mathcal{B} \subset \mathcal{P}$. Assume that $\alpha \in \mathcal{P}$ and $\alpha \notin \mathcal{B}$ i.e. $(x)\alpha \notin B$. The normal subgroup $(x)\alpha E_\Delta(G) + B$ is E_Δ-invariant. Namely, for all $f \in E_\Delta(G)$ and $t \in End_\Delta(G)$ we have

$$((x)\alpha f + b)\, t = (x)\alpha f t + (b)t + \delta \in (x)\alpha E_\Delta(G) + B,$$

because $\delta \in \Delta \subseteq B$ and $b,\ (b)t \in B$. Since B is a maximal E_Δ-invariant subgroup of $(G,+)$, then $(x)\alpha E_\Delta(G) + B = G$. Thus, there exist $f \in E_\Delta(G)$ and $b \in B$ such that $(x)\alpha f + b = x$. The map $h \colon G \to G$ with $h = -\alpha f + e$ belongs to $E_\Delta(G)$. Since $(x)h = -(x)\alpha f + x = b - x + x = b \in B$ we hawe $h \in \mathcal{B}$, i,e. $h \in \mathcal{P}$. Also, $\alpha f \in \mathcal{P}$. Hence $e = (\alpha f + h) \in \mathcal{P}$ and $\mathcal{P} = E_\Delta(G)$. Therefore, $\mathcal{B}$ is a miximal right ideal of $E_\Delta(G)$.

Example 3. Let $E_\Delta(Z_4)$ be a near-ring of Δ-endomorphisms of the group $(Z_4,+)$ (table 1). The subgroup $\Delta = \{0,2\}$ is a maximal E_Δ-invariant subgroup of $(Z_4,+)$. For $x = 3 \notin \Delta$ the set

$$\mathcal{B} = \{f \in E_\Delta(Z_4) \colon (3)f \in \Delta\} = \{f_0, f_3, f_7, f_8, f_{12}, f_{13}, f_{14}, f_{15}\}$$

is a maximal right ideal of $E_\Delta(Z_4)$.

Theorem 3.9 *Let $B \neq G$ be a sum of all minimal monzero E_Δ-invariant subgroups of a finite group $(G,+)$. If $\Delta \subseteq B$ is a fully invariant subgroup of $(G,+)$ then the nil radical of $E_\Delta(G)$ is nonzero.*

Proof. Let B_i $(i \in I)$ be a collection of all minimal nonzero E_Δ-invariant subgroups of $(G,+)$ and let $\mathcal{A}(B_i)$ be annihilator ideals of the subgroups B_i $(i \in I)$.

We prove first that $\cap_i \mathcal{A}(B_i)$ is nonzero. Suppose, if possible $\cap_i \mathcal{A}(B_i) = (0)$. By using the Proposition 2.4 it follows that $E_\Delta(G)$ contains no nonzero right $E_\Delta(G)$-subgroup consisting of nilpotent elements. Thus, by Theorem 3 of [3] $E_\Delta(G)$ is a direct sum of minimal nonzero $E_\Delta(G)$-subgroups. Hence, by Proposition 3.5 we obtain $\Im(E_\Delta(G)) \subseteq B$. In particular, for identitety map $e \in E_\Delta(G)$ we have $G = (G)e \subseteq B$, i.e. $G = B$. But this contradictory to the supposition that $G \neq B$. Therefore $\cap_i \mathcal{A}(B_i) \neq (0)$. Since the nil radical is the sum of all nil ideals and $\cap_i \mathcal{A}(B_i)$ is nonzero nil ideal, it follows that the nil radical of $E_\Delta(G)$ is nonzero.

