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ON FIXED POINT THEOREMS OF MAIA TYPE

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1. In this note we present some variants of the following result of Maia [10]: *Let X be a non-empty set endowed in with two metrics ρ, σ, and let f be a mapping of X into itself. Suppose that $\rho(x,y) \leq \sigma(x,y)$ in X, X is a complete space and f is continuous with respect to ρ, and $\sigma(fx, fy) \leq k \cdot \sigma(x,y)$ for all x, y in X, where $0 \leq k < 1$. Then, f has a unique fixed point in X.*

This theorem (cf. also [18], [11], [4], [12], [17]) generalizes the Banach fixed-point principle and is connected with Bielecki's method [1] of changing the norm in the theory of differential equations. Our results follow as a consequence of two metrics, of two transformations [3] and of the generalized metric space concept ([8], [9]).

2. Let $(E, \|\cdot\|)$ be a Banach space, let S be a normal cone in E (see e.g. [6]) and let $\preceq$ denote the partial order in E generated by the cone S. Suppose that X is a non-empty set and a function $d_E\colon X \times X \to S$ satisfying for arbitrary elements x, y, z in X the following conditions:

(A 1) $d_E(x,y) = \theta$ if and only if $x = y$ (θ denotes the zero of the space E);

(A 2) $d_E(x,y) = d_E(y,x)$;

(A 3) $d_E(x,y) \preceq d_E(x,z) + d_E(z,y)$

Then, this function d_E is called the *generalized metric* in X.

Further, let us put $d^+(x,y) = \|d_E(x,y)\|$ for x and y in X. If every d^+-Cauchy sequence in X is d^+-convergent (i.e., $\lim_{p,q\to\infty} d^+(x_p, x_q) = 0$ for a sequence (x_n) in X, implies the existence of an element x_0 in X such that $\lim_{n\to\infty} d^+(x_n, x_0) =$), then (X, d_E) is called [6] a *generalized complete metric space.*

Moreover, in this paper we shall use the notations of $\mathcal{L}^*$-space, the $\mathcal{L}^*$-product of $\mathcal{L}^*$-spaces and a continuous mapping of $\mathcal{L}^*$-space into $\mathcal{L}^*$-space (see e.g. [7]).

3. Let E, S and $\preceq$ be as above. In this section suppose we are given:

L – a bounded positive linear operator of E into itself with the spectral radius $r(L)$ less than one (see e.g. [6]);

X, A – two non-empty sets;

ρ_E, σ_E – two generalized metrics in X such that $\rho_E(x,y) \preceq C \cdot \sigma_E(x,y)$ for all x, y in X, where C is a positive constant;

T – a transformation from A to X such that $(T[A], \rho_E)$ s a generalized complete metric space[1].

Modifying the reasoning from [6, Th. II. 6. 2], we obtain the following result:

Proposition 1. *Let (X, ρ_E) be a generalized complete metric space, let $f\colon X \to X$ be a continuous mapping with respect to ρ^+, and let $\sigma_E(fx, fy) \preceq L(\sigma_E(x,y))$ for all x, y in X. Then f has a unique fixed point ξ in X. Moreover, if $x_0 \in X$ and $x_n = fx_{n-1}$ for $n \geq 1$, then:*

(i) $\lim_{n\to\infty} \|\rho_E(x_n, \xi)\| = 0$,

(ii) $\|\rho_E(x_m, \xi)\| \leq N \cdot C \cdot \|L^m u\|$ *for all $m \geq 0$, where N is same constant and u is a solution of equation $u = \sigma_E(x_0, fx_0) + Lu$ in the space E (see [6, Th. I. 2. 2]).*

Now, we shall prove

Proposition 2. *Let (X, ρ_E) be a generalized complete metric space, let $f_m\colon X \to X$ $(m = 0, 1, \dots)$ be continuous mappings with respect to ρ^+, and let $\sigma_E(f_m x, f_m y) \preceq L(\sigma_E(x,y))$ for all x, y in X. Denote by $\xi_m (m = 0, 1, \dots)$ a unique fixed point of f_m, and suppose that* $\lim_{n\to\infty} \|\sigma_E(f_n x, f_0 x)\| = 0$ *for every x in X. Then* $\lim_{n\to\infty} \|\rho_E(\xi_n, \xi_0)\| = 0$.

