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A CONVERGENCE THEOREM FOR A METHOD FOR SIMULTANEOUS DETERMINATION OF ALL ZEROS OF A POLYNOMIAL

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Abstract. As it is well-known the Newton-Raphson method is closely connected with the Taylor polynomial. Using this connection the Ostrowski' fundamental existence theorem for Newton-Raphson method [3], [4] can be proved in an very natural way [6]. The S.B. Prešić's method [7] for simultaneuus determination of all roots of polynomial can be obtained using the interpolation formulae of Newton and Lagrange [5]. We use that fact in the convergence theorem which we prove in this paper. We note that the convergence conditions depend only on the intial points of roots, their distances and on the degree of polynomial.

Let

$$p(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0 \quad (n \geq 2) \tag{1}$$

be a complex polynomial. The S.B. Prešić iteration formulae for simultaneous determination all roots of the polynomial (2) are the following:

$$\begin{aligned} a_{i+1} &= a_i - \frac{p(a_i)}{(a_i - b_i)(a_i - c_i)\cdots(a_i - s_i)} \\ b_{i+1} &= b_i - \frac{p(b_i)}{(b_i - a_i)(b_i - c_i)\cdots(b_i - s_i)} \qquad (i = 0, 1, \dots) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ s_{i+1} &= s_i - \frac{p(s_i)}{(s_i - a_i)(s_i - b_i)\cdots(s_i - r_i)}. \end{aligned} \tag{2}$$

We prove now a convergence theorem for the method (2) which parallels to Ostrowski's fundamental existence theorem for Newton-Raphson method.

THEOREM. *Let $a_0, b_0, \dots, s_0$ be different complex numbers satisfying the following conditions:*

$$\begin{aligned} &A(0)\frac{|-p(a_0)|}{|a_0 - b_0||a_0 - c_0|\cdots|a_0 - s_0|} \leq \frac{|a_0 - b_0||a_0 - c_0|\cdots|a_0 - s_0|}{\sigma n M(0) n^{n-2}} \\ &B(0)\colon \frac{|-p(b_0)|}{|b_0 - a_0||b_0 - c_0|\cdots|b_0 - s_0|} \leq \frac{|b_0 - a_0||b_0 - c_0|\cdots|b_0 - s_0|}{\sigma n M(0)^{n-2}} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &S(0)\colon \frac{|-p(s_0)|}{|s_0 - a_0||s_0 - b_0|\cdots|s_0 - r_0|} \leq \frac{|s_0 - a_0||s_0 - b_0|\cdots|s_0 - r_0|}{\sigma n M(0)^{n-2}} \end{aligned}$$

where $M(0) = \max\{|a_0 - b_0|,\ |a_0 - c_0|, \dots, |r_0 - s_0|\}$. *Form, starting with* $a_0, b_0, \dots, s_0$, *the sequences*

$$(a_i), (b_i), \dots, (s_i) \tag{3}$$

by the recurrence formulae (2). *Then all of the sequence* (3) *converge, i.e. there exist complex numbers* $a, b, \dots, s$ *such that*

$$\lim_{i\to\infty} a_i = a,\ \ \lim_{i\to\infty} b_i = b, \dots,\ \ \lim_{i\to\infty} s_i = s \tag{4}$$

and $a, b, \dots s$ *are all roots of the polynomial* (1). *Moreover, consider the circles*

$$K(a_0) = \{x: |x - a_1| \le |a_1 - a_0|\}, \dots, K(s_0) = \{x: x - s_1| \le |s_1 - s_0|\} \tag{5}$$

and let (x_i) *be any of the sequences* (3) *with* $\lim_{i\to\infty} x_i = I$. *Then* I *and all* x_i *lie in the circle* $K(x_0)$.