PROPOSITION 3.10. *Let Δ be a minimal fully invariant subgroup of a finite group $(G,+)$ and let $\mathcal{N}$ be any nilpotent $E_\Delta(G)$-subgroup of $E_\Delta(G)$. If the normal subgroup $\mathcal{W}$ of the group $(E_\Delta(G),+)$, generated by the set $E_\Delta(G)\mathcal{N}$, contains the relative defect of the subset $E_\Delta(G)\mathcal{N}$ with respect to $E_\Delta(G)$, then $\mathcal{W}$ is a nilpotent ideal of $E_\Delta(G)$.*

PROOF. By Proposition 1.2 $\mathcal{W}$ is an ideal of $E_\Delta(G)$. Since $\mathcal{N}$ is a right $E_\Delta(G)$-subgroup of $E_\Delta(G)$ and $E_\Delta(G)$ has identity, the elements of $\mathcal{W}$ have the form $w = \sum_i (f_i \pm h_i n_i - f_i)$, $(f_i,\ h_i \in E_\Delta(G),\ n_i \in \mathcal{N})$. If $x \in G$, $x \neq 0$, and $n \in \mathcal{N}$, then E_Δ-invariant subgroup of $(G,+)$ generated by $(x)n$ is properly contained in the E_Δ-invariant subgroup generated by x. Indeed, let X be E_Δ-invariant subgroup generated by x and let Y be E_Δ-invariant subgroup generated by $(x)n$. Clearly $Y \subseteq X$. Let us suppose that $Y = X$. Then there exists $f \in E_\Delta(G)$ such that $(x)nf = x$ and, we have a contradiction, because $nf \in \mathcal{N}$ and $\mathcal{N}$ is a nilpotent $E_\Delta(G)$-subgroup. Thus $Y \subset X$.

Let $B = \sum_k B_k$ be a sum of all minimal E_Δ-invariant subgroups of $(G,+)$ and let $w = \sum_i (f_i \pm h_i n_i - f_i) \in \mathcal{W}$, $(f_i, h_i \in E_\Delta(G),\ n_i \in \mathcal{N})$. Then there exists a positive integer p such that $(x)w^p \in B$, because every fully invariant subgroup generated by $(x)h_i n_i$ is properly contained in the fully invariant subgroup generated by $(x)h_i$. Thus,

$$\begin{aligned}(x)w^{p+1} = ((x)w^p)w &= \left(\sum_k b_k\right) w \\ &= \sum_i \left[\left(\sum_k b_k\right) f_i \pm \left(\sum_k b_k\right) h_i n_i - \left(\sum_k b_k\right) f_i\right].\end{aligned}$$

By Proposition 2.5 B is E_Δ-invariant subgroup, i.e.

$$\left(\sum_k b_k\right) h_i n_i = \left(\sum_k b_k'\right) n_i, \quad (b_k, b_k' \in B_k).$$

Let $n_i = \sum_j (\pm t_{ij})$, $(t_{ij} \in End_\Delta(G))$, then

$$\begin{aligned}\left(\sum_k b_k\right) h_i n_i = \left(\sum_k b_k'\right) n_i = \left(\sum_k b_k'\right) \sum_j (\pm t_{ij}) = \\ = \sum_j \pm \left(\sum_k b_k'\right) t_{ij} = \sum_j \pm \left(\sum_k (b_k') t_{ij}\right) + \delta,\ (\delta \in \Delta).\end{aligned}$$

The elements of different minimal E_Δ-invariant subgroups B_k commute elementwise. Thus

$$\left(\sum_k b_k\right) h_i n_i = \sum_k \left[(b_k{}') \sum_j (\pm t_{ij})\right] + \delta = \sum_k (b_k{}') n_i + \delta.$$

Therefore

$$(x)w^{p+1} = \sum_i \left[\left(\sum_k b_k\right) f_i \pm \left(\sum_k (b_k{}')n_i + \delta\right) - \left(\sum_k b_k\right) f_i\right].$$

By Proposition 2.4, $n_i \in \mathcal{A}(B_k)$ for all k and hence $(x)w^{p+1} \in \Delta$. Thus, there exist $\delta', \delta'' \in \Delta$ such that

$$\begin{aligned}(x)w^{p+2} &= ((x)w^{p+1})w = (\delta')w = (\delta') \sum_i (f_i \pm h_i n_i - f_i) = \\ &= \sum_i [(\delta')f_i \pm (\delta')h_i n_i - (\delta')f_i] = \\ &= \sum_i [(\delta')f_i \pm (\delta'')n_i - (\delta')f_i] = 0.\end{aligned}$$

Thus, every element $w \in \mathcal{W}$ is nilpotent. Because G is finite it follows that $\mathcal{W}$ is nilpotent.