Proof. Consider the linear equation $u = \sigma_E(\xi_0, f_n\xi_0) + Lu$ $(n = 1, 2, \dots)$ with the unique solution u_n in E (see [6, Th. I. 2. 2]). By Proposition 1 we obtain $\|\rho_E(\xi_n, \xi_0)\| \leq N \cdot C \cdot \|u_n\|$ for $n \geq 1$, where N is constant.

Let $\varepsilon > 0$ by such that $r(L) + \varepsilon < 1$. Further, let us denote by $\|\cdot\|_\varepsilon$ the norm equivalent to $\|\cdot\|$ such that $\|L\|_\varepsilon \leq \varepsilon + r(L)$ (see [6, p. 15]) ($\|L\|_\varepsilon$ is the norm of L generated by $\|\cdot\|_\varepsilon$). We have

$$\|u_n\|_\varepsilon \leq \|\sigma_E(f_n\xi_0, f_0\xi_0)\|_\varepsilon + \|Lu_n\|_\varepsilon \leq \|\sigma_E(f_n\xi_0, f_0\xi_0)\|_\varepsilon + (r(L) + \varepsilon)\|u_n\|_\varepsilon$$

for $n \geq 1$. Since $\lim_{n\to\infty} \|\sigma_E(f_n\xi_0, f_0\xi_0)\|_\varepsilon = 0$, so $\lim_{n\to\infty} \|u_n\|_\varepsilon \leq (\varepsilon + r(L)) \cdot \lim_{n\to\infty} \|u_n\|_\varepsilon$, and consequently $\lim \|\rho_E(\xi_n, \xi_0)\| = 0$.

Theorem 1. *Let $H\colon A \to X$ be a mapping such that $H[A] \subset T[A]$ and $\sigma_E(Hx, Hy) \preceq L(\sigma_E(Tx, Ty))$ for all x, y in A. Suppose that* $\lim_{n\to\infty} \|\rho_E(Hx_n, Hx)\| = 0$ *for every sequence (x_n) in A with* $\lim_{n\to\infty} \|\rho_E(Tx_n, Tx)\| = 0$ *Then:*

(i) *for every u in $T[A]$ the set $H[T_{-1}u]$ contains only one element*[2]*;*

(ii) *there exists a unique element ξ in $T[A]$ such that $H[T_{-1}\xi] = \xi$, and every sequence of successive approximations $u_{n+1} = H[T_{-1}u_n]$ $(n = 1, 2, \dots)$ is ρ^+-convergent to ξ;*

[1] $T[A]$ denotes the image of the set A by the transformation T

[2] $T_{-1}u$ denotes the inverse image of u under T

(iii) $Hx = Tx$ *for all* x *in* $T_{-1}\xi$*;*

(iv) *if* $Hx_i = Tx_i$ $(i = 1, 2)$*, then* $Tx_1 = Tx_2$*.*

Proof. Let us put $fz = H[T_{-1}z]$ for z in $T[A]$. Obviously, $fz \in T[A]$ for all z in $T[A]$. If $v_i \in fz$ $(i = 1, 2)$, then $v_i = Hx_i$ with $Tx_i = z$. Hence $\theta \preceq \sigma_E(v_1, v_2) \preceq L(\sigma_E(Tx_1, Tx_2)) = \theta$ and $v_1 = v_2$. Therefore, $H[T_{-1}z]$ contains only one element.

It can be easily seen that the mapping f of $T[A]$ into itself is continuous with respect to ρ^+. Indeed, let $z_n \in T[A]$ for $n \geq 1$ and let $\lim_{n\to\infty} \|\rho_E(z_n, z_0)\| = 0$. Then there exist $x_m \in T_{-1}z_m$ $(m = 0, 1, \dots)$ such that $fz_m = Hx_m$. We have $\|\rho_E(Hx_n, Hx_0)\| = \|\rho_E(fz_n, fz_0)\|$ for $n \geq 1$, and consequently $\lim_{n\to\infty} \|\rho_E(fz_n, fz_0)\| = \lim_{n\to\infty} \|\rho_E(Hx_n, Hx_0)\| = 0$.

