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A MERCERIAN THEOREM FOR SLOWLY VARYING SEQUENCES

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Abstract. The purpose of this note is to investigate a Mercerian problem for triangular matrix transformations of slowly varying sequences. A statement of this type for the nonnegative arithmetical means M_p, was recently proved by S. Aljančić [1], using the evaluation of the inverse of the associated Mercerian transformation. In this note a corresponding result is proved for nonnegative triangular matrix transformations satisfying a certain condition, which can be applied to the arithmetical means M_p $p_n \geq 0$, the Cesàro transformation C_α of order α, $0 < \alpha \leq 1$, other Nörlund transformations N_p, $p_n > 0$ and (p_{n+1}/p_n) nondecreasing, as well as to some other standard methods. The proof is based on the properties rather than on the evaluation, of the inverse of the associated Mercerian transformation.

1. Let A denote a matrih transformation and A_{kn} the entries of its matrix. For a sequence s let As denote the transformed sequence whenever it exists, and $(As)_n$ its terms. We say that A is triangular if $A_{nk} = 0$ for $k < n$ an we say that A is normal if it is triangular and $A_{nn} \neq 0$ for all n. For a triangular matrix transformation A let $A_n = \sum_{k=0}^n A_{nk}$ and let us say that A is normalized if $A_n = 1$ for all n. If A is normal then it is invertable and its A^{-1} is also normal: if in addition A is normalized then clearly A^{-1} is normalized also.

A sequence s, $s_n > 0$ for all n, is slowly varying in the sence of Karamata [3] if $\lim_{n\to\infty} s_{[tn]}/s_n = 1$ for all $t > 0$. Let $\mathcal{L}$ denote the set of all slowly varying sequences. We say that a real matrix transformation A is $\mathcal{L}$-permanent if $\lim_{n\to\infty}(As)_n/s_n$ exists and is $\neq 0$ for every $s \in \mathcal{L}$. M. Vuilleumier in [3] gave a characterization of $\mathcal{L}$-permanent metrix transformations which reduces to the following statement for triangular matrix transformations:

THEOREM A. *A triangular matrix transformation A is $\mathcal{L}$-permanent if and only if*

i) $\lim\limits_{n\to\infty} A_n = \alpha$, $\alpha \neq 0$ *and*

ii) $\sum\limits_{k=1}^{n} |A_{nk}|(k+1)^{-\delta} = 0(1)(n+1)^{-\delta} (n \to \infty)$ *for some* $\delta > 0$.

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Under these condition $\lim_{n\to\infty}(As)_n/s_n = \alpha$ *for every* $s \in \mathcal{L}$.

In what follows we will be concerned only with real triangular matrix transformations.

For a sequence p such that $P_n = \sum_{k=0}^{n} p_k \neq 0$ for all n let M_p and N_p be defined by:

$$(M_p)_{nk} = p_k/P_n \text{ for } k \le n \text{ and } (M_p)_{nk} = 0 \text{ for } k > n$$
$$(N_p)_{nk} = p_{n-k}/P_n \text{ for } k \le n \text{ and } (N_p)_{nk} = 0 \text{ for } k > n.$$

M_p and N_p are called the aritmetical mean and the Nörlund transformation respectively. In the special case when $p_n = \varepsilon_n^{\alpha-1}$ where $\varepsilon_n^{\alpha} = \binom{n+\alpha}{n}$ $\alpha > -1$, N_p is the Cesàro transformation C_α of order α.

If A is normalized and the condition ii) of Theorem A holds, $B = I + \lambda A$ where I is the identity and λ real, $\lambda \neq -1$, then $B_n = 1 + \lambda$ and

$$\sum_{k=0}^{n} |B_{nk}|(k+1)^{-\delta} \le (1+|\lambda|)\sum_{k=0}^{n} |A_{nk}|(k+1)^{-\delta} = 0(1)(n+1)^{-\delta}$$

and therefore by Theorem A, $s \in \mathcal{L}$ implies $\lim_{n\to\infty}(Bs)_n/s_n = 1+\lambda$. Moreover for such A, $s \in \mathcal{L}$ implies $Bs \in \mathcal{L}$ whenever Bs is positive. Note that also by above, if $\lambda > -1$ then $s \in \mathcal{L}$ implies Bs is eventually positive and that $\lambda > -1$ is a necessary condition in order that $S \in \mathcal{L}$ implies $Bs \in \mathcal{L}$.