THEOREM 3.11. *Let Δ be a minimal fully invariant subgroup of a finite group $(G,+)$ and let $\mathcal{N}$ be any nilpotent $E_\Delta(G)$-subgroup of $E_\Delta(G)$. If the normal subgroup ω of the group $(E_\Delta(G),+)$ generated by the set $E_\Delta(G)\mathcal{N}$ contains the relative defect of the subset $E_\Delta(G)\mathcal{N}$ with respect to $E_\Delta(G)$, then the nil radical $\eta(E_\Delta(G))$ coincides with the radical $J_2(E_\Delta(G))$.*

PROOF. By Proposition 3.10 $\mathcal{N} \subseteq \eta(E_\Delta(G))$, because the nil radical $\eta(E_\Delta(G))$ is the sum of all nil ideals. Thus, $E_\Delta(G)/\eta(E_\Delta(G))$ contains no nonzero nilpotent right $E_\Delta(G)$-subgroups. By using two theorems of Blackett ([3], Thms 1 and 2) it follows that every minimal right ideal of $E_\Delta(G)/\eta(E_\Delta(G))$ contains an idempotent element. By Beidleman [1], a proper ideal B of a near-ring R is called a strong radical-ideal of R if and only if every nonzero right ideal R/B contains a minimal right ideal which contains an idempotent element. Hence, $\eta(E_\Delta(G))$ is a strong radical-ideal of $E_\Delta(G)$. The following step in the proof is the same as that of ([1], Th. 8).

If the group $(G,+)$ is equal to the sum of its minimal fully invariant subgroups, then as an immediate consequence of Proposition 3 of [1], $J_2(E(G)) = (0)$, where $E(G)$ is an endomorphism near-ring. However, this is not true for near-ring $E_\Delta(G)$ if $(G,+)$ is equal to the sum its minimal E_Δ-invariant subgroups, where Δ is a proper minimal E_Δ-invariant subgroup of $(G,+)$. For example, the group $(Z_6,+)$

is a direct sum of a minimal E_Δ-invariant subgroups $B_1 = \Delta = \{0, 3\}$ and $B_2 = \{0, 2, 4\}$, but the radical

$$J_2(E_\Delta(Z_6)) = \mathcal{D} = \{f_0, f_9, F_{12}, f_{14}\} \neq (0),$$

where $\mathcal{D}$ is the defect of the near-ring $E_\Delta(Z_6)$ (table 2). In general, let $(G, +)$ be a direct sum of minimal E_Δ-invariant subgroups, where Δ is a proper E_Δ-invariant subgroup and let $\mathcal{D}$ be the defect of the near-ring $E_\Delta(G)$. Is it $J_2(E_\Delta(G)) = \mathcal{D}$? The answer is connected to the posiibility that every Δ-endomorphism f of $(G, +)$ can be uniqeuly expressed in the form $f = h + \delta$, where $h \in E(G)$ and $\delta \in \mathcal{D}$.

4. Embeddings of near-ring with defect into some $E_\Delta(G)$

The problem of embedding the near-rings with the defect of distributivity is not easy. The following results refer to the particular case and generalize corresponding results for distributively generated near-ring (see [7]).

By using the technique of "right multiplicator" we have.

