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ON AN INEQUALITY OF P.M. VASIĆ and R.R. JANIĆ

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0. In [1], P.M. Vasić and R.R. Janić have given generalization of the inequality by Z. Opial ([2], see also [3, p. 351]. Their result, in not so rigorous from, is as follows.

THEOREM A. *Let* $p_i (i = 1, \dots, 2n+1)$ *and* $x_i \in [a,b] = I$ $(i = 1, \dots, 2n+1)$ $\left(\frac{\sum_{i=1}^{2k+1} p_i x_i}{\sum_{i=1}^{2k+1} p_i} \in I, k = 1, \dots, n\right)$, *be such real numbers, that it is for every* $k = 1, \dots, n$

$$1^\circ\ p_1 > 0,\ p_{2k} \leqq 0,\ p_{2k} + p_{2k+1} \leqq 0,\ \sum_{i=1}^{2k} p_i \geq 0,\ \sum_{i=1}^{2k+1} p_i > 0;$$

$$2^\circ\ x_{2k} \leqq x_{2k+1},\ \sum_{i=1}^{2k} p_i (x_i - x_{2k+1}) \geqq 0;$$

then, for every convex function f *on* I, *the following inequality holds*

$$\sum_{i=1}^{2n+1} p_i f(x_i) \leqq \left(\sum_{i=1}^{2n+1} p_i\right) f\left(\frac{\sum_{i=1}^{2n+1} p_i x_i}{\sum_{i=1}^{2n+1} p_i}\right). \tag{1}$$

If f is concave, the reverse inequality holds.

If we change the conditions 1° and 2° in Theorem A, we shall show that the following similar results hold:

(α. 1) If the condition 1° holds and the revese inequalites hold in conditions 2°, then, for every convex function f on I, (1) is valid. If f is concave, the reverse inequality in (1) is valid.

(α. 2) If, instead of conditions 1° and 2°, the following ones hold

3° $p_1 > 0$, $p_{2k+1} \geqq 0$, $p_{2k} + p_{2k+1} \geqq 0$, $\sum_{i=1}^{2k} p_i \geqq 0$; $\sum_{i=1}^{2k+1} p_i > 0$;

4° $x_{2k} \leqq x_{2k+1}$, $\sum_{i=1}^{2k-1} p_i(x_i - x_{2k}) \leqq 0$;

then, for every convex function f, the reverse inequality in (1) holds. If f is concave, the inequality (1) holds.

(α. 3) If 3° holds and the reverse inequalitios hold in 4°, then, for every convex function f, the reverse inequality in (1) holds. If f is concave, the inequality (1) holds.

In their proof, P.M. Vasić and R.R. Janić started from the Jensen-Steffensen inequality, in the form postulated by Steffensen [3, p. 109], for $n = 3$. In this form we have a nondecreasing sequence of points. However, the Jensen. Steffensen inequality is valid in the same form for a nonincreasing sequence of points (see, for instance, [4, Theorem A]) and (α. 1) can be proved by complete analogy. If we apply directly the method mathematical induction, given in the proof of P.M. Vasić and R.R. Janić, on the Jensen-Steffensen inequality for $n = 3$, we get (α. 3).

Remark 1. Result (α. 3) is generalization of the inequality by G. Szegö [5] (see also [3, p. 112])

1. We can use Theorem A, $(\alpha, 1)$, $(\alpha, 2)$ and $(\alpha, 3)$, by analogy to Ch.O. Imoru [6], in order to obtain various conditions for which the well-known inequality from Fuchs's generalization [3, p. 165] of the Majorization theorem [3, p. 164] is valid. Denoting by

$$c_k = \sum_{i=1}^{k-1} b_i(x_i - y_i).$$

Then, the following theorem is valid:

Theorem 1. *Let the numbers $b_1 \geqq \cdots \geqq b_n > 0$, and $x_i, y_i \in I$ $(i = 1, \ldots, n)$ $(0 \in I;\ x_{k+1} + c_{k+1}/b_{k+1} \in I,\ k = 1, \ldots, n-1)$ satisfy the conditions*

$$\text{(A)} \qquad y_k \leqq x_{k+1}, \quad (k = 1, \ldots, n;\ x_{n+1} \equiv 0);$$

$$\text{(B)} \qquad \sum_{i=1}^{k} b_i x_i \geqq \sum_{i=1}^{k} b_i y_i, \quad (k = 1, \ldots, n-1);$$

$$\text{(C)} \qquad \sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} b_i x_i.$$

Then, for every convex function on I, the followinng inequality holds

$$\text{(2)} \qquad \sum_{i=1}^{n} b_i f(x_i) \leqq \sum_{i=1}^{n} b_i f(y_i).$$

If f is concave, the reverse inequality holds.

