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M – PARANORMAL OPERATORS

S.C. Arora[1] *and Ramesh Kumar*

Abstract. V. Istratescu has recently defined M-paranormal operators on a Hilbert space H as: An operator T is called M-paranormal if for all $x \in H$ with $\|x\| = 1$,

$$\|T^2x\| \geqq \frac{1}{M}\|Tx\|^2$$

We prove the following results:

1. T is M-paranormal if and only if $M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$ for all $\lambda > 0$.
2. If a M-paranormal operator T double commutes with a hyponormal operator S, then the product TS is M-paranormal.
3. If a paranormal operator T doble commutes with a M-hyponormal operator, then the product TS is M-paranormal.
4. If T is invertible M-paranormal, then T^{-1} is also M-paranormal.
5. If $ReW(T) \leq 0$, where $W(T)$ denotes the numerical range of T, then T is M-paranormal for $M \geq 8$.
6. If a M-paranormal partial isometry T satisfies $\|T\| \leq \frac{1}{M}$, then it is subnormal.

Introduction

Let H be a complex Hilbert Space and $B(H)$, the set of all bounded operators on H. B.L. Wadhawa in [9] introduced the class of M-hyponormal operators: An operator T in $B(H)$ is said to be M-hyponormal if there exists a real number $M > 0$ such that

$$\|(T - zI)^*x\| \leq M\|(T - zI)x\|$$

for each x in H and for each complex number z. V. Istratescu in [7] has studied some structure theorems for a subclass of M-hyponormal operator. The following definition of M-paranormal operators also apprears in [7].

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DEFINITION: An operator T in $B(H)$ is said to be M-paranormal if for all $x \in H$ with $\|x\| = 1$,

$$\|T^2x\| \geq \frac{1}{M}\|Tx\|^2$$

If $M = 1$, the class of M-paranormal operators becomes the class of paranormal operators as studied by Ando [1] and Furuta [4]. The purpose of the present paper is to study certain properties of M-paranormal operators.

1. We begin with a charaterization of M-paranormal operators in the following way;

THEOREM 1.1: *A bounded linear operator T is M-paranormal if and only if*

$$M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$$

for all $\lambda > 0$.

PROOF We know that for positive numbers b and c, $c - 2b\lambda + \lambda^2 \geq 0$ for all $\lambda > 0$ if and only if $b^2 \leq c$. Let $b = \|Tx\|^2$ and $c = M^2\|T^2x\|^2$, $\|x\| = 1$. Then T is M-paranormal if and only if $b^2 \leq c$. This means that T is M-paranormal if and only if $M^2\|T^2x\|^2 - 2\lambda\|Tx\|^2 + \lambda^2 \geq 0$ for each $\lambda > 0$ and for each x with $\|x\| = 1$. This proves the assertion.

Equivalently, putting $A = (TT^*)^{1/2}$ and $B = (T^*T)^{1/2}$ we see that T is M-paranormal if and only if $M^2AB^2A - 2\lambda A^2 + \lambda^2 \geq 0$ for each $\lambda > 0$.

COROLLARY 1.2: *Let T be a weighted shift with weights $\{\alpha_n\}$. Then T is M-paranormal if and only if*

$$|\alpha_n| \leq M|\alpha_{n+1}|$$

for each n.

It can easily be seen by simple computations that if T is M-hyponormal, then it is M-paranormal. However the converse need not be true. Indeed if $\{e_n\}$ is an orthonormal basis for a separable Hilbert space and if T is a weighted bilateral shift defined as

$$Te_n = \frac{1}{2^{|n|}}e_{n+1}$$

for each n, that T is not M-hyponormal for any $M > 0$ [8, Corollary 5] but by Corollary 1.2, T is M-paranormal for any $M \geq 2$. We also notice that T is not a paranormal operator. Again a compact paranormal operator is normal [6, Theorem 2]. However the operator T shows that this result is not valid for M-paranormal operators if $M > 1$.