PROPOSITION 4.1. *Let (R, S) be a near-ring with the defect D. If $A(R) = (0)$, then R embeos in $E_D(R)$.*

PROPOSITION 4.2. *Let R be a near-ring such that $R = A(R) \oplus B$, where B is an ideal of R. Let $D \neq R$ be the defect of distributivity of R. Then D is the defect of the near-ring B.*

PROOF. Since $B \simeq R/A(R)$ it follows that B is a near-ring with the defect D'. On the other hand $A(R) = \{a \in R : ra = 0$, for all $r \in R\}$, i.e. $A(R)$ is a near-ring with the defect $D'' = (0)$. By Proposition 1.3 R is a near-ring with the defect $D = D' \oplus D'' = D'$.

THEOREM 4.3. *Let (R, S) be a near-ring with the defect $D \neq R$ and let R be a direct sum of ideals which include $A(R)$, where $A(R)$ is finite. Then there exist the group $(G, +)$ and its normal subgroup Δ such that R embeds in $E_\Delta(G)$.*

PROOF. Let $R = A(R) \oplus B$. By Proposition 3 of [7], $A(R)$ embeds in some $E(G_1)$. Bu Lemma 2 of [7], $A(B) = (0)$. Since D is a defect of B (Proposition 4.2), it follows that B embeds in $E_D(B)$ (Proposition 4.1). Thus, R embeds in $\mathcal{R} = E(G_1) \oplus E_D(B)$, whereby multiplication on $\mathcal{R}$ is componentwise. Let $\mathcal{D}$ be a defect of the near-ring $E_D(B)$. Then, by Proposition 1.3 it follows that $\mathcal{R}$ is a near-ring with defect $\mathcal{D} \neq \mathcal{R}$, because the defect of $E(G_1)$ is zero. The nearring $\mathcal{R}$ contains identity $e = (e_1, e_2)$, where $e_1 \in E(G_1)$ and $e_2 \in E_D(B)$ are identity mappings, thus $A(\mathcal{R}) = (0)$. Hence by Proposition 4.1 $\mathcal{R}$ embeds in $E_D(\mathcal{R})$. Consequently, there exist the group $(G, +)$ and its normal subgroup Δ such that R embeds in $E_\Delta(G)$.

DEFINITION. Let $(R, +)$ be a direct sum of the subgroups $(A, +)$ and $(B, +)$. Let $(A, +, .)$ and $(B, +, .)$ be two subnear-rings of the nearring

TABLE 1.

The Δ-endomorphisms of $(Z_4, +)$ for $\Delta = \{0, 2\}$.

The group $(E_\Delta(Z_4), +)$ and the semigroup $(E_\Delta(Z_4), \circ)$· Δ-endomorphisms

0123	+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_0 = 0000	0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
f_1 = 0123	1	1	3	0	2	7	12	13	9	11	14	8	15	6	5	4	10
f_2 = 0321	2	2	0	3	1	14	13	12	4	10	7	15	8	5	6	9	11
f_3 = 0202	3	3	2	1	0	9	6	5	14	15	4	11	10	13	12	7	8
f_4 = 0103	4	4	7	14	9	3	8	15	2	6	0	12	13	11	10	1	5
f_5 = 0121	5	5	12	13	6	8	3	0	11	9	13	7	14	2	1	10	4
f_6 = 0323	6	6	13	12	5	15	0	3	10	4	8	14	7	1	2	11	9
f_7 = 0222	7	7	9	4	14	2	11	10	0	13	1	6	5	15	8	3	12
f_8 = 0220	8	8	11	10	15	6	9	4	13	0	5	2	1	14	7	12	3
f_9 = 0301	9	9	14	7	4	0	15	8	1	5	3	13	12	10	11	2	6
f_{10} = 0101	10	10	8	15	11	12	7	14	6	2	13	3	0	9	4	5	1
f_{11} = 0303	11	11	15	8	10	13	14	7	5	1	12	0	3	4	9	6	2
f_{12} = 0200	12	12	6	5	13	11	2	1	15	14	10	9	4	0	3	8	7
f_{13} = 0002	13	13	5	6	12	10	1	2	8	7	11	4	9	3	0	15	14
f_{14} = 0020	14	14	4	9	7	1	10	11	3	12	2	5	6	8	15	0	13
f_{15} = 0022	15	15	10	11	8	5	4	9	12	3	6	1	2	7	14	13	0