Proof. Let $M(i) = \max\{|a_i - b_i|, |a_i - c_i|, \dots, |r_i - s_i|\}(i = 0, 1, \dots)$ and let $A(i), \dots, S(i)(i = 1, 2, \dots)$ be the following formulae

$$\begin{array}{c} \dfrac{|-p(a_i)|}{|a_i - b_i|\cdots|a_i - s_i|} \le \dfrac{|a_i - b_i|\cdots|a_i - s_i|}{\sigma n M(i)^{n-2}} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{|-p(s_i)|}{|s_i - a_i|\cdots|s_i - r_i|} \le \dfrac{|s_i - a_i|\cdots|s_i - r_i|}{\sigma n M(i)^{n-2}} \end{array} \tag{6}$$

respectively. If (x_i) is any of the sequences (3), i.e.

$$x_{i+1} = x_i - \frac{p(x_i)}{(x_i - y_i)\cdots(x_i - v_i)} \quad (i = 0, 1, \dots) \tag{7}$$

where $x_i, y_i, \dots v_i$ is a ayclic permutation of $a_i, b_i, \dots, s_i$, let $X(i)(i = 0, 1, \dots)$ be the corresponding formula (6), i.e. $X(i)$ is the formula

$$\frac{|-p(x_i)|}{|x_i - y_i|\cdots|x_i - v_i|} \le \frac{|x_i - y_i|\cdots|x_i - v_i|}{\sigma n M(i)^{n-2}}. \tag{8}$$

We prove that the sequence (x_i) converges to a root I of $p(x)$ such that I and all x_i lie in the circle $K(x_0)$. To prove that it suffices to prove the following implications $(i = 0, 1, \dots)$:

(α_i) $A(i) \wedge \cdots \wedge S(i) \wedge \operatorname{Dif}(a_i, \dots, s_i) \Rightarrow \operatorname{Dif}(a_{i+1}, \dots, s_{i+1})$

(β_i) $A(i) \wedge \cdots \wedge S(i) \wedge \operatorname{Dif}(a_i, \dots, s_i) \Rightarrow \left(|x_{i+2} - x_{i+1}| \le \dfrac{1}{3}|x_{i+1} - x_i|\right)$

(γ_i) $A(i) \wedge \cdots \wedge S(i) \wedge \operatorname{Dif}(a_i, \dots, s_i) \Rightarrow X(i + 1)$

where Dif $(x, y, \dots, v)$ denotes that all $x, y, \dots, v$ are different numbers, i.e. this is the conjunction:

$$x \neq y \wedge x \neq z \wedge \cdots \wedge u \neq v.$$

Namely, suppose that (α_i), (β_i), (γ_i) are proved for all $i = 0, 1, \dots$ and that $A(0), \dots, S(0)$, Dif $(a_0, \dots, s_0)$ are satisfied. Then, by induction on i it follows immediately

$$(\forall i) \ \mathrm{Dif}\,(a_i, \dots, s_i) \tag{9}$$

$$(\forall i) X(i) \tag{10}$$

where $X(i)$ is any of the formulae (6). Using (9) and (10) from (β_i) we obtain immediatelly

$$(\forall i) \left(|x_{i+2} - x_{i+1}| \leq \frac{1}{3} |x_{i+1} - x_i| \right) \tag{11}$$

Thus we have a sequence of circles

$$K(x_0) \supseteq K(x_1) \supseteq K(x_2) \supseteq \dots$$

with the radius of $K(x_{i+1})$ at most equal to one-third the radius of $K(x_i)$, where

$$K(x_i) = \{x \colon |x - x_{i+1}| \leq |x_{i+1} - x_i|\} \quad (i = 0, 1, \dots) \tag{12}$$

We know that such a sequence converges to a point I. Since each of the circles lies in $K(x_0)$, and $K(x_0)$ is closed, all $x_i (i = 0, 1, \dots)$ and I lie in $K(x_0)$.

To prove that the limit points $a, b, \dots, s$ of the sequences (3) respectively are all roots of $p(x)$ we proceed in the following way. The formulae (2) are equivalent to the identity [6]:

(13)

$$\begin{aligned} &(x - a_{n+1})(x - b_n) \cdots (x - s_n) + (x - a_n)(x - b_{n+1}) \cdots (x - s_n) + \cdots \\ &+ (x - a_n)(x - b_n) \cdots (x - s_{n+1}) - (n-1)(x - a_n)(x - b_n) \cdots (x - s_n) = p(x). \end{aligned}$$

