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WEAK CONVERGENCE OF THE SEQUENCE OF SUCCESSIVE APPROXIMATIONS FOR PARA–NONEXPANSIVE MAPPINGS

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Let T be a mapping from a normed linear space X into itself and let $F(T)$ be the set of all its fixed points. T is said to be nonexpansive if

$$\|T(x) - T(y)\| \leq \|x - y\|$$

for all x, y in X, and quasi-nonexpansive if

$$\|T(x) - z\| \leq \|x - z\|$$

for all $x \in X$ and $z \in F(T)$. We shall say that T is para-nonexpansive if

$$\|T(x) - T(y)\| \leq \max\left\{\frac{\|x - T(x)\| \cdot \|y - T(y)\|}{\|x - y\|},\ \ \|x - y\|\right\}$$

for all $x, y \in X$, $x \neq y$. T is called asymptotically regular if

$$\|T^{n+1}(x) - T^n(x)\| \to 0 \text{ as } n \to \infty$$

for all $x \in X$.

In this paper we study the weak convergence of the sequence of successive approximations for a para-nonexpansive mapping. In this connection it is interesting to note that a mapping may be para-nonexpansive without being nonexpansive. For example, let $T: R \to R$, where R is the set of real numbers, be defined as follows:

$$T(x) = \begin{cases} 1, & x \neq 1 \\ \dfrac{1}{2}, & x = 1. \end{cases}$$

It is easy to check that T is para-nonxepansive. However, as T is discontinuous it cannot be nonexpansive.

A normed linear space X is said to possess the property (P) if for any sequence $\{x_n\}$ in X converging weakly to x_0,

$$\liminf_{n\to\infty} \|x_n - x\| > \liminf_{n\to\infty} \|x_n - x_0\|,$$

for all $x \neq x_0$.

1. In order to prove our main result, we shall need the following lemmas:

LEMMA 1.1. *(Opial [4, Lemma 1]). Every Hilbert space has property (P).*

Although (P) does not hold in all uniformly convex spaces, it is interesting to note that there exist nonreflexive spaces which satisfy (P) [2, p. 24].

LEMMA 1.2. *Let T be a para-nonexpansive self-mapping on a closed convex set C of a Hilbert space such that $F(T)$ is nonempty. Then $F(T)$ is colosed and convex.*

PROOF. Let z be a limit point of $F(T)$ and $\{z_n\}$ be a sequence of distinct points of $F(T)$ converging to z. Then

$$\begin{aligned}
\|z - T(z)\| &= \lim_{n\to\infty} \|z_n - T(z)\| \\
&= \lim_{n\to\infty} \|T(z_n) - T(z)\| \\
&\leq \lim_{n\to\infty} \max\left\{ \frac{\|z_n - T(z_n)\| \cdot \|z - T(z)\|}{\|z_n - z\|}, \|z_n - z\| \right\} \\
&= \lim \|z_n - z\| \\
&= 0.
\end{aligned}$$

Thus $z \in F(T)$.

To prove that $F(T)$ is convex, let $x, y \in F(T)$, $x \neq y$ and $\lambda \in (0,1)$. Taking $z = \lambda x + (1-\lambda)y$, it is easy to check that

$$\|x - y\| = \|x - T(z)\| + \|T(z) - y\| = \|x - z\| + \|z - y\| \tag{1.1}$$

and

$$\begin{cases} (1-\lambda)\|x - y\| = \|x - T(z)\| = \|x - z\|, \\ \qquad\quad \lambda\|x - y\| = \|y - T(z)\| = \|y - z\|. \end{cases} \tag{1.2}$$

Therefore

$$\|x - T(z)\| = \frac{(1-\lambda)}{\lambda} \|T(z) - y\|.$$

Also in view of (1.1) $T(z)$ lies on the line segment joining x and y. Since X is strictly convex,

$$(x - T(z)) = p(T(z) - y)$$

for some $p > 0$. Consequently $p = \frac{1-\lambda}{\lambda}$, and hence

$$\begin{aligned} x - T(z) &= \frac{(1-\lambda)}{\lambda}(T(z) - y), \\ \Rightarrow T(z) &= \lambda x + (1-\lambda)y \\ &= z. \end{aligned}$$

Thus $F(T)$ is convex.