∘	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	0	2	1	3	9	5	6	7	15	4	10	11	13	12	14	8
3	0	3	3	0	0	3	3	3	3	0	0	0	0	0	3	3
4	0	4	9	3	4	10	11	3	12	9	10	11	12	13	0	13
5	0	5	6	3	10	5	6	7	7	11	10	11	3	0	14	14
6	0	6	5	3	11	5	6	7	14	10	10	11	0	3	14	7
7	0	7	7	0	0	7	7	7	7	0	0	0	0	0	7	7
8	0	8	8	0	0	8	8	8	8	0	0	0	0	0	8	8
9	0	9	4	3	9	10	11	3	13	4	10	11	13	12	0	12
10	0	10	11	3	10	10	11	3	3	11	10	11	3	0	0	0
11	0	11	10	3	11	10	11	3	0	10	10	11	0	3	0	3
12	0	12	12	0	0	12	12	12	12	0	0	0	0	0	12	12
13	0	13	13	0	0	13	13	13	13	0	0	0	0	0	13	13
14	0	14	14	0	0	14	14	14	14	0	0	0	0	0	14	14
15	0	15	15	0	0	15	15	15	15	0	0	0	0	0	15	15

The near-ring $E_\Delta(Z_4)$ has the defect $\mathfrak{D} = \{f_0, f_3, f_{12}, f_{13}\}$.

TABLE 2.

The Δ-endomorphisms of $(Z_6, +)$ for which the subgroups $B_1=\Delta=\{0, 3\}$ ane $B_2=\{0, 2, 4\}$ are invariant. The group $(E_\Delta(Z_6), +)$.

Δ-endomorphisms

012345	+	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$_0=000000$	0	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$f_1=012345$	1	1	2	3	4	5	0	10	11	7	8	9	6	15	16	17	18	19	20	21	14	13	22	23	12
$f_2=024024$	2	2	3	4	5	0	1	9	6	11	7	8	10	18	19	20	21	14	13	22	17	16	23	12	15
$f_3=030303$	3	3	4	5	0	1	2	8	10	6	11	7	9	21	14	13	22	17	16	23	20	19	12	15	18
$f_4=042042$	4	4	5	0	1	2	3	7	9	10	6	11	8	22	17	16	23	20	19	12	13	14	15	18	21
$f_5=054321$	5	5	0	1	2	3	4	11	8	9	10	6	7	23	20	19	12	13	14	15	16	17	18	21	22
$f_6=012045$	6	6	10	9	8	7	11	2	0	5	4	3	1	16	15	22	19	18	23	14	21	12	17	20	13
$f_7=054021$	7	7	11	6	10	9	8	0	4	3	2	1	5	20	23	18	13	12	21	16	15	22	19	14	17
$f_8=042342$	8	8	7	11	6	10	9	5	3	2	1	0	4	17	22	15	20	23	18	13	12	21	16	19	14
$f_9=030003$	9	9	8	7	11	6	10	4	2	1	0	5	3	14	21	12	17	22	15	20	23	18	13	16	19
$f_{10}=024324$	10	10	9	8	7	11	6	3	1	0	5	4	2	19	18	23	14	21	12	17	22	15	20	13	16
$f_{11}=000300$	11	11	6	10	9	8	7	1	5	4	3	2	0	13	12	21	16	15	22	19	18	23	14	17	20
$f_{12}=000003$	12	12	15	18	21	22	23	16	20	17	14	19	13	0	11	9	1	6	8	2	10	7	3	4	5
$f_{13}=000303$	13	13	16	19	14	17	20	15	23	22	21	18	12	11	0	3	6	1	4	10	2	5	9	8	7
$f_{14}=030000$	14	14	17	20	13	16	19	22	18	15	12	23	21	9	3	0	8	4	1	7	5	2	11	6	10
$f_{15}=012342$	15	15	18	21	22	23	12	19	13	20	17	14	16	1	6	8	2	10	7	3	9	11	4	5	0
$f_{16}=012042$	16	16	19	14	17	20	13	18	12	23	22	21	15	6	1	4	10	2	5	9	3	0	8	7	11
$f_{17}=042345$	17	17	20	13	16	19	14	23	21	18	15	12	22	8	4	1	7	5	2	11	0	3	6	10	9
$f_{18}=024021$	18	18	21	22	23	12	15	14	16	13	20	17	29	2	10	7	3	9	11	4	8	6	5	0	1
$f_{19}=024321$	19	10	14	17	20	13	16	21	15	12	23	22	18	10	2	5	9	3	0	8	4	1	7	11	6
$f_{20}=054024$	20	29	13	16	19	14	17	12	22	21	18	15	23	7	5	2	11	0	3	6	1	4	10	9	8
$f_{21}=030300$	21	21	22	23	12	15	18	17	19	16	13	20	14	3	9	11	4	8	6	5	7	10	0	1	2
$f_{22}=042045$	22	22	23	12	15	18	21	20	14	19	16	13	17	4	8	6	5	7	10	0	11	9	1	2	3
$f_{23}=054324$	23	23	12	15	18	21	22	13	17	14	19	16	20	5	7	10	0	11	9	1	6	8	2	3	4