PROOF. Let, in Theorem A, be $x_{2n+1} = 0$ and $p_{2n+1} = 1 - \sum_{i=1}^{2n} p_i$. Then, from (1) we get

$$\sum_{i=1}^{2n} p_i f(x_i) + \left(1 - \sum_{i=1}^{2n} p_i\right) f(0) \leqq f\left(\sum_{i=1}^{2n} p_i x_i\right). \tag{3}$$

For $k = n$, from $1°$, we get

$$P_{2n} \leqq 0, \ \sum_{i=1}^{2n-1} p_i \geqq 1, \ \sum_{i=1}^{2n} p_i \leqq 0. \tag{$(1°)'$}$$

Using the substitutions: $x_{2k-1} \to x_k$, $x_{2k} \to y_k$, $p_{2k-1} \to b_k > 0$, $p_{2k} \to -b_k$, (3) becomes

$$\sum_{i=1}^{n} b_i f(x_i) - \sum_{i=1}^{n} b_i f(y_i) + f(0) \leqq f\left(\sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} b_i y_i\right) \tag{4}$$

and using (C) we get (2).

On the other hand, form $1°$ $(k = 1, \dots, n)$ and $(1°)'$ $(k = n)$ we get $b_{k+1} \leqq b_k$, $b_n \geqq 1$, e.i. $b_1 \geqq b_2 \geqq \cdots \geqq b_n \geqq 1$, and from $2°$ we get (A) and (B).

One can casily conclude that the condition $b = 1$ can be replaced by the condition $b_n > 0$. Namely, when $0 < b_n < 1$, the weights ${b_k}' = b_k/b_n$ satisfy the conditions for which (2) is valid, so multiplying with b_n (i.e. with the previous weights) we can see that (2) is also valid for b_k.

We get the following similar results if we use $(\alpha.\ 1)$, $(\alpha.\ 2)$ or $(\alpha.\ 3)$, instead of Theorem A, in proving a previous theorem.

$(\beta.\ 1)$ If Theorem 1 the condition (C) holds and the reverse inequalities holds in conditions (A) and (B) then, for every convex function f, (2) is valid. If f is concave, the reverse inequality holds.

$(\beta.\ 2)$ Let the real numbers $0 < b_1 \leqq \cdots \leqq b_n$ and $x_i, y_i \in I$ $(i = 1, \dots, n)$ $(0 \in I; x_{k+1} + c_{k+1}/b_{k+1} \in I,\ k = 1, \dots, n-1)$ satisfy the conditions (A) and (C) as conditions (B) with reverse inequalities. Then, for every convex function f, the reverse inequality in (2) is valid. If f is concave, then the inequality (2) holds.

$(\beta.\ 3)$ If in $(\beta.\ 2)$ the conditions (B) and (C) hold and the reverse inequalities hold in conditions (A), then, for every convex function f, the reverse inequality in (2) is valid. If f is concave, then the inequality (2) holds.

If, instead of (C), the following condition is valid

$$\sum_{i=1}^{n} b_i x_i \geqq \sum_{i=1}^{n} b_i y_i, \tag{D}$$

then, for nonincreasing convex function f

$$f\left(\sum_{i=1}^{n} b_i x_i - \sum_{i=1}^{n} b_i y_i\right) \leqq f(0)$$

and from (4) follows (2). Hence the following theorem is valid:

THEOREM 2. *Let the real numbers $b_1 \geqq \cdots \geqq b_n > 0$ and $x_i, y_i \in I$, $(i = 1, \dots, n)$ ($c_n \in I$ if $b_n \geqq 1$ and $c_n/b_n \in I$ if $b_n < 1$; $x_{k+1} + c_{k+1}/b_{k+1} \in I$, $k = 1, \dots n-1$) satisfy the conditions* (A), (B) *and* (D). *Then, for every nonincreasing convex function f on I, the inequality* (2) *is valid. If f is nondecreasing concave, the reverse inequality holds.*

We get by analogy

(γ. 1) If in Theorem 2 the reverse inequalities hold for conditions (A), (B) and (D), then, for every nondecreasing convex function f, (2) is valid. If f is nonicreasing concave, the reverse inequality holds.