We shall denote by $L(x_0)$, the set of weak limit points of the sequence $\{T^n(x_0)\}$.

THEOREM 1.1. *Let C be a closed convex subset of a Hilbert space X and let T be a para-nonexpansive self-mapping on C with a nonempty $F(T)$. If for any $x_0 \in C$, $L(x_0) \subset F(T)$, then $\{T^n(x_0)\}$ converges weakly to a point in $F(T)$.*

PROOF. If $T^k(x_0) \in F(T)$ for some k, then there is nothing to prove. So let $T^k(x_0) \notin F(T)$ for all k. Now for every $y \in F(T)$,

$$\begin{aligned} \|T^{n+1}(x_0) - y\| &= \|T(T^n(x_0)) - T(y)\| \\ &\le \max\left\{ \frac{\|T^n(x_0) - T^{n+1}(x_0)\| \cdot \|y - T(y)\|}{\|T^n(x_0) - y\|} \|T^n(x_0) - y\| \right\} \\ &= \|T^n(x_0) - y\|, \;\; n = 0, 1, 2, \dots . \end{aligned}$$

Therefore the sequence $\{\|T^n(x_0) - y\|\}$ is nonincresaing, and hence there exists $r(y) \ge 0$ such that

$$\lim_{n\to\infty} \|T^n(x_0) - y\| = r(y).$$

Now for any $d \ge 0$, let

$$F_d = \{y \in F(T) : r(y) \le d\}.$$

It follows by Lemma 1.2 that, if d is large enough, F_d is a closed, convex and bounded subset of $F(T)$. Since X is reflexive, we can find a smallest d^* such that F_{d^*} is nonempty. By strict convexity of X, F_{d^*} will have only one element z. We assert that $\{T^n(x_0)\}$ converges to z weakly. If not, by the reflexivity of X and the boundedness of $\{T^n(x_0)\}$, there exists a subsequence $\{T^{n_i}(x_0)\}$ converging weakly to y different from z. By hypothesis $y \in F(T)$. Since X satisfies (P),

$$d^* = r(z) = \liminf_{i\to\infty} \|T^{n_i}(x_0) - z\| > \liminf_{i\to\infty} \|T^{n_i}(x_0) - y\| = r(y),$$

which contradicts the choise of d^*.

COR. 1.1. *Let C be a closed convex set in a Hilbert space X and let T be a nonexpansive self-mapping on C such that $F(T)$ is nonempty. If for any $x_0 \in C$, $L(x_0) \subset F(T)$, then $\{T^n(x_0)\}$ converges weakly to a point in $F(T)$.*

If T is a nonexpansive asymptotically regular self-mapping defined over a closed convex set C in a Hilbert space X, then

$$L(x_0) \subset F(T). \text{ For, if } \{T^n(x_0)\} \supset \{T^{n_i}(x_0)\} \to z (\neq T(z))$$

weakly, then by the property (P)

$$\begin{aligned} \liminf_{i\to\infty} \|T^{n_i}(x_0) - z\| &< \liminf_{i\to\infty} \|T^{n_i}(x_0) - T(z)\| \\ &= \liminf_{i\to\infty} \|T^{n_i+1}(x) - T(z)\| \\ &\leq \liminf_{i\to\infty} \|T^{n_i}(x_0) - z\|, \end{aligned}$$

a contradiction. Therefore $z = T(z)$, and hence $L(x_0) \subset F(T)$. Thus we have (Opial [4, Theorem 1]).

COR. 1.2. *Let C be a closed convex set in a Hilbert space X and let $T: C \to C$ be a nonexpansive asymptotically regular mapping with $F(T)$ nonempty. Then for any $x_0 \in C$, $\{T^n(x_0)\}$ is weakly convergent to an element of $F(T)$.*

REMARK. All the results proved so far remain true even if the space is reflexive and strictly convex in which the property (P) holds. Moreover these results remain valid even if we take the mappings to be quasi-nonexpansive.