Further, it is easy to verify that $\sigma_E(fu, fv) \preceq L(\sigma_E(u, v))$ for all u, v in $T[A]$. Consequently, applying Proposition 1 the proof of (ii) is completed.

Obviously, (iii) holds and we omit the proof. Now, we prove (iv): Suppose that $Hx_i = Tx_i$ $(i = 1, 2)$ and $Tx_i \neq Tx_2$. Then, $\sigma_E(Tx_1, Tx_2) \preceq L(\sigma_E(Tx_1, Tx_2))$ and $-\sigma_E(Tx_1, Tx_2) \notin S$. Therefore, by theorem II. 5. 4 from [6. p. 81], we obtain $r(L) \geq 1$. This contradiction completes our proof.

Using Theorem 1 and Proposition 2 we obtain the following

Theorem 2. *Let* $H_m : A \to X$ $(m = 0, 1, \dots)$ *be mappings with* $H_m[A] \subset T[A]$ *and* $\sigma_E(H_m x, H_m y) \preceq L(\sigma_E(Tx, Ty))$ *for all* x, y *in* A. *Further, suppose that* $\lim_{n\to\infty} \|\rho_E(H_m x_n, H_m x)\| = 0$ *for every sequence* (x_n) *in* A *with* $\lim_{n\to\infty} \|\rho_E(Tx_n, Tx)\| = 0$.

Let $\xi_m (m = 0, 1, \dots)$ *be an element in* $T[A]$ *such that* $H_m[T_{-1}\xi_m] = \xi_m$. *Assume that* $\lim_{n\to\infty} \|\sigma_E(H_n x, H_0 x)\| = 0$ *for every* x *in* A. *Then* $\lim_{n\to\infty} \|\rho_E(Ty_n, Ty_0)\| = 0$*, where* $y_m \in T_{-1}\xi_m$ *for* $m \geq 0$.

4. M. Krasnoselskii [5] has given the following version of well-known result of Schauder: *If* W *is a non-empty bounded closed convex subset of a Banach space,* f *is a contraction and* g *is completely continuous on* W *with* $fx + gy \in W$ *for all* x, y *in* W*, then the equation* $fx + gx = x$ *has a solution in* W.

Now, we give a modification and some generalization of this Krasnoselskii's result.

Let $(E, \|\cdot\|)$ be a Banach space, let S be a cone in E with the partial order $\preceq$ such that if $\theta \preceq x \preceq y$ then $\|x\| \leq \|y\|$, and let L be as in Sec. 3. Further, let X be a vector space endowed with two generalized norms $|||\cdot|||_i : X \to S$ $(i = 1, 2)$ (see [6, p. 94]) such that $|||x|||_1 \preceq C \cdot |||x|||_2$ for all x in X. Denote: ρ_E, σ_E–generalized metrics in X generated by $|||\cdot|||_1$ and $|||\cdot|||_2$, respectively.

Theorem 3. *Let* K *be a non-empty convex subset of* X*, let* (K, ρ^+) *be a complete space and let* Q*,* F *be transformations with the values in* K *defined on* K *and* $K \times K$ *respectively. Assume, moreover, that the following condition holds:*

(i) $Q : (K, \rho^+) \to (K, \rho^+)$ *is continuos,* $Q[K]$ *is a conditionally compact set with respect to* σ^+ *and* $|||F(u, y) - F(v, y)|||_2 \preceq |||Qu - Qv|||_2$ *for all* u, v, y *in* K*;*

(ii) $|||F(x,y) - F(x,z)|||_2 \preceq L(|||y-z|||_2)$ *for all* x, y, z *in* K;

(iii) *for every* x *in* K *the function* $y \mapsto F(x,y)$ *of* K *into itself is continuous with respect to* ρ^+.

Then there exists a point x *in* K *such that* $F(x,x) = x$.

Proof. Consider the mapping $y \mapsto F(x,y)$ (x is fix in K) of K into itself. By Proposition 1, there exists exactly one u_x in K such that $F(x,u_x) = u_x$. Now define an operator V as $x \mapsto u_x$.