Embry [3] has established that an operator T is subnormal if and only if

$$\sum_{i,j=0}^{n} (T^{i+j}x_i,\ T^{i+j}x_j) \geq 0$$

for all finite collection of vectors $x_0, x_1, \ldots, x_n$ in H. Using this characterization, we find out the condition under which a M-paranormal operator becomes subnormal.

THEOREM 1.3: *If a M-paranormal partial isometry T satisfies $\|T\| \leq \frac{1}{M}$, then it is subnormal.*

PROOF: Since T is a partial isometry, $TT^*T = T$ [5, Corollary 3, Problem 98], Also T being M-paranormal, therefore by Theorem 1.1

$$M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 \geq 0$$

for each $\lambda > 0$. Using $TT^*T = T$ we obtain

$$M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2 T^*T = T^*T[M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2]T^*T \geq 0$$

This is true for each $\lambda > 0$ and hence for $\lambda = 1$,

$$M^2T^{*2}T^2 - T^*T \geq 0$$

This means

$$\|Tx\|^2 \leq M^2\|T^2x\|^2 \leq M^2\|T\|^2\|Tx\|^2 \leq \|Tx\|^2$$

since $\|T\| \leq \frac{1}{M}$. This shows

$$T^*T = M^2T^{*2}T^2$$

which on repeated use yields $T^*T = M^{2(n-1)}T^{*n}T^n$ for each $n \geq 1$. Now, let $x_0, x_1, \ldots, x_n$ be a finite collection of vectors in H

$$\begin{aligned} M^{4n} \sum_{i,j=0}^{n} (T^{i+j}x_i, T^{i+j}x_j) &= \sum_{i,j=0}^{n} M^{4n-2(i+j-1)}(M^{2(i+j-1)}T^{*i+j}T^{i+j}x_i, x_j) \\ &= \sum_{i,j=0}^{n} M^{[2n+1-i-j]}(T^*Tx_i, x_j) \end{aligned}$$

Since T^*T is a projection [5, Problem 98], we obtain

$$\begin{aligned} M^{4n} \sum_{i,j=0}^{n} (T^{i+j}x_i, T^{i+j}x_j) &= \sum_{i,j=0}^{n} M^{2[2n+1-i-j]}((T^*T)^{i+j}x_i, (T^*T)^{i+j}x_j) \\ &= M^{2(2n+1)}(x_0, x_0) + M^{4n} \sum_{\substack{i,j \\ i,j=1}} ((T^*T)x_i, (T^*T)x_j) \\ &+ M^{2(2n-1)} \sum_{\substack{i,j \\ i,j=2}} ((T^*T)^2x_i, (T^*T)^2x_j) + \cdots + \\ &+ M^2 \sum_{\substack{i,j \\ i,j=2n}} ((T^*T)^{2n}x_i, (T^*T)^{2n}x_j) \end{aligned}$$

As $M \geq 1$, we get that

$$M^{2(2n+1)}(x_0, x_0) \geq M^{4n}(x_0, x_0)$$

Thus

$$\begin{aligned} M^{2(2n+1)}(x_0, x_0) + M^{4n} \sum_{\substack{i,j \\ i,j=1}} ((T^*T)x_i, T^*T)x_j) \\ \geq M^{4n}(x_0, x_0) + M^{4n} \sum_{\substack{i,j \\ i,i=1}} ((T^*T)x_i, (T^*T)x_j) \\ = M^{4n} \sum_{i,j=0}^{1} \big((T^*T)^{i+j}x_i, (T^*T)^{i+j}x_j\big) \geq 0, \end{aligned}$$

since T^*T being self-adjoint is subnormal. Again

$$M^{4n} \sum_{i,j=0}^{1} ((T^*T)^{i+j}x_i, (T^*T)^{i+j}x_j) \geq M^{2(2n-1)} \sum_{i,j=0}^{1} ((T^*T)^{i+j}x_i), (T^*T)^{i+j}x_i).$$

