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SOME FIXED POINT THEOREMS IN BOOLEAN ALGEBRA

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A Boolean transformation is a mapping

$$F: D \to B^n$$

where $D \leq B^n$ is a set of solutions of Boolean equations $d_j(X) = 0 \;\; j = 1, \dots, k$. F is defined as follows

$$F(X) = (f(X), \dots, f_n(X)), \quad X = (x_1, \dots, x_n),$$

each f_i is a Boolean function $f_i: B^n \to B, \;\; i = 1, \dots, \;\; n, \;\; B = \{0, 1\}$.

DEFINITION 1. X is a fixed point of a Boolean transformation F with restriction $d_j(X) = 0, \;\; j = 1, \dots, k$, if and only if

$$F(X) = X \text{ and } d_j(X) = 0, \quad (j = 1, \dots, k). \tag{1}$$

X is a fixed point of a Boolean transformation F with restrictions d_j $(j = 1, \dots, k)$ if X is a solution of the system of the Boolean equations:

$$f_i(X) = x_i, \;\; i = 1, \dots, n; \;\; d_j = (X) = 0, \;\; j = 1, \dots, k, \tag{1'}$$

which follows from the definitions.

LEMMA 1. *The system of Boolean equations* (1′) *is equivalent to the Boolean equation:*

$$\bigcup_{j=1}^{k} d_j(X) \cup \bigcup_{i=1}^{n} (x_i \bar{f}_i(X) \cup \bar{x}_i f_i(X = 0)) \tag{2}$$

PROOF. It is know that Boolean equations $f = g$ and $\bar{f}g \cup \bar{f}g = 0$ are equivalent. If this property is applied to the system (1) it will become an equivalent system of Boolean equations:

$$\begin{aligned} x_i\bar{f}_i(X) \cup \bar{x}_i f_i(X) &= 0, \qquad i = 1, \dots , n \\ d_j(X) &= 0 \qquad j = 1, \dots , k. \end{aligned} \tag{3}$$

Furthermore, we know that a system of Boolean equations $g_i = 0$, $i = 1, \dots , m$ is equivalent to a Boolean equation $\bigcup_{i=1}^{m} g_i = 0$, if this property is applied to the system (3) it will become the Boolean equation (2) which is equivalent with the system (3).

LEMMA 2. *The system of Boolean equations* (2) *has a solution if and only if*

$$\prod_{\alpha\in B^n} \left[\bigcup_{j=1}^{n} d_j(\alpha) \cup \bigcup_{i=1}^{n} (\alpha_i \bar{f}_i(\alpha) \cup \bar{\alpha}_i f(\alpha)) \right] = 0, \tag{4}$$

where $\alpha = (\alpha_1, \dots , \alpha_n)$.

PROOF. Let $f(x)$ be the left side of the equation (2). It is known that every Boolean function can be written in a disjunctive cannonical form. Therefore, the equation (2) can be written in the following form:

$$f(x_1, \dots , x_n) = \bigcup_{(\alpha_1,\dots,\alpha_n)\in B^n} f(\alpha_1, \dots , \alpha_n) x_1^{\alpha_1} \cdots x_n^{\alpha_n} = 0, \tag{5}$$

where $x^0 = \bar{x}$, $x^1 = x$.

The equation (5) has a solution if and only if

$$\bigcap_{\alpha\in B^n} f(\alpha) = 0. \tag{6}$$

The lemma is proved.

Directly from the lemmas 1 and 2 we have the following theorem.

THEOREM 1. *A Boolean transformation* $f(x) = x$ *with the restrictions* $d_j(x) = 0$ $(j = 1, \dots , k)$ *has a solution if and only if the condition* (4) *is fulfilled.*

The matrix $A = [a_{ij}]$, $i, j = 1, \dots , n$, *is a Boolean matrix when the elements* a_{ij} *are Boolean terms from the Boolean algebra* $(\{0, 1\}, \cup, \cdot, -)$. *The linear Boolean transformation is a linear operator*

$$F(X) = AX \tag{7}$$

where $X = [x_{ij}]$

$$A = [a_{ij}], \quad i, j = 1, \dots , n.$$

The matrix product is

$$AX = \left[\bigcup_{k=1}^{n} a_{ik}x_{kj}\right].$$

Let A be a given matrix. The problem is to find a matrix X, such that

$$X = AX. \tag{8}$$

The Boolean matrix equation (8) *is equivalent to the system of Boolean equations*

$$\bigcup_{k=1}^{n} a_{ik}x_{kj} = x_{kj}, \quad i,j = 1,\dots,n. \tag{9}$$

If we replace (9) *in* (2) *follows:*

Consequence 1. The system of Boolean equations (9) is equivalent to the Boolean equations:

$$\bigcup_{i,j=1}^{n}\left(\bar{x}_{ij}\bigcup_{k=1}^{n} a_{ik}x_{kj} \cup x_{ij}\prod_{k=1}^{n}(\bar{a}_{ik}\cup\bar{x}_{kj})\right) = 0 \tag{10}$$