This operator V maps continuously (K,ρ^+) into itself. Indeed, let (x_n) be a sequence in K such that $\rho^+(x_n,x_0) \to 0$ as $n \to \infty$. Let us put $f_m x = F(x_m, x)$ $(m = 0,1,\dots)$ for x in K. The conditions (i) and (ii) imply that all the assumptions of the Proposition 2 are satisfied. Therefore, f_m has a unique fixed point ξ_m and $\rho^+(\xi_n,\xi_0) \to 0$ as $n \to \infty$, *so we are done.*

Now we are going to show that $V[K]$ is conditionally compact with respect to ρ^+: Let (x_n) be a sequence in K, and let $y_n = F(x_n, u_{x_n})$ for $n \geq 1$. Let $\varepsilon > 0$ be such that $r(L)+\varepsilon < 1$, let $\|\cdot\|_\varepsilon$ be the norm equivalent to $\|\cdot\|$ with $\|L\|_\varepsilon \leq r(L)+\varepsilon$, and let us put $\sigma^+_\varepsilon(x,y) = \||||x-y|||_2\|_\varepsilon$ for x, y in K. We have

$$\begin{aligned}\||||y_i - y_j|||_2\|_\varepsilon &\leq \|L(|||y_i-y_j|||_2) + |||Qx_i - Qx_j|||_2\|_2\|_\varepsilon \leq \\ &\leq (r(L)+\varepsilon)\||||y_i-y_j|||_2\|_\varepsilon + \||||Qx_i - Qx_j|||_2\|_\varepsilon.\end{aligned}$$

hence

$$(1-(r(L)+\varepsilon))\cdot\||||y_i - y_j|||_2\|_\varepsilon \leq \||||Qx_i - Qx_j|||_2\|_\varepsilon$$

for every i, $j \leq 1$. Suppose that (Qx_n) is a σ^+-Cauchy sequence. Then, (Qx_n) is a σ^+_ε-Cauchy sequence and consequently (y_n) is ρ^+-convergent in K.

By application of the Schauder fixed point theorem, our proof is completed.

Remark. The above theorem will remain true if (i) is repleaced by the following condition: Q is continuous and $Q[K]$ is a conditionally compact set with respect to ρ^+, and $|||F(u,y) - F(v,y)|||_2 \preceq |||Qu - Qv|||_1$ for all u, v, y in K.

5. Let us remark applications and further results can be obtained if the concept of a generalized metric space in the Luxemburg sense [9] (not every two points have necessarily a finite distance) will be used. Cf. [13]–[17]. How, we give some application of Theorem 2 (in the cose of) to functional equations.

In this section, let $(\mathbf{R}^k, \|\cdot\|)$ denote the k-dimensional Euclidean space, let $E = \mathbf{R}^k$, and let $S = \{(t_1, t_2 \dots, t_k) \in \mathbf{R}^k : t_i \geq 0 \text{ for } 1 \leq i \leq k\}$. Then, $(x_1, x_2, \dots, x_k) \preceq (y_1, y_2, \dots, y_k)$ if we have $x_i \leq y_i$ for every $1 \leq i \leq k$.

Suppose that $J = [0,\infty)$, $K_{ij} \geq 0$ $(i,j = 1,2,\dots,k)$ are constants, and $p: J \to J$ is a locally bounded function. Let us denote by:

A – the set of continuous functions $(x_1, x_2, \dots, x_k)$ from J to $\mathbf{R}^k$ such that $x_1(t) = 0(\exp(p(t)))$ $(1 \leq i \leq k)$ for every t in J;

X – the set of bounded continuous functions from J to $\mathbf{R}^k$;

Λ – the metric space with the metric δ;

$\mathcal{F}$ – the set of continuous functions $(f_1, f_2, \ldots, f_k)$ from $J \times \mathbf{R}^k \times \Lambda$ into $\mathbf{R}^k$ satisfying the following conditions:

$$| f_i(t, t_1, \ldots, t_k, \lambda) - f_i(t, s_1, s_2, \ldots, s_k, \lambda) | \leq \sum_{j=1}^{k} K_{ij} \mid t_j - s_j \mid$$

$(1 \leq i \leq k)$ for every t in J, t_j, s_j in $\mathbf{R}^k$ and λ in Λ; $f_i(t, \theta, \lambda) = 0(\exp(p(t)))$ $(1 \leq i \leq k)$ for fixed λ in Λ and every t in J (θ denotes the zero of space $\mathbf{R}^k$).