Taking the limit we obtain the identity

$$p(x) = (x - a)(x - b) \cdots (x - s)$$

wherefrom we conclude that $a, b, \dots, s$ are all roots of $p(x)$. It remains to prove that for all $i = 0, 1, \dots$ the implications (α_i), (β_i), (γ_i) hold. First of all assume that the following conjuction

$$A(i) \wedge \cdots \wedge S(i) \wedge \mathrm{Dif}\,(a_i, \dots, s_i) \tag{14}$$

holds, wherefrom we deduce the inequalities:

(i) $$|x_{i+1}-x_i| \le \alpha |x_i - y_i|$$

(ii) $$|x_{i+1}-y_{i+1}| \le (1+2\alpha)|x_i - y_i|$$

(iii) $$|x_{i+1}-y_{i+1}| \ge (1-2\alpha)|x_i - y_i|$$

where α denotes the number $\frac{1}{\sigma n}$, x_i and y_i are any two different elements in $a_i, \ldots, s_i$ and x_{i+1}, y_{i+1} are the corresponding $(i+1)^{th}$ iteration points defined by (2).

Proof of (i)

$$\begin{aligned}
|x_{i+1}-x_i| =& \frac{|-p(x_i)|}{|x_i-y_i|\cdots|x_i-v_i|} \\
& \text{(By (7), i.e. by definition of } (x_i)\text{)} \\
=& \frac{\alpha|x_i-y_i|\cdots|x_i-v_i|}{M(i)^{n-2}} \\
& \text{(By hypothesis } X \text{ (i))} \\
\le& \frac{\alpha|x_i-y_i|M(i)^{n-2}}{M(i)^{n-2}} \\
& \text{(By definition of } M \text{ (i))} \\
=& \alpha|x_i-y_i|
\end{aligned}$$

Proof of (ii):

$$\begin{aligned}
|x_{i+1}-y_{i+1}| =& |x_{i+1}-x_i+x_i-y_i+y_i-y_{i+1}| \\
\le& |x_{i+1}-x_i|+|x_i-y_i|+|y_{i+1}-y_i| \\
\le& \alpha|x_i-y_i|+|x_i-y_i|+\alpha|y_i-x_i| \\
& \text{(Using the inequality (i))} \\
=& (1+2\alpha)|x_i-y_i|
\end{aligned}$$

Proof of (iii):

$$\begin{aligned}
|x_{i+1}-y_{i+1}| =& |x_{i+1}-x_i+x_i-y_i+y_i-y_{i+1}| \\
\ge& ||x_i-y_i|-|x_{i+1}-x_i+y_i-y_{i+1}|| \\
\ge& |x_i-y_i|-|x_{i+1}-x_i+y_i-y_{i+1}| \\
\ge& |x_i-y_i|-|x_{i+1}-x_i|-y_{i+1}-y_i| \\
\ge& |x_i-y_i|-\alpha|x_i-y_i|-\alpha|y_i-x_i| \\
=& (1-2\alpha)|x_i-y_i|.
\end{aligned}$$

We note that if the assumption (14) holds then from (ii) it follows immediately

$$M(i+1) \leq (1+2\alpha)M(i) \tag{15}$$

We prove now the implication (α_i), (β_i), (γ_i),

Proof of (α_i): Assume that the hypothesis (14) holds. Let x_{i+1}, y_{i+1} be any two different elements in $\{a_{i+1}, \dots, s_{i+1}\}$. We prove that they are different numbers. By the inequality (iii) we have

$$|x_{i+1} - y_{i+1}| \geq \left(1 - \frac{2}{\sigma n}\right)|x_i - y_i| > 0$$

since both $|x_i - y_i|$, $1 - \frac{2}{\sigma n}$ are positive if $n \geq 2$.