As an immediate consequence we have the following result due to Petryshyn and Williamson [5, Theorem 4.4].

THEOREM A. *Let C be a closed convex subset of a strictly convex reflexive Banach space X having property (P) and if $T: C \to X$ be quasi-nonexpansive such that $F(T)$ is nonempty. Suppose that*

(i) *T is asymptotically regular at a point x_0 and $T^n(x_0) \in C$ for all $n \geq 1$*

(ii) *if $\{T^{n_i}(x_0)\} \to s$ (weakly) in C and $(I - T)T^{n_i}(x_0) \to 0$ then $z = T(z)$. Then $\{T^n(x_0)\}$ converges weakly to a point in $F(T)$.*

We remark that the condition of continuity imposed on T in Theorem 4.4 of [5] is redundant.

2. In this section we study the weak convergence in strictly convex reflexive Banach spaces with a weakly continuous duality mapping, see [1]. It is known that every strictly convex reflexive Banach space with a weakly continuous duality mapping possesses the property (P) [3]. Therefore if C is a closed convex set in a strictly convex reflexive Banach space X with a weakly continuous duality mapping, if $T: C \to C$ is para-nonexpansive with a nonempty $F(T) \supset L(x_0)$, then $\{T^n(x_0)\}$ converges weakly to a point in $F(T)$. In case we take T nonexpansive or quasi-nonexpansive the result still holds. This, in particular, includes the results of Opial [4, Theorem 2] and Petryshyn and Williamson [5. Theorem 4.5] as special cases.

Although the sequences of iterates of a para-nonexpansive mapping T may not converge weakly to fixed point of T, the sequence of iterates of a mapping T_λ

where $T_\lambda = \lambda I + (1-\lambda)T$ for some $\lambda \in (0,1)$, may weakly converge to a fixed point of T. It is known that for a nonexpansive mapping $T\colon C \to C$, where C is a closed convex subset of a Banach space X, T_λ is also nonexpansive for every $\lambda \in (0,1)$. But this is not the case if T is a para-nonexpansive mapping. In fact T_λ may fail to be para-nonexpansive for any $\lambda \in (0,1)$. Take $T\colon R \to R$, where R is the set of real numbers, such that

$$T(x) = \begin{cases} 1, & x \neq 1 \\ \dfrac{1}{2}, & x = 1 \end{cases}$$

and take $x = 11/10$ and $y = 1$. However, as in the case of a nonexpansive mapping, $F(T) = F(T_\lambda)$ and if the space is uniformly convex and $F(T)$ is nonempty then T_λ is asymptotically regular.

Lastly we state without proof the following

THEOREM 2.1. *Let C be a closed convex set in a uniformly convex Banach space with a weakly continuous duality mapping and let $T\colon C \to C$ be a para-nonexpansive maping with $F(T)$ nonempty. If for some $x_0 \in C$, $L_\lambda(x_0)$, the set of weak limit points of $\{T_\lambda^n(x_0)$ is contained in $F(T_\lambda)$, then $\{T_\lambda^n(x_0)\}$ converges weakly to a ponit in $F(T)$.*

REFERENCES

[1] F.E. Browder, *Fixed point theorems for nonlinear semicontractive mappings in Banach spaces,* Arch. Rational Mech. Anal. 21 (1966), 259–269.

[2] E. Lami Dozo, *Operateurs nonexpansifs, P-compacts et proprietes geometriques de la norma,* doctrol dissertations, Universite Libre de Bruxelles, 1970.

[3] J.P. Gossez and E.Lami Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings,* Pac J. Math. 40 (1972), 565–573.

[4] Zdzislaw Opial, *Weak convergence of the sequence of successive saproximations for nonexpansive mappings,* Bull. Amer. Math. Soc. 73 (1967), 591–597.

[5] W.V. Petryshyn and T.W. Williamson, jr., *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mapping,* J. Math. Anal. and Appl. 43 (1973), 459–497.

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