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# A NOTE ON THE TWO CARDINAL PROBLEM

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Let $T$ be a a theory in a countable language $L$ with 1-placed predicate symbol $U$ and let $\mathfrak{A} = \langle A, V, \dots \rangle$ be a model for $L$, $V$ being interpretation of predicate symbol $U$. We say that $\mathfrak{A}$ is an $(\alpha, \beta)$ model if $\mid A \mid = \alpha$ and $\mid V \mid = \beta$. If $T$ has an $(\alpha, \beta)$ model, we say that $T$ admits the pair of cardinals $(\alpha, \beta)$. Given a theory $T$, the question is which pairs $(\alpha, \beta)$ does $T$ admit. Theorems 1 through 5 are from (1) (consequtively, Proposition 3.2.11, Theorem 4.3.10, Corollary 4.3.11, Theorem 6.5.11).

THEOREM 1. *Let $T$ be a theory in a countable language $L$, and let $\alpha, \beta, \gamma$ range over infinite cardinals. Then:*

(i) *if $T$ admits $(\alpha, \beta)$ then $T$ admits $(\gamma, \beta)$ for all $\gamma$ such that $\beta \leq \gamma \leq \alpha$.*

(ii) *if $T$ admits $(\alpha, \beta)$ then $T$ admits all $(\gamma, \gamma)$.*

(iii) *for each $n \in \omega$, there is theory $T$ such that $T$ admits every $(\aleph_n(\alpha), \alpha)$ and $T$ does not admit any $(\aleph_n(\alpha)^+, \alpha)$.*

(iv) *for each $n \in \omega$, there is a theory $T$ such that $T$ admits every $(\aleph_n(\alpha), \alpha)$ and $T$ does not admit any $(\aleph_{n+1}(\alpha), \alpha)$.*

THEOREM 2. *If a countable theory $T$ admits $(\alpha, \beta)$ with $\alpha > \beta \geq \omega$, then $T$ admits the pair $(\omega_1, \omega)$.*

THEOREM 3. *If a theory $T$ admits $(\alpha, \beta)$ and $\beta \geq \omega$, then for all cardinals $\gamma$, $T$ admits $(\alpha^\gamma, \beta^\gamma)$. In fact every $(\alpha, \beta)$ model has an elementary extension, which is an $(\alpha^\gamma, \beta^\gamma)$ model.*

COROLLARY 4. *Assume the* GCH. *Suppose $\alpha \geq \alpha' \geq \beta' \geq \beta \geq \omega$ and $\|L\| \leq \alpha'$. Then every theory $T$ in $L$ which admits $(\alpha, \beta)$, admits $(\alpha', \beta')$.*

THEOREM 5. *Let $L$ have a 1-placed predicate symbol $U$.*

(i) *if $\alpha^\omega \geq \beta' \geq \beta^\omega$ and $\alpha \geq \beta \geq \omega$, then every $(\alpha, \beta)$ model has a complete extension which is an $(\alpha^\omega, \beta')$ model.*

(ii) *suppose* $\alpha \geq \alpha' \geq \beta' \geq \beta^\omega$, $\beta \geq \omega$ *and* $\alpha' \geq \|L\|$. *Then every theory which admits* $(\alpha,\beta)$, *admits* $(\alpha',\beta')$.

*From the above it is clear that, given a pair* $(\alpha,\beta)$ *which a theory* $T$ *admits, interesting would be to get other pairs* $(\gamma,\delta)$ *admitted by* $T$, *such that* $\alpha \geq \gamma \geq \delta \geq \beta$ *does not hold, with* $\gamma$ *and* $\delta$ *being as far apart as possible.*

*Theorem* 3 *is obtained using ultrapowers of* $(\alpha,\beta)$ *model, over an regular ultrafilter with the cardinal* $\gamma$ *as index set. Similarly, it follows that if a theory* $T$ *admits the pair* $(\alpha,\beta)$, *then* $T$ *admits the pair*

$$\left(\left|\prod_D \alpha\right|,\ \left|\prod_D \beta\right|\right),$$

*for any ultrafilter* $D$. *The following theorem is obtained generalizing the proof of Theorem* 3.

THEOREM 6. *If* $T$ *admits pairs* $(\alpha_i,\beta_i)$, *where* $i \in I$ *and* $D$ *is an ultrafilter over* $I$ *then* $T$ *admits the pair*

$$\left(\left|\prod_D \alpha_i\right|,\ \left|\prod_D \beta_i\right|\right).$$

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*Hence, if using ultraproducts in two cardinal problem, the goal would be to keep* $|\prod_D \alpha_i|$ *and* $|\prod_D \beta_i|$ *as far apart as possible.*

*The following theorem is from* (2).