Hence

$$\begin{aligned} M^{2(2n+1)}(x_0, x_0) + M^{4n} \sum_{\substack{i,j \\ i+j=1}} ((T^*T)x_i, (T^*T)x_j) \\ + M^{2(2n-1)} \sum_{\substack{i,j \\ i+j=2}} ((T^*T)^2x_i, (T^*T)x_j) \\ \geq M^{2(2n-1)} \sum_{i,j=0}^{1} ((T^*T)^{i+j}x_i, (v^*T)^{i+j}x_j) \\ + M^{2(2n+1)} \sum_{\substack{i,j \\ i+j=2}} ((T^*T)^2x_i, (T^*T)^2x_j) \\ = M^{2(2n-1)} \sum_{i,j=0}^{2} ((T^*T)^{i+j}x_i, (T^*T)^{i+j}x_j) \geq 0. \end{aligned}$$

Continuing in this way, we would have

$$M^{4n} \sum_{i,j=0}^{n} (T^{i+j}x_i, T^{i+j}x_j) \geq M^2 \sum_{i,j=0}^{n} ((T^*T)^{i+j}x_i, (T^*T)^{i+j}x_j) \geq 0,$$

This gives

$$\sum_{i,j=0}^{n} (T^{i+j}x_i, T^{i+j}x_j) \geq 0$$

Hence T is subnormal.

COROLLARY 1.4: *Every paranormal partial isometry is subnormal.*

Our next result appears in [2] for general Banach Algebras. We are giving its proof here for operators on Hilbert space.

THEOREM 1.5: *If $ReW(T) \leq 0$, where $W(T)$ denotes the numerical range of T, then T is M-paranormal for $M \geq 8$.*

PROOF: We shall prove that

$$\|T^2x\| \geq \delta[\|\delta x + Tx\| - \delta\|x\|]$$

for each $x \in H$ and for each $\delta \geq 0$. Let $y = \delta x + Tx$. If $y = 0$, the required inequality is obviously true. Hence suppose that $y \neq 0$. Let $z = \frac{y}{\|y\|}$. Now

$$\begin{aligned}\|y\| = \|y\|\|z\|^2 = \|y\|\left(\frac{y}{\|y\|}, z\right) \\ = (y, z) = (\delta x + Tx, z) \\ = \delta(x, z) + (Tx, z).\end{aligned}$$

Hence

$$\begin{aligned}\|y\|(Ty, z) = (Ty, z) = \delta(Tx, z) + (T^2x, z) \\ = \delta\|y\| - \delta^2(x, z) - +(T^2x, z).\end{aligned}$$

By hypothesis $Re(Tz, z) \leq 0$. Hence

$$\begin{aligned}\|T^2x\| \geq |(T^2x, z)| \geq -Re(T^2x, z) \geq \delta\|y\| - \delta^2(x, z) \\ \geq \delta\|y\| - \delta^2\|x\| \\ = \delta(\|\delta x + Tx\| - \delta\|x\|).\end{aligned}$$

Now

$$\|Tx\| - \delta\|x\| \leq |\,\|Tx\| - \delta\|x\|\,| \leq \|Tx + \delta x\|$$

Using this we get

$$\begin{aligned}\|T^2x\| \geq \delta(-\delta\|x\| + \|Tx\| - \delta\|x\|) \\ = \delta(\|Tx\| - 2\delta\|x\|)\end{aligned}$$

If $\|x\| = 1$ and $\delta = \|Tx\|/4$, we obtain

$$\|T^2x\| \geq \frac{\|Tx\|^2}{8}.$$

§ 2: In this section we discuss the conditions under which, the sum, the product and the inverse (if it exists) of M-paranormal operators become M-paranormal. The question of inverse can be readily answered.