Proof of (β_i): We assume again that (14) holds. Diving $p(x)$ with $(x - x_i)\cdots(x - v_i)$ at the point $x = x_{i+1}$, it is easy to obtain the following identity

$$\begin{aligned} p(x_{i+1}) &= (x_{i+1} - x_i)(x_{i+1} - y_i)\cdots(x_{i+1} - v_i) + \\ &+ \frac{p(x_i)}{(x_i - y_i)(x_i - z_i)\cdots(x_i - v_i)}(x_{i+1} - y_i)(x_{i+1} - z_i)\cdots(x_{i+1} - v_i) \\ &+ \frac{p(y_i)}{(y_i - x_i)(y_i - z_i)\cdots(y_i - v_i)}(x_{i+1} - y_i)(x_{i+1} - z_i)\cdots(x_{i+1} - v_i) + \cdots \\ &+ \frac{p(v_i)}{(v_i - x_i)(v_i - y_i)\cdots(v_i - u_i)}(x_{i+1} - x_i)(x_{i+1} - y_i)\cdots(x_{i+1} - u_i). \end{aligned} \tag{16}$$

Using (7) the preceding identity becomes

(17)

$$\begin{aligned} p(x_{i+1}) &= \frac{p(y_i)}{(y_i - x_i)(y_i - z_t)\cdots(y_i - v_i)}(x_{i+1} - x_i)(x_{i+1} - z_i)\cdots(x_{i+} - v_i) + \cdots \\ &+ \frac{p(v_i)}{(v_i - x_i)(v_i - y_i)\cdots(v_i - u_i)}(x_{i+1} - x_i)(x_{i+1} - y_i)\cdots(x_{i+1} - u_i). \end{aligned}$$

Starting from (17) we deduce the inequality

$$|p(x_{i+1})| \leq |x_{i+1} - x_i| \cdot (n-1)\alpha(1+\alpha)^{n-2}|x_i - y_i||x_i - z_i|\cdots|x_i - v_i| \tag{18}$$

in the following way:

$$\begin{aligned}
|p(x_{i+1})| &\leq \frac{|-p(y_i)|}{|y_i - x_i||y_i - z_i|\cdots|y_i - v_i|}|x_{i+1} - x_i||x_{i+1} - z_i|\cdots|x_{i+1} - v_i| + \cdots \\
&+ \frac{|-p(v_i)|}{|v_i - x_i||v_i - y_i|\cdots|v_i - u_i|}|x_{i+1} - x_i||x_{i+1} - y_i|\cdots|x_{i+1} - u_i| \\
&= |x_{i+1} - x_i|[|y_{i+1} - y_i||x_{i+1} - z_i|\cdots|x_{i+1} - v_i| + \cdots \\
&\qquad + |v_{i+1} - v_i||x_{i+1} - y_i|\cdots|x_{i+1} - u_i|] \\
&\text{(By definition (2) of the sequence } (a_i), \ldots, (s_i)) \\
&\leq |x_{i+1} - x_i|[\alpha|x_i - y_i|(1 + 2\alpha)|x_i - z_i|\cdots(1 + 2\alpha)|x_i - v_i| + \cdots \\
&\qquad + \alpha|x_i - v_i|(1 + 2\alpha)|x_i - y_i|\cdots(1 + 2\alpha)|x_i - u_i|] \\
&\text{(Using the equalities (i) and (ii))} \\
&= |x_{i+1} - x_i| \cdot (n-1)\alpha(1 + \alpha)^{n-2}|x_i - y_i||x_i - z_i|\cdots|x_i - v_i|
\end{aligned}$$

Further, from (iii) it follows immediately

$$\begin{aligned}
&|x_{i+1} - y_{i+1}||x_{i+1} - z_{i+1}|\cdots|x_{i+1} - v_{i+1}| \geq \\
&\qquad \geq (1 - 2\alpha)^{n-1}|x_i - y_i||x_i - z_i|x_i - v_i|.
\end{aligned} \tag{19}$$

From (18) and (19) we obtain

$$|x_{i+2} - x_{i+1}| \leq \frac{(n-1)\alpha(1+\alpha)^{n-2}}{(1-2\alpha)^{n-1}}|x_{i+1} - x_i| \tag{20}$$

since $1 - 2\alpha > 0$ for $\alpha = \frac{1}{\sigma n}$ and $n \geq 2$.