The set A admits a norm $|||\cdot|||$ defined as $|||x||| = \sup\{\exp(-p(t)) \cdot |x(t)| \colon t \geq 0\}$. In X we define the generalized metric d_E as follows: for each $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_k)$ write $d_E(x, y) = (\|x_1 - y_1\|, \|x_2 - y_2\|, \ldots, \|x_k - y_k\|)$, where $\|\cdot\|$ denotes the usual supremum norm in the space of bounded continuous functions on J. Obviously, (X, d_E) is a generalized complete metric space.

We shall deal with the set $\mathcal{F}$ as an $\mathcal{L}^*$-space endowed with convergence: $\lim_{n\to\infty}(f_1^{(n)}, f_2^{(n)}, \ldots, f_k^{(n)} = (f_1^{(0)}, f_2^{(0)}, \ldots, f_k^{(0)})$ if an only if

$$\lim_{n\to\infty} \sup\{\exp(-p(t)) \cdot |f_i^{(n)}(t, u, \lambda) - f_i^{(0)}(t, u, \lambda)| \colon (t, u) \in J \times \mathbf{R}^k\} = 0$$

for every λ in Λ and evry $1 \leq i \leq k$. Moreover, $\mathcal{F} \times \Lambda$ be the $\mathcal{L}^*$-product of the $\mathcal{L}^*$-spaces $\mathcal{F}$, Λ.

Further, suppose that $h \colon J \to J$ is a continuous function, there exists a constant $q > 0$ such that $\exp(p(h(t))) \leq q \cdot \exp(p(t))$ for all t in J, and $[q \cdot K_{ij}]$ $(1 \leq i, j \leq k)$ is a non-zero matrix with

$$\begin{vmatrix} 1 - qK_{11} & -qK_{12} & \cdots & -qK_{1i} \\ -qK_{21} & 1 - qK_{22} & \cdots & -qK_{2i} \\ \cdots & \cdots & \cdots & \cdots \\ -qK_{i1} & -qK_{i2} & \cdots & 1 - qK_{ii} \end{vmatrix} > 0$$

for every $i = 1, 2, \ldots, k$.

Under these conditions we have the following theorem:

For an arbitary F in $\mathcal{F}$ and λ in Λ there exists a unique function $x_{(F'\lambda)}$ in A such that

$$x_{(F'\lambda)}(t) = F(t, x_{(F'\lambda)}(h(t)), \lambda)$$

for every $t \geq 0$. Moreover, if there exists functions α, β from J to J such that $\alpha(t) = 0(\exp(p(t)))$ for $t \geq 0$, $\beta(t) \to 0$ as $t \to 0_+$ and

$$|f_i(t, u, \lambda) - f_i(t, u, \mu)| \leq \alpha(t) \cdot \beta(\delta(\lambda, \mu)) \quad (1 \leq i \leq k)$$

for all $(f_1, f_2, \ldots, f_k) \in \mathcal{F}$, $t \geq 0$, $u \in \mathbf{R}^k$ and λ, μ in Λ, then the function

$$(F, \lambda) \mapsto x_{(F'\lambda)}$$

maps continuously $\mathcal{L}^*$-space $\mathcal{F} \times \Lambda$ into Banach space A.

Proof. Let $m = 0, 1, \dots$ Let $F^{(m)} = (f_1^{(m)}, \dots, f_k^{(m)}) \in \mathcal{F}$ and $\lambda_m \in \Lambda$ be such that $\lim_{n\to\infty} F^{(n)} = F^{(0)}$ and $\lim_{n\to\infty} \delta(\lambda_n, \lambda_0) = 0$. For each x in A, define:

$$\begin{aligned}
(Tx)(t) &= \exp(-p(t)) \cdot x(t),\\
(H_m x)(t) &= \exp(-p(t)) \cdot F^{(m)}(t, x(h(t)), \lambda_m)
\end{aligned}$$

on J.