THEOREM 7. *If* $D$ *is* $(\alpha,\beta)$ *regular and* $\nu, k$ *are cardinals such that*

$$\alpha \leq \nu \leq \beta,\ \ \alpha \leq k \leq \beta \text{ and}$$

$k^\nu = k$, *then*

(i) $|\prod_D k| \geq 2^\beta$ *and*

(ii) $|\prod_D k| = |\prod_D k|^\nu$.

COROLLARY 8

(iii) *If* $(cf\gamma)^{\overset{cf\gamma}{\smile}} = cf\gamma$ *then* $|\prod_D cf\gamma|^{cf\gamma} = |\prod_D cf\gamma|$.

(iv) *If* $\gamma$ *is strongly inaccessible or* ($\gamma = \lambda^+$ *and* $2^\lambda = \lambda^+$) *then*

$$\left|\prod_D \gamma\right| = 2^\gamma.$$

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*This indicates that the ultrafilter $D$ above should be as irregular as possible.*

*To obtain nonregular ultrafilters, large cardinal axioms are used. In fact, the smaller the cardinal over which nonregular ultrafilter is constructed, the stronger large cardinal property is used.*

THEOREM 9. *Let $k$ be a measurable cardinal and let $D$ be an normal ultrafiletr over $k$. Let $T$ be a theory in a countable language.*

(i) *if $T$ admits pairs $(\alpha^+, \alpha)$ for a set of $\alpha'$s in $D$ then $T$ admits $(k^+, k)$.*

(ii) *if $T$ admits $(2^\alpha, \alpha)$ for a set of $\alpha'$s in $D$, then $T$ admits the pair $(2^k, k)$.*

(iii) *if $T$ admits $(\alpha_i, \beta_i)$ with $\alpha_i$ being unlimited almost everywhere and $\beta_i$ being limited almost everywhere, with, say $\lambda$ then $T$ admits the pair $(2^k, \lambda)$.*

PROOF. Using method of 4.3 of (1).

THEOREM 10. *Let $k$ be a measurable cardinal with normal ultrafilter $D$ and let $D^+$ be a $k$ complete uniform ultrafilter over $k^+$; let $f = \langle \alpha_i : i \in k \rangle$ and $g = \langle \beta_i : i \in k \rangle$ be such that $g <_D id \leq_D f$, $id$ being identity function on $k$. If a theory $T$ admits pairs $(\alpha_i, \beta_i)$, $i \in k$, then there is $\lambda < k$ such that $T$ admits the pair $(2^{k+}, \lambda)$.*

PROOF. From theorem 9, it follows that $T$ admits $(2^k, \lambda)$. Taking ultrapower modulo filter $D^+$ (which is $(k,k)$-regular and $(k^+, k^+)$-regular), we have, applying Theorem 7, that

$$\left|\prod_{D+} 2^k\right| = \left|\prod_{D+} {k^+}^{k+}\right| = \left|\prod_{D+} k^+ \underbrace{\phantom{k}}^{k+}\right| \underbrace{\phantom{k}}^{k+} = 2^{k+}.$$

The assertion then follows from Theorem 6 and Theorem IX.

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With assumption of huge cardinals Magidor (3) obtained the following result.

THEOREM 11. *If there is a huge cardinal then*

(i) *there is an uniform ultrafilter $D$ over $\omega_2$ such that $|\prod_D \omega| \leq \omega_2$*

(ii) *there is an uniform ultrafilter $E$ over $\omega_3$ which is not $(\omega_3, \omega_1)$ regular and such that*

$$\left|\prod_E \omega_1\right| \leq \omega_3.$$

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*Applying Corrollary 8, Theorem 5 and Theorem 6 we can conclude.*

COROLLARY 12

(i) *if there is a filter* $D$ *over* $\omega_2$ *like in* (i) *of Theorem* 11 *and* $2^{\omega_1} = \omega_2$ *(which is true in the model of* (3)) *then*

*if* $T$ *admits the pair* $(\omega_2, \omega)$ *then* $T$ *admits* $(2^{\omega_2}, \omega_2)$.

(ii) *if there is a filter* $E$ *over* $\omega_3$, *like in* (ii) *of Theorem* 11 *and* $2^{\omega_2} = \omega_3$ *(which is true in the model of* (3)) *then*

*if* $T$ *admits the pair* $(\omega_3, \omega_1)$ *then* $T$ *admits* $(2^{\omega_3}, \omega_3)$.

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*The proof of Theorem* 11 *is general giving rise to similar filters over the other cardinals, which can be used similarly as above.*

## REFERENCES

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