It remains to prove the inequality

$$\frac{(n-1)\alpha(1+\alpha)^{n-2}}{(1-2\alpha)^{n-1}} \leq \frac{1}{3} \tag{21}$$

where $\alpha = \frac{1}{\sigma n}$. Indeed, by the Bernoulli's inequality we have for $n \geq 2$

$$\begin{aligned}
\frac{(n-1)\alpha(1+\alpha)^{n-2}}{(1-2\alpha)^{n-1}} &= \frac{(n-1)\alpha}{1+\alpha} \cdot \frac{1}{\left(1 - \frac{3\alpha}{1+\alpha}\right)^{n-1}} \\
&\leq \frac{(n-1)\alpha}{1+\alpha} \cdot \frac{1}{1 - \frac{3\alpha(n-1)}{1+\alpha}} \\
&= \frac{n-1}{3n+4} \\
&< \frac{1}{3}.
\end{aligned}$$

Proof of (γ_i): Assume that (14) holds. The formula $X(i+i)$ wich we are going to prove is equivalent to:

$$\frac{|x_{i+2}-x_{i+1}|M(i+1)^{n-2}}{\alpha|x_{i+1}-y_{i+1}||x_{i+1}-z_{i+1}|\cdots|x_{i+1}-v_{i+1}|} \leq 1. \tag{22}$$

To prove this we first deduce the following inequality chain:

$$\frac{|x_{i+1}-x_{i+1}|M(i+1)^{n-2}}{\alpha|x_{i+1}-y_{i+1}||x_{i+1}-z_{i+1}|\cdots|x_{i+1}-v_{i+1}|}$$
$$\leq \frac{1}{3}\cdot\frac{(1+2\alpha)^{n-2}}{(1-2\alpha)^{n-1}\alpha}\cdot\frac{|x_{i+1}-x_i|M(i)^{n-2}}{|x_i-y_i||x_i-z_i|\cdots|x_i-v_i|}$$

(Using the inequlities (iii), (15), and (20))

$$\leq \frac{1}{3(1+2\alpha)}\cdot\frac{1}{\left(1-\frac{4\alpha}{1+2\alpha}\right)^{n-1}}$$

(Since by the assumtion $X(i)$ the inequality

$$\frac{|x_{i+1}-x_i|M(i)^{n-2}}{\alpha|x_i-y||x_i-z_i|\cdots|x_i-v_i|} \leq 1$$

holds.)

wherefrom, by the Bernoulli's inequality, we get (with $\alpha=\frac{1}{\sigma n}$)

$$\frac{|x_{i+2}-x_{i+1}|M(i+1)^{n-2}}{\alpha|x_{i+1}-y_{i+1}||x_{i+1}-z_{i+1}|\cdots|x_{i+1}-v_{i+1}|}$$
$$\leq \frac{1}{3(1+2\alpha)}\cdot\frac{1}{1-\frac{4(n-1)\alpha}{1+2\alpha}}$$
$$= \frac{n}{n+3}$$
$$< 1$$

what completes the proof of (γ_i).

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REFERENCES

[1] Cauchy, A., *Sur la détermination approximative des racines d'une équation algébrique ou transcendante,* Oeuvres completes (II) 4, Gauthieer – Villars, Paris, 1899 pp. 573–609.

[2] Henrici, P. *Uniformly convergent algorithms for the simultaneous determination of all zeros of a polynomial,* Studies in Numer. Anal. (1969), pp. 1 – 8.

[3] Ostrowski, A., *Sur la convergence et l'estimation des erreurs, dans quelques procédés résolution des equations numériques,* Collection of Papers in Memory of D.A. Grave, Moskva, 1940, pp. 213 – 234.

[4] Ostrowski, A., *Solution of Equations and System of Equations,* Academic Press Neq York & London, 1966.

[5] Prešić M.D., *Un procédé itératif pour déterminer k zeros d'un polynome,* C. R. Acad. Sc. Paris, 273 (1971), pp. 446 – 449.

[6] Prešić, M.D., *On Ostrowski's fundamental existence theorem,* Publ. Inst. Math., Beograd, t. 24 (38), 1978, pp. 125 – 132.

[7] Prešić, S.B., *Un procédé itératif pour la factorisation des polynomes,* C. R. Acad. Sc. Paris, 262 (1966), pp, 862 – 863.

[8] Prešić, S.B., *Jedan interativni postupak za faktorizaciju polinoma,* Mat. vesnik, Beograd, 5 (20), 1068, pp. 205 – 216.