The semigroup $(E_\Delta(Z_6), \circ)$. The near-ring $E_\Delta(Z_6)$ has the defect

$\mathcal{D} = \{f_0, f_9, f_{12}, f_{14}\}$.

$\circ$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
2	0	2	4	0	2	4	2	4	2	0	4	0	0	0	0	2	2	2	4	4	4	0	2	4
3	0	3	0	3	0	3	0	0	3	0	3	3	0	3	0	3	0	3	0	3	0	3	0	3
4	0	4	2	0	4	2	4	2	4	0	2	0	0	0	0	4	4	4	2	2	2	0	4	2
5	0	5	4	3	2	1	7	6	10	9	8	11	14	21	12	19	18	23	16	15	22	13	20	17
6	0	6	2	9	4	7	6	7	4	9	2	0	0	12	14	16	16	22	18	18	20	14	22	20
7	0	7	4	9	2	6	7	6	2	9	4	0	14	14	12	18	18	20	16	16	22	12	20	22
8	0	8	2	11	4	10	4	2	8	0	10	11	0	11	0	8	4	8	2	10	2	11	4	10
9	0	9	0	9	0	9	0	0	9	0	9	9	0	9	0	9	0	9	0	9	0	9	0	9
10	0	10	4	11	2	8	2	4	10	0	8	11	0	11	0	10	2	10	4	8	4	11	2	8
11	0	11	0	11	0	11	0	0	11	0	11	11	0	11	0	11	0	11	0	11	0	11	0	11
12	0	12	0	0	0	12	0	12	12	0	12	12	0	12	0	12	0	12	0	12	0	12	0	12
13	0	13	0	13	0	13	0	0	13	0	13	13	0	13	0	13	0	13	0	13	0	13	0	13
14	0	14	0	14	0	14	0	0	14	0	14	14	0	14	0	14	0	14	0	14	0	14	0	14
15	0	15	2	3	4	23	16	20	8	14	10	11	0	11	14	15	16	18	2	10	20	21	4	23
16	0	16	2	14	4	20	16	20	4	14	2	0	0	0	14	16	16	4	2	2	20	14	4	20
17	0	17	2	13	4	19	22	18	8	12	10	11	12	13	0	8	4	17	18	19	2	11	22	10
18	0	18	4	12	2	22	18	22	2	12	4	0	0	0	12	18	18	18	4	4	22	12	2	22
19	0	19	4	13	2	17	18	22	10	12	8	11	0	11	12	19	18	10	4	8	22	13	2	17
20	0	20	4	14	2	16	20	16	2	14	4	0	14	14	0	2	2	20	16	16	4	0	20	4
21	0	21	0	21	0	21	0	0	21	0	21	21	0	21	0	21	0	21	0	21	0	21	0	21
22	0	22	2	12	4	18	22	18	4	12	2	0	12	12	0	4	4	22	18	18	2	0	22	2
23	0	23	4	21	2	15	20	16	10	14	8	11	14	21	0	10	2	23	16	15	4	11	20	8

$(R,+,.)$ with the defect D. A multiplication on R is D-componentwise if for all $a,\ a' \in A$ and $b,\ b' \in B$ there exists $d \in D$ such that $(a+b)(a'+b') = aa' + bb + d$. We say that R is a D-direct sum of thee subnear-rings A and B.