The purpose of this note is to investigate the converse statement, namely to find sufficient conditions in order that for A normalized and such that ii) of Theorem A holds, $Bs \in \mathcal{L}$ for λ real, $\lambda > -1$, implies $\lim_{n\to\infty} s_n/(Bs)_n = 1/(1+\lambda)$ and therefore $s \in \mathcal{L}$ whenever s is positive. This mercerian type question, for the arithmetical means, raised in a recent by S. Aljančić in [1]. The following result was proved in [1]:

THEOREM B. *If* $p_0 > 0$ *and* $p_n \ge 0$, *for* $n = 1, 2, \dots,$

$$\sum_{k=0}^{n} p_k(k+1)^{-\delta} = 0(1)P_n(n+1)^{-\delta}(n\to\infty) \text{ for some } \delta > 0$$

and $B = I + \lambda M_p$ *where* $\lambda > -1$, *then* $Bs \in \mathcal{L}$ *implies* $\lim s_n/(Bs)_n = 1/(1+\lambda)$ *and consequently* $s \in \mathcal{L}$ *if* s *is positive.*

The proof of this theorem in [1] is based on the evaluation of the inverse transformation of $B = I + \lambda M_p$.

Here we will prove a statement of the type mentioned above for nonnegative normalized transformation A, which can be applied to the arithmetical means M_p with $p_0 > 0$ and $p_n \ge 0$, the Casàro transformation C_α of order α, $0 < \alpha \le 1$, other Nörlund transformations N_p with $p_n > 0$ and (p_{n+1}/p_n) nondecreasing, as well as to some other standard methods. Our proof will be based on the properties of the inverse of $B = I + \lambda A$, rather than on the evaluation of B^{-1}.

2. Theorem 1. *Let A normalized, nonnegative i.e. $A_{nk} \geq 0$,*

$$A_{n0} > 0 \text{ and } A_{n+1,i}A_{nk} \leq A_{ni}A_{n+1,k} \text{ for all } n,\ 0 \leq k \leq i \leq n \tag{2.1}$$

$$\sum_{k=0}^{n} A_{nk}(k+1)^{-\delta} = 0(1)(n+1)^{-\delta}(n \to \infty) \text{ for some } \delta > 0 \tag{2.2}$$

and $B = I + \lambda A$ where $\lambda > -1$, then $Bs \in \mathcal{L}$ implies $\lim_{n\to\infty} s_n/(Bs)_n = 1/(1+\lambda)$ and consequently $s \in \mathcal{L}$, if s is positive.

To prove the theorem we will need the following lemma:

Lemma 1. i) If A is normal, nonnegative and (2.1) holds then $A^{-1}_{nk} \leq 0$ for $k < n$.

ii) If A is normal, $A^{-1}_{nk} \leq 0$ for $k < n$ and $A_{nn} > 0$ then $A_{nk} \geq 0$ for $k \leq n$.

Proof. The statement ii) is a part of Theorem II. 16 in [2]. Although the statement i) is only a little sharper result than Lemma II. 5 in [2] we give the proof here for completeness.

Clearly $A^{-1}_{10} = -A_{10}A^{-1}_{00}/A_{11} > 0$.

So suppose that $A^{-1}_{mk} \leq 0$ for $m \leq n$ and $k \leq m$. Now by (2.1) $A_{ni} = 0$ implies $A_{n+1,i} = 0$. For $k < n+1$ let k_n be the smallest integer i such that $A_{ni} \neq 0$, $k \leq i \leq n$, which exists since $A_{nn} \neq 0$. Then for $k < n+1$ we have

$$\begin{aligned} 0 = \sum_{i=k}^{n+1} A_{n+1,i}A^{-1}_{ik} &= A_{n+1,n+1}A^{-1}_{n+,k} + \sum_{i=k_n}^{n} A_{n+1,i}A^{-1}_{i\alpha} \geq \\ &\geq A_{n+1,n+1}A^{-1}_{n+1,k} + \frac{A_{n+1,\alpha_n}}{A_{nk_n}} \sum_{i=k_n}^{n} A_{in}A^{-1}_{ik} \end{aligned}$$

since $A_{n+1,i} \leq A_{ni}A_{n+1,k_n}/A_{nk_n}$ by (2.1). Therefore

$$0 \geq A_{n+1,n+1}A^{-1}_{n+1,k} + \frac{A_{n+1,k_n}}{A_{nk_n}} \sum_{i=k}^{n} A_{ni}A^{-1}_{ik} \geq A_{n+1,n+1}A^{-1}_{n+1,k}$$

so that $A^{-1}_{n+1,k} \leq 0$ and the conclusion follows by induction.

Proof of Theorem 1. First $A = I + \lambda A$ is normal for every $\lambda > -1$ by the assumptions that A is normalized and nonnegative. Namely if $A_{nn} - 0$ then clearly $B_{nn} = 1$ and if $A_{nn} \neq 0$ then

$$0 < A_{nn} \leq \sum_{k=0}^{n} A_{nk} = 1 \text{ implies } \lambda > -1 \geq -1/A_{nn}$$

so that $B_{nn} = 1 + \lambda A_{nn} > 0$. Thus B^{-1} exists for every $\lambda > -1$.

We will show now that

$$\lim_{n\to\infty} B_n^{-1} = 1/(1+1) \tag{2.3}$$

and

$$\sum_{k=0}^{n} |B_{nk}^{-1}|(k+1)^{-\delta} = 0(1)(n+1)^{-\delta}(n\to\infty) \text{ for some } \delta > 0 \tag{2.4}$$

both hold and therefore that the conclusion follows by Theorem A.

Now $B_n = 1 + \lambda A_n = 1 + \lambda$ so that

$$\sum_{k=0}^{n} B_{nk}^{-1}(1+\lambda) = \sum_{k=0}^{n} B_{nk}^{-1} B_k = (B^{-1}B)_n = 1 \text{ for all } n$$

and hence (3.3) holds. It remains to verify (2.4).

We suppose first that $\lambda \geq 0$. Clearly B is nonnegative.

If $\lambda > 0$ then $A_{n0} > 0$ implies $B_{n0} > 0$. Moreover from (2.1) it follows that also

$$B_{n+1,i}B_{nk} \geq B_{ni}B_{n+1,k} \text{ for all } n \text{ and } 0 \leq k \leq i \leq n.$$

Thus by Lemma 1 statement i) we conclude that $B_{nk}^{-1} \leq 0$ for $k < n$ if $\lambda > 0$. Now if $\lambda = 0$ then $B = I$. Therefore for all $\lambda \geq 0$

$$-B_{nk}^{-1} = B_{nn}^{-1} \sum_{i=k}^{n-1} B_{ni}B_{ik}^{-1} \leq B_{nn}^{-1}B_{nk}B_{kk}^{-1} \leq B_{nk} \text{ for } k < n$$

so that

$$\begin{aligned}\sum_{k=0}^{n} |B_{nk}^{-1}|(k+1)^{-\delta} &= -\sum_{k=0}^{n-1} B_{nk}^{-1}(k+1)^{-\delta} + B_{nn}^{-1}(n+1)^{-\delta} \leq \\ &\leq \lambda \sum_{k=0}^{n-1} A_{nk}(k+1)^{-\delta} + (n+1)^{-\delta} = 0(1)(n+1)^{-\delta}\end{aligned}$$

by the assumption (2.2)

Suppose now that $-1 < \lambda < 0$. Clearly $B_{nk} \leq 0$ for $k < n$ and $B_{nn} = 1 + \lambda A_{nn} > 0$ as it was shown before. Thus by Lemma 1 statement ii) it follows that $B_{nk}^{-1} \geq 0$ for $k \leq n$.

For $\delta \in R$, real numbers, let us define

$$M_n(\delta) = \sum_{k=0}^{n} A_{nk} \left(\frac{n+1}{k+1}\right)^{\delta}.$$

Then clearly for each n, M_n is nondecreasing and convex on R as a liniar combination of such functions. Let $M(\delta) = \sup_n M_n(\delta)$ whenever it exists. Since (2.2) holds for some $\delta_0 > 0$, it also holds for all δ, $\delta < \delta_0$. Thus M is defined on $(-\infty, \delta_0)$ and is nondecreasing and convex thereon. Hence M is continuous on every closed subinterval of $(-\infty, \delta_0)$ and therefore $\lim_{\delta \to 0^+} M(\delta) = 1$.[1] Since $(1-\lambda)/2(-\lambda) > 1$ for $-1 < \lambda < 0$ the later implies that there exists $\delta > 0$ such that $M_n(\delta) < (1-\lambda)/2(-\lambda)$ for all n and therefore

$$\sum_{k=0}^{n} A_{nk}(k+1)^{-\delta} < \frac{1-\lambda}{2(-\lambda)}(n+1)^{-\delta} \text{ for all } n \tag{2.5}$$

We will show now that for this δ

$$\sum_{k=0}^{n} |B_{nk}^{-1}|(k+1)^{-\delta} = \sum_{k=0}^{n} B_{nk}^{-1}(k+1)^{-\delta} < \frac{2}{1+\lambda}(n+1)^{-\delta} \text{ for all } n \tag{2.6}$$

Clearly

$$\sum_{i=0}^{n} B_{ni} \sum_{k=0}^{i} B_{ik}^{-1}(k+1)^{-\delta} = \sum_{k=0}^{n} (BB^{-1})_{nk}(k+1)^{-\delta} = (n+1)^{-\delta}$$

and therefore

$$B_{nn} \sum_{k=0}^{n} B_{nk}^{-1}(k+1)^{-\delta} = (n+1)^{-\delta} - \sum_{i=0}^{n-1} B_{ni} \sum_{k=0}^{i} B_{ik}^{-1}(k+1)^{-\delta}. \tag{2.7}$$

Since $B_{00}^{-1} = 1/(1+\lambda A_{00}) = 1/(1+\lambda)$, (2.6) clearly holds for $n = 0$. We proceed by induction and assume that (2.6) holds for $0, 1, \ldots, n-1$. Then by (2.5) and (2.7) we have

$$\begin{aligned} B_{nn} \sum_{k=0}^{n} B_{nk}^{-1}(k+1)^{-\delta} &< (n+1)^{-\delta} + (-\lambda) \sum_{i=0}^{n-1} A_{ni} \frac{2}{1+\lambda}(i+1)^{-\delta} < \\ &< (n+1)^{-\delta} + \frac{2(-\lambda)}{1+\lambda} \frac{1-\lambda}{2(-\lambda)}(n+1)^{-\delta} - \frac{2(-\lambda)}{1+\lambda} A_{nn}(n+1)^{-\delta} = \\ &= \left(1 + \frac{1-\lambda}{1+\lambda} - \frac{2(-\lambda)}{1+\lambda} A_{nn}\right)(n+1)^{-\delta} = -\frac{2}{1+\lambda} B_{nn}(n+1)^{-\delta}. \end{aligned}$$

Therefore (2.6) holds for all n and consequently (2.4) is also true for $-1 < \lambda < 0$.

Corollary 1. Theorem B

[1] A similar argument is used in the proof of Theorem B in [1].

Corollary 2. Let $p_n > 0$ for all n, (p_{n+1}/p_n) nondecreasing,

$$\sum_{k=0}^{n} p_{n-k}(k+1)^{-\delta} = 0(1)P_n(n+1)^{-\delta}(n \to \infty) \text{ for some } \delta > 0$$

and $B = I + \lambda N_p$ where $\lambda > -1$, then $Bs \in \mathcal{L}$ implies $\lim_{n\to\infty} s_n/(Bs)_n = 1/(1+\lambda)$ and consequently $s \in \mathcal{L}$, if s is positive.

Corollary 3. Let $0 < \alpha \leq 1$ and $B = I + \lambda C_\alpha$ where $\lambda > -1$. Then $Bs \in \mathcal{L}$ implies $\lim_{n\to\infty} s_n/(Bs)_n = 1/(1+\lambda)$ and consequently $s \in \mathcal{L}$ if s is positive.

Remark. In Theorem 1 and the corollaries $Bs \in \mathcal{L}$ implies that s is eventually positive.

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