THEOREM 2.1: *If T is invertible M-paranormal operator then T^{-1} is also M-paranormal.*

PROOF: We have
$$M\|T^2x\| \geq \|Tx\|^2$$
for each x with $\|x\| = 1$. This can be replaced by
$$\frac{M\|x\|}{\|Tx\|} \geq \frac{\|Tx\|}{\|T^2x\|}$$
for each $x \in H$. Now replace x by $T^{-2}x$, then
$$M\|x\|\,\|T^{-2}x\| \geq \|T^{-1}x\|^2.$$
for each x in H. This shows that T^{-1} is M-paranormal.

The sum of two M-paranormal even commuting or double commuting (A and B are said to be double commuting if A commutes with B and B^*) operators may not be M-paranormal as can be seen by the following example

EXAMPLE 2.2: Let
$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } S = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
be operators on 2-dimensional space. Then T and S are both $\sqrt{\mathbf{2}}$ – paranormal while $T + S$ is not so.

That the product of two M-paranormal commuting (even double commuting) may not be M-paranormal is illustrated by the following considerations.

Let T be any M-paranormal operator. We claim that $T \otimes I$ and $I \otimes T$ are both M-paranormal. This can be seen by using the fact that the tensor product of two positive operators is positive and the following computations.
$$\begin{aligned} M^2[(T \otimes I)^*]^2(T \otimes I)^2 - 2\lambda(T \otimes I)^*(T \otimes I) + \lambda^2(I \otimes I) \\ = [M^2T^{*2}T^2 - 2\lambda T^*T + \lambda^2] \otimes I. \end{aligned}$$
Now $T \otimes T = (T \otimes I)\ (I \otimes T)$. Thus to prove our assertion we find an example of a M-paranormal operator T such that $T \otimes T$ is not M-paranormal. Suppose that H is a 2-dimensional Hilbert space. Let K be the direct sum of a denumerable copies of H. Let A and B be any two positive operators on H. Define an operator $T = T_{A'B'n}$ on K as
$$T\langle x_1, x_2, \ldots, \rangle = \langle 0, Ax_1, Ax_2, \ldots, A_n, Bx_{n+1}, Bx_{n+2}\ldots\rangle,$$
we can compute to find that T is M-paranormal iff $M^2AB^2A - 2\lambda A^2 + \lambda^2 \geq 0$ for each $\lambda > 0$. Set
$$C = \begin{bmatrix} M & M \\ M & 2M \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 2 \\ 2 & 8 \end{bmatrix}.$$

Then both C and D are positive and for $\lambda > 0$

$$M^2D - 2\lambda C + \lambda^2 = \begin{bmatrix} (M-\lambda)^2 & 2M(M-\lambda) \\ 2M(M-\lambda) & (2M-\lambda)^2 + 4M^2 \end{bmatrix}.$$

This operator is also seen to be positive. Now let $A = C^{\frac{1}{2}}$ and $B = (C^{-1/2}$ $DC^{-1/2})^{1/2}$. Taking $T = T_{A'B'n}$ as mentioned above, we find that T is M-paranormal. We claim that $T \otimes T$ is not M-paranormal. Let if possible

$$M^2[(T \otimes T)^*]^2(T \otimes T)^2 - 2\lambda(T \otimes T)^*(T \otimes T) + \lambda^2(I \otimes I) \geq 0$$

for each $\lambda > 0$. Putting $\lambda = 1$, we get that

$$M^2[T^{*2}T^2 \otimes T^{*2}T^2] - 2[T^*T \otimes T^*T] + I \otimes I \geq 0.$$

Thus the compression of this operator to the canonical image of $H \otimes H$ in $K \otimes K$ is also positive. But the compression coincides with

$$M^2(D \otimes D) - 2(C \otimes C) + I \otimes I = \begin{bmatrix} 1-M^2 & 0 & 0 & 2M^2 \\ 0 & 4M^2+1 & 2M^2 & 12M^2 \\ 0 & 2M^2 & 4M^2+1 & 12M^2 \\ 2M^2 & 12M^2 & 12M^2 & 56M^2+1 \end{bmatrix}$$

which is not positive.