(γ. 2) Let the real numbers $0 < b_1 \leqq \cdots \leqq b_n$ and $x_i, y_i \in I$ $(i = 1, \dots, n)$ ($c_n \in I$ if $b_n \leq 1$ and $c_n/b_n \in I$ if $b_n > 1$; $x_{k+1} + c_{k+1}/b_{k+1} \in I$, $k = 1, \dots, n-1$) satisfy conditions (A) as conditions (B) and (D) with reverse inequalities. Then, for every nonicreasing convex function f, the reverse inequality in (2) is valid. If f is nonicreasing concave, then (2) holds.

(γ. 3) If in (γ. 2) the conditions (B) and (D) hold and the reverse inequalities hold in conditions (A), then for every nondecreasing convex function f, the reverse inequality in (2) is valical. If f is noninereasing concave, then (2) holds.

2. Lj.R. Stanković and I.B. Lacković ([7]) proved the following result:

THEOREM B. *Let a and b be nonnegative real numbers and let $a+b \leqq c$. Then for every convex functions $x \mapsto f(x)$ defined defined for all $x \geqq 0$, the following inequality holds*

$$f(a) + f(b+c) \geqq f(a+b) + f(c). \tag{5}$$

If the function f is concave the above inequality is reversed.

Let $a_i (i = 1, \dots, 2n+1)$ be nonnegative real numbers. We shall prove the following generalization Theorem B:

THEOREM 3. *If $a_1 \geqq a_3 \geqq \cdots \geqq a_{2n+1}$ then, for every convex function f on $[0, \infty)$ the following inequality holds*

$$\begin{aligned} f(a_1 + a_2) + \cdots + f(a_{2n-1} + a_{2n}) + f(a_{2n+1}) \geqq \\ \geqq f(a_1) + f(a_2 + a_3) + \cdots + f(a_{2n} + a_{2n+1}). \end{aligned} \tag{6}$$

If f is concave, the reverse inequality holds.

PROOF. Let, in (β. 1), be: $b_i \equiv 1$ $(i = 1, \dots, n)$; $n = n+1$; $x_1 = a_1$, $x_2 = a_2 + a_3, \dots, x_{n+1} = a_{2n} + a_{2n+1}$; $y_1 = a_1 + a_2, \dots, y_n = a_{2n+-1} + a_{2n}, y_{n+1} = a_{2n+1}$. Then, from (2), we get (6)

REMARK 2. From Theorem 3, for $n = 1$, we get that (5) is valid if $a \leqq c$.

By means of complet analogy, substituting: $b_i \equiv 1$ $(i = 1, \dots, n)$; $n = n+1$; $x_1 = a_1 + a_2, \dots, x_n = a_{2n-1} + a_{2n}$, $x_{n+1} = a_{2n+1}$; $y_1 = a_2 + a_3, \dots, y_n = a_{2n} + a_{2n+1}$. $y_{n+1} = a_1$; from (β. 1) and (β. 3) we get the following similar results:

(δ. 1) If $a_2 \geqq a_4 \geqq \cdots \geqq a_{2n}$ and $a_{2k+1} \geqq a_1$ $(k = 1, \dots, n)$, then, for every convex function f on $[0, \infty)$, the reverse inequality in (6) holds. If f is concave, then (6) holds.

(δ. 2) If $a_2 \geqq a_0 \geqq \cdots \geqq a_{2n}$ and $a_{2k+1} \leqq a_1$ $(k = 1, \dots, n)$ then, for every convex function f on $[0, \infty)$, (6) holds. If f is concave the revese inequality holds.

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