The replacement of (10) in (4) gives:

Consequence 2. The system of Boolean equations (9) has a solution if and only if

$$\prod_{\alpha\in B^{n^2}}\left[\bigcup_{i,j=1}^{n}\left(\bar{\alpha}_{ij}\bigcup_{k=1}^{n} a_{ik}\alpha_{kj} \cup \alpha_{ij}\prod_{k=1}^{n}(\bar{a}_{ik}\cup\bar{\alpha}_{kj})\right)\right] = 0, \tag{11}$$

where

$$\alpha = (\alpha_{11},\dots,\alpha_{1n},\alpha_{21},\dots,\alpha_{2n},\dots,\alpha_{n1},\dots,\alpha_{nn}).$$

Directly from the Consequence 2. the statement follows:

Theorem 2. *Let A be a given matrix. The matrix equation $X = AX$ has a solution if and only if the condition* (11) *is fulfilled.*

Lemma 3. The Boolean matrix equation $AX = X$ always has at least one solution. That is

$$X = 0.$$

Theorem 3. *The Boolean matrix equation $AX = X$ has a singular solution $x = 0$ if and only if the matrix A is of the form:*

$$\begin{vmatrix} 0\dots0 & a_{1k} & 0\dots0 \\ \dots & \dots & \dots \\ 0\dots0 & a_{k-1k} & 0\dots0 \\ 0\dots0 & 0 & 0\dots0 \\ 0\dots0 & a_{k+1k} & 0\dots0 \\ \dots & \dots & \dots \\ 0\dots0 & a_{nk} & 0\dots0 \end{vmatrix} \tag{12}$$

where $a_{1k}, \dots, a_{k-k}, a_{k+1k}, \dots, a_{nk}$ *are the elements of the set* $\{0,1\}$.

Proof. Let A be a matrix of the form (12). If we replace it in the Boolean matrix equation $AX = X$ we get a system of Boolean equations:

$$\text{(13)} \qquad \begin{array}{llll} \text{for} \quad i \neq k, & a_{ik}x_{kj} = x_{ij}, & i,j = 1,\dots,n \\ \text{for} \quad i = k & 0 = x_{ij} & j = 1,\dots,n. \end{array}$$

From (13) it follows that $x_{ij} = 0$, $i,j = 1,\dots,n$, i.e. $X = 0$.

Suppose that the matrix A^* is not of the form (12) and $X = 0$ is a singular solution, i.e. there exist some elements $a_{i,ji} \neq 0$ which are not in the k-th column.

Let us replace the matrix A^* in the Boolean matrix equation $AX = X$. We get a system of Boolean equations:

$$\text{(14)} \qquad \begin{array}{lllll} \text{for } i \neq k, & i \neq i_1, & a_{ik}x_{ki} = x_{ij}, & i,j = 1,\dots,n \\ \text{for } i \neq k, & i = i_1, & a_{i_1 j}x_{i,j} \cup a_{i_1 j}x_{ij}x_{i_1,j}, & j = 1,\dots,n \\ \text{for } i = k & & 0 = x_{ij}, & j = 1,\dots,n. \end{array}$$

The system of Boolean equations (14) is equivalent to the system of Boolean equations

$$\text{(15)} \qquad \begin{array}{lllll} \text{for } i \neq k & i = i_1, & x_{ij} = 0, & i,j = 1,\dots,n \\ \text{for } i \neq k, & i \neq i_1, & a_{i_1 j}x_{i_1 j} = x_{i_1 j}, & j = 1,\dots,n \\ \text{for } i = k & & 0 = x_{ij}, & j = 1,\dots,n. \end{array}$$

Zero matrix is not the only solution of the system (15). It is contradiction.

So the theorem is proved.

Lemma 4. $A \cup X = X$ (where $A \cup X = [a_{ij} \cup x_{ij}]$ $i,j = 1,\dots,n$) if and only if $X = A \cup P$, P is an arbitrary Boolean $n \times n$ matrix.