Let $E_\Delta(G)$ be a Δ-endomorphism near-ring with the defect $\mathcal{D}$. For some idempotent $e \in E_\Delta(G)$ let $\mathcal{A}$ be the subgroup of $(E_\Delta(G),+)$ generated by $\{s - es\colon s \in End_\Delta(G)\}$ and $\mathcal{M}$ be the subgroup of $(E_\Delta(G),+)$ generated by $\{es\colon s \in End_\Delta(G)\}$.

THEOREM 4.4. *Let $G = B \oplus C$ be a direct sum of E_Δ-invariant subgroups B and C, where B is summand and Δ is a subset of one of the summands. If e is the projection map $e\colon G \to B$ and $\mathcal{AM} \subseteq \mathcal{D}$, then $E_\Delta(G)$ is the $\mathcal{D}$-direct sum of the subnear-rings $\mathcal{A}$ and $\mathcal{M}$, where $\mathcal{D}$ is the defect of $E_\Delta(G)$.*

PROOF. The projection map $e\colon G \to B$ is an endomorphism of $(G,+)$. The idempotent $e \in End(G)$ is a right identity for $\mathcal{M}$. Hence, $\mathcal{M}$ is a subnear-ring of $E_\Delta(G)$. Also, by Corollary 2.3 of [11] it follows that $\mathcal{A}$ is an ideal of $E_\Delta(G)$. Because B and C commute elementwise and B is E_Δ-invariant abelian summand, it follows that the decomposition $E_\Delta(G) = \mathcal{A} + \mathcal{M}$ has $\mathcal{M}$ in the additive center of $E_\Delta(G)$, i.e. semidirect sum $\mathcal{A} + \mathcal{M}$ is direct.

We shall now prove that the multiplication on $E_\Delta(G)$ is $\mathcal{D}$-componentwise. Let $a, a' \in \mathcal{A}$ and $m, m' \in \mathcal{M}$, where $a' = s' - es'$, $m = et$, $m' = et'(s', t, t' \in End_\Delta(G))$. Then

$$
\begin{aligned}
(a+m)(a'+m') &= (a+m)(s'-es') + (a+m)et' \\
&= (a+m)s' - (a+m)es' + (a+m)et' = \\
&= as' + ms' + \delta_1 - (aes' + mes' + \delta_2) + aet' + met' + \delta_3 = \\
&= as' - aes' + aet' + ms' - mes' + met' + \delta = \\
&= aa' + am' + ma' + mm' + \delta \\
&= aa' + mm' + \delta',\ \ (\delta_1, \delta_2, \delta_3, \delta, \delta' \in \mathcal{D})
\end{aligned}
$$

because $ma' = et(s' - es') = ets' - etes' = 0$ and $\mathcal{AM} \subseteq \mathcal{D}$.

For example, if for an idempotent of the near-ring $E_\Delta(Z_6)$ with the defect $\mathcal{D} = \{f_0, f_9, f_{12}, f_{14}\}$ (table 2) we take the map $e = f_3\colon G \to B = \{0,3\}$ then, $E_\Delta(Z_6)$ is a $\mathcal{D}$-direct sum of the subnear-rings

$$\mathcal{A} = \{f_0, f_2, f_4, f_6, f_7, f_9, f_{11}, f_{12}, f_{13}, f_{14}, f_{16}, f_{18}, f_{20}, f_{22}\}$$

and $\mathcal{M} = \{f_0, f_3\}$

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