$Q_r$-SEMIGROUPS

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T. E. Nordahl, in [5], considered the commutative $Q$-semigroups. C. S. H. Nagore, [4] extended Nordahl’s results on quasi-commutative semigroups. A. Cherubini-Spoletini and A. Varisco consider Putcha’s $Q$-semigroups. The definition of weakly commutative semigroup has given by M. Petrich in [6]. Here, we give the definition of $Q_r$-semigroup i.e. a semigroup in which every proper right ideal is a power joined semigroup and we give as well some characterizations of weakly commutative $Q_r$-semigroups, (Theorem 3.1.).

In section 1, we characterize semilattices of groups. In section 2, we consider archimedean weakly commutative semigroups. A weakly commutative semigroup which does not have prime ideals is characterized by Theorem 2.1. This theorem is a generalization of G. Thierrin’s result, [11]. By Theorem 2.3, are characterized weakly commutative semigroups with an idempotent which are archimedean. This theorem is an extension of T. Tamura and N. Kimura’s result in [10]. G. Thierrin and G. Thomas in [12], too, give a characterization for these semigroups. In section 3, we give the definition of $Q_r$-semigroup. This notion is another generalization of the notion of power joined semigroup. The description of weakly commutative $Q_r$-semigroups is given by Theorem 3.1.

For undefined notions we refer to [2] and [7].

1. Semilattices of groups

Here, we will characterize the semilattices of groups using the notion of weakly commutative semigroup.

Definition 1.1. [6]. A semigroup $S$ is weakly commutative if for every $a, b \in S$ there exist $x, y \in S$ and $n \in \mathbb{N}$ such that

$$ (ab)^n = xa = by. $$
Denote with \( \pi \) the class of all weakly commutative semigroups.

**Theorem 1.1.** Let \( S \) be a semigroup. Then \( S \) is a semilattice of groups if and only if \( S \in \pi \) and \( S \) is a (left, right, intra-) regular.

**Proof.** Let \( S \) be a regular semigroup. Then for every \( a \in S \) there exists \( x \in S \) such that \( a = axa \). Hence, \( a = (ax)^n a \), for every \( n \in N \). As \( S \in \pi \), then for \( a \) and \( x \) there exist \( m \in N \) and \( z \in S \) such that \( (ax)^m = za \), so \( a = (ax)^n a = za^2 \in Sa^2 \). Hence, \( S \) is a left regular semigroup. Similarly, we have that \( S \) is a right regular semigroup. By Theorem 12. [8] we have that \( S \) is a semilattice of groups.

The converse follows by Theorem 12. [8].

**Corollary 1.1** Let \( S \) be a semigroup. Then \( S \) is a (left, right) simple and \( S \in \pi \) if and only if \( S \) is a group.

### 2. Archimedean Semigroups

**Definition 2.1.** [10]. A semigroup \( S \) is left (right) archimedean is for every \( a,b \in S \) there exist \( x,y \in S \) and \( n \in N \) such that \( a^n = xb \), \( b^n = ya \). \((a^n = bx, b^n = ay)\). \( S \) is an archimedean semigroup if for every \( a,b \in S \) there exist \( x,u,y,v \in S \) and \( n \in N \) such that \( a^n = xby \), \( b^n = uav \).

**Lemma 2.1.** Let \( S \in \pi \). Then, the following conditions are equivalent:

(i) \( S \) is left archimedean,

(ii) \( S \) is right archimedean,

(iii) \( S \) is archimedean.

**Proof.** (i) \( \Rightarrow \) (ii). Let for every \( a, b \in S \) exist \( x, y \in S \) and \( n \in N \) such that

\[
a^n = xb, \quad b^n = ya.
\]

As \( S \in \pi \), then for \( x \) and \( b \) there exist \( m \in N \) and \( z, u \in S \) such that

\[
(xb)^m = bz = ux.
\]

Similarly

\[
(ya)^k = av = wy
\]

for some \( k \in N \) and \( v, w \in S \). From (2.1) and (2.2) we have that

\[
a^{nm} = (xb)^m = bz.
\]

From (2.1) and (2.3) we have

\[
b^k = (ya)^k = av.
\]
From (2.4) and (2.5) it follows
\[ a^{nmk} = (bz)^k, \quad b^{nmk} = (av)^m. \]

Hence, \( S \) is a right archimedean semigroup. Similarly to the previous, it can be proved that (iii) \( \Rightarrow \) (i), (ii) \( \Rightarrow \) (iii) follows immediately.

**Corollary 2.1.** A weakly commutative archimedean semigroup has one idempotent at most.

**Lemma 2.2.** Let \( S \) be a weakly commutative archimedean semigroup. Then every semiprime ideal from \( S \) is two-sided.

**Proof.** Let \( S \in \pi \) and \( R \) be a right ideal of \( S \) and \( R \) is semiprime. For arbitrary \( a \in R, b \in S \) there exist \( x, y \in S \) and \( n \in N \) such that \((ba)^n = ax \in R\), hence \( ba \in R \). Similarly, for a left ideal of \( S \).

**Theorem 2.1.** Let \( S \) be a semigroup. Then \( S \) is weakly commutative and \( S \) does not have proper prime ideals if and only if \( S \) is a left and right archimedean semigroup.

**Proof.** Let \( S \) be a weakly commutative semigroup that does not have proper prime ideals. Let \( \langle a \rangle \) be a cyclic semigroup generated by \( a \in S \). Denote with \( S_a \) the set of all \( x \in S \) such that they divide from the left side some element from \( \langle a \rangle \). The set \( S_a \) is non-empty since \( \langle a \rangle \subset S_a \). The set \( S_a \) a subsemigroup of \( S \). For \( x, y \in S_a, u, w \in S^1 \) and \( h \in N \) such that \( wz = a^h \), \( wy = a^h \) and exist \( v \in S^1 \) and \( k \in N \) such that \( yv = a^k \), (Lemma 2.1.), so \( u(xy)v = a^{h+k} \), (Lemma 2.1.). Hence, \( xy \in S_a \). Take \( S \setminus S_a \neq \emptyset \) and \( z \in S \setminus S_a, a \in S \). The element \( az \) is not in \( S_a \), (if \( az \in S_a \), then there exist \( u \in S \) such that \( uaz \in \langle a \rangle \), so \( z \in S_a \), which is impossible). Hence, \( az \in S \setminus S_a \), so \( S \setminus S_a \) is a left ideal of \( S \). Since \( S_a