For $x = (x_1, x_2, \dots x_k) \in A$ and $t \geq 0$ we obtain

$$\begin{aligned}
|(H_m x)(t)| &\leq (|F^{(m)}(t, x(h(t)), \lambda_{(m)}) - F^{(m)}(t, \theta, \lambda_m)| + \\
&\quad + |F^{(m)}(t, \theta, \lambda_m)|) \cdot \exp(-p(t)) \leq \\
&\leq \left(\sum_{j=1}^{k} \sum_{j=1}^{k} K_{ij} |x_j(h(t))| + |F^{(m)}(t, \theta, \lambda_m) \right) \cdot \exp(-p(t) \leq \\
&\leq (c_1 \cdot \exp(p(h(t))) + c_2 \cdot \exp(p(t))) \cdot \exp(-p(t)) \leq c_1 q + c_2
\end{aligned}$$

with some constants c_1, c_2, and therefore H_m maps A into X. Further, it can be easily seen that $T[A] = X$ and $H_m[A] \subset T[A]$.

We observe [2] that the operator L generated by the matrix $[q \cdot K_{ij}]$ is a bounded positive linear operator with the spectral radius less than 1. For $x = (x_1, \dots, x_k)$, $y = (y_1, \dots, y_k)$ in A and $t \geq 0$ we have

$$\exp(-p(t)) \cdot |f_i^{(m)}(t, x(h(t)), \lambda_m) - f_i^{(m)}(t, y(h(t)), \lambda_m| \leq$$

$$\leq \left(\sum_{j=1}^{k} K_{ij} \cdot \sup_{t \geq 0} \exp(-p(t)) |x_j(t) - y_j(t)| \right) \cdot \exp(-p(t)) \cdot \exp(p(h(t))) \leq$$

$$\leq q \cdot \sum_{j=1}^{k} K_{ij} \cdot \sup_{t \geq 0} \exp(-p(t)) \cdot |x_j(t) - y_j(t)|,$$

$$d_E(H_m x, H_m y) = (\sup_{t\geq 0} \exp(-p(t)) \cdot |f_1^{(m)}(t, x(h(t)), \lambda_m) - f_1^{(m)}(t, y(h(t)), \lambda_m)|,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sup_{t\geq 0} \exp(-p(t)) \cdot |f_k^{(m)}(t, x(h(t)), \lambda_m) - f_k^{(m)}(t, y(h(t)), \lambda_m)|),$$

$$L(d_E(Tx, Ty)) = \left(q \cdot \sum_{j=1}^{k} K_{1j} \cdot \sup_{t\geq 0} \exp(-p(t)) \cdot |x_j(t) - y_j(t)|, \dots \right.$$

$$\left. \dots, q \cdot \sum_{j=1}^{k} k_{kj} \cdot \sup_{t\geq 0} \exp(-p(t)) \cdot |x_j(t) - y_j(t)| \right)$$

and therefore $d_E(H_m x, H_m y) \preceq L(d_E(Tx, Ty))$.

Let us fix x in A. For $t \geq 0$, $1 \leq i \leq k$ and $n \geq 1$ we get

$$|f_i^{(n)}(t, x(h(t)), \lambda_n) - f_i^{(0)}(t, x(h(t)), \lambda_0)| \leq \alpha(t) \cdot \beta(\delta(\lambda_n, \lambda_0)) + \\ + |f_i^{(n)}(t, x(h(t)), \lambda_0) - f_i^{(0)}(t, x(h(t)), \lambda_0)|$$

hence

$$\sup_{t \geq 0} \exp(-p(t)) |f_i^{(n)}(t, x(h(t)), \lambda_n) - f_i^{(0)}(t, x(h(t)), \lambda_0)| \leq c \cdot \beta(\delta(\lambda_n, \lambda_0)) + \\ + \sup\{\exp(-p(t)) |f_i^{(n)}(t, u, \lambda_0) - f_i^{(0)}(t, u, \lambda_0)| : (t, u) \in J \times \mathbf{R}^k\}$$

with some constant c, and it follows

$$\lim_{n \to \infty} \sup_{t \geq 0} \exp(-p(t)) |f_i^{(n)}(t, x(h(t)), \lambda_n) - f_i^{(0)}(t, x(h(t)), \lambda_0)| = 0.$$

Finally, $\|d_E(H_n x, H_0 x)\| \to 0$ as $n \to \infty$.

This proves that the theorem 1 and 2 is applicable to the mappings T, $H_m (m = 0, 1, \dots)$, and the proof is finished.

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