Proof. The Boolean matrix equation $A \cup X = X$ is equivalent to a system of Boolean matrices:

$$\text{(16)} \qquad a_{ij} \cup x_{ij} = x_{ij}, \quad i,j = 1,\dots,n.$$

The system (16) is equivalent to the system of Boolean equations:

$$\text{(17)} \qquad a_{ij}\bar{x}_{ij} = 0, \quad i,j = 1,\dots,n.$$

It is known that the equations $AX \cup B\bar{X} = 0$, $X = \bar{A}P \cup B\bar{P}, P \in \{0,1\}$ are equivalent. If we use this property in (16) we get the system of Boolean equations.

$$\text{(18)} \qquad x_{ij} = a_{ij} \cup p_{ij}, \quad p_{ij} \in \{0,1\}, \quad i,j = 1,2,\dots,n.$$

This system is equivalent to the system (16) therefore, from (18) there follows that

$$X = A \cup P, \text{ where} P \text{ is an arbitrary Boolean matrix.}$$

LEMMA 5. $A \odot X = X \quad (A \odot X = [a_{ij}, x_{ij}], \quad i,j = 1,\dots,n)$ *if and only of* $X = A \odot P$, P *is an arbitrary Boolean* $n \times n$ *matrix.*

PROOF. The Boolean matrix $A \odot A = X$ is equivalent to the system of Boolean equations

$$a_{ij}x_{ij} = x_{ij}, \quad i,j = 1,\dots,n. \tag{19}$$

The system (19) is equivalent to the system of Boolean equations

$$\bar{a}_{ij}x_{ij} = 0, \quad i,j = 1,\dots,n. \tag{20}$$

Let us aply to (20) the property that the equations $AX \cup B\bar{X} = 0$, $X = \bar{A}P \cup B\bar{P}$, $P \in \{0,1\}$ are equivalent. We get the system of Boolean equations:

$$x_{ij} = a_{ij}p_{ij} \quad p_{ij} \in \{0,1\}, \quad i,j = 1,\dots,n. \tag{21}$$

Therefore, the system (21) is equivalent to the system (19). From (21) there follows that:

$$X = A \odot P, \quad P \text{ is an arbitrary Boolean matrix.}$$

LEMMA 6. $A * X = X$ *(where* $A * P = [\bar{a}_{ij}x_{ij} \cup a_{ij}\bar{x}_{ij}]$ $i,j = 1,\dots,n)$ *if and only if* $X = A * P$*,* P *is an abritrary Boolean matrix.*

PROOF. The Boolean matrix $A * P = X$ is equivalent to the system of Boolean equations

$$\bar{a}_{ij}x_{ij} \cup a_{ij}\bar{x}_{ij} = x_{ij}, \quad i,j = 1,\dots,n. \tag{22}$$

This system is equivalent to the system of Boolean equations

$$a_{ij}x_{ij} \cup a_{ij}\bar{x}_{ij} = 0 \quad i,j =,\dots,n. \tag{23}$$

The system (23) is equivalent to the system of Boolean equations:

$$x_{ij} = \bar{a}_{ij}p_{ij} \cup a_{ij}\bar{p}_{ij}, \quad i,j = 1,\dots,n, \quad p_{ij} \in \{0,1\},$$

i.e. $X = A * P$, P is an arbitrary Boolean matrix.

THEOREM 4. *The Boolean matrix equation* $A \oplus X = X$ *(where*

$$A \oplus X = \left[\bigcup_{k=1}^{n} e_{ik}\bar{x}_{kj} \cup \bar{a}_{ik}x_{kj}\right] \quad i,j = 1,\dots,n)$$

for a given matrix A has a solution if and only if

$$\prod_{\alpha\in B^{n^2}}\left(\bigcup_{i,j=1}^{n}\left(\alpha_{ij}\bigcup_{k=1}^{n}(\bar{a}_{ik}\alpha_{kj}\cup a_{ik}\bar{\alpha}_{kj})\cup\alpha_{ij}\prod_{k=1}^{n}(a_{ik}\cup\bar{\alpha}_{kj})(\bar{\alpha}_{ik}\cup\alpha_{kj})\right)\right)=0$$

where $\alpha=(\alpha_{11},\dots,\alpha_{1n},\alpha_{21},\dots,\alpha_{2k},\dots,\alpha_{n1},\dots,\alpha_{nn})$.

THEOREM 5. *The Boolean matrix equations* $A\circ X=X$ *(where*

$$A\circ X=\left[\bigcup_{k=1}^{n}(a_{ik}\cup x_{kj})\right].\quad i,j=1,\dots,n)$$

for a given matrix A has a solution if and only if

$$\prod_{\alpha\in B^{n^2}}\left[\bigcup_{i,j=1}^{n}\left(\bar{\alpha}_{ij}\bigcup_{k=1}^{n}(a_{ik}\cup\alpha_{kj})\cup\alpha_{ij}\prod_{k=1}^{n}\bar{a}_{ik}\bar{\alpha}_{kj}\right)\right]=0$$

where $\alpha=(\alpha_{11},\dots,\alpha_{1n},\alpha_{21},\dots,\alpha_{2n},\dots,\alpha_{n1},\dots,\alpha_{nn})$.

The proof of Theorem 4. and Theorem 5. directly follows form Lemma 2.

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