THEOREM 2.3: *If a M-paranormal operator T double commutes with a hyponormal operator S, then the product TS is M-parnormal.*

PROOF: Let $\{E(t)\}$ be the resolution of the identity for the self-adjoint operator S^*S. By hypothesis T^*T and $T^{*2}T^{*2}$ both commute with every $E(t)$. Since S is hyponormal, $S^*S \geq SS^*$. Hence for each $\lambda > 0$

$$\begin{aligned} &M^2[(TS)^*]^2(TS)^2 - 2\lambda(TS)^*(TS) + \lambda^2 \\ &\quad = M^2(T^{*2}T^2)(S^{*2}S^2) - 2\lambda(T^*T)(S^*S) + \lambda^2 \\ &\quad \geq M^2T^{*2}T^2(S^*S)^2 - 2\lambda(T^*T)(S^*S) + \lambda^2 \\ &\quad = \int_0^\infty (t^2M^2T^{*2}T^2 - 2\lambda tT^*T + \lambda^2)dE(t) \\ &\quad \geq 0, \end{aligned}$$

since T is M-paranormal. Hence TS is M-paranormal by Theorem 1.1.

If S is a M-hyponormal operator, then $M^2S^*S \geq SS^*$ [9]. Now if T is any operator double commuting with S, then

$$M^2[(TS)^*]^2(TS)^2 - 2\lambda(vS)^*(TS) + \lambda^2 \geq T^{*2}T^2(S^*S)^2 - 2\lambda(T^*T)(S^*S) + \lambda^2$$

for each λ. Using this and arguing as in Theorem 2.3, we can prove the following.

THEOREM 2.4: *If a paranormal operator T double commutes with a M-hyponormal operator S, then TS is M-paranormal.*

With suitable modifications in the proof of [1, Theorem 3], the following can be easily established.

THEOREM 2.5: *Let T and S be double commuting operators. Let one of T and S be paranormal and other be M-paranormal. Then the product TS is M-paranormal if there are a self-adjoint operator A and bounded positive Borel functions $f(t)$ and $g(t)$ such that*

$$(f(t) - f(s))(g(t) - g(s)) \geq 0, \quad (-\infty < t, s < \infty),$$

and one of the following holds.

(a) $f(A) = T^*T$ *and* $g(A) = S^*S$,

(b) $f(A) = T^{*2}T^2$ *and* $g(A) = S^*S$,

(c) $f(A) = T^{*2}T^2$ *and* $g(A) = S^{*2}S^2$.

REMARK 2.6: Motivated by M-power class considered by Istratescu [7], we consider the subclass S of M-paranormal operators satisfying

$$\|T^n x\|^2 \leq M\|T^{2n}x\|$$

for each $n \geq 1$ and for all $x \in H$ with $\|x\| = 1$. We con easily prove the following:
(i) If $T \in S$, then the spectral radius r_T of T satisfies

$$\frac{1}{M}\|T\| \leq r_T.$$

(ii) If $T \in S$ and is invertible, then $T^{-1} \in S$.
(iii) If $T \in S$ and $z \in \rho(T)$, the resolvent set of T, then

$$\|(T - z)^{-1}\| \leq \frac{M}{d(z, \sigma(T))}$$

(iv) If $T \in S$ and is quasinilpontent then $T = 0$.
(v) If $T \in S$, then the set

$$M_T = \{x \colon \|T^n x\| \leq M\|x\|, \ \ n = 1, 2, \ldots\}$$

is a closed invariant subspace for T and also for all operators commuting with T.

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Department of Mathematics.
Hans Raj College,
University of Delhi,
Delhi – 11007, India.

Department of Mathematics
S.G.T.B. Khalsa College,
University of Delhi,
Delhi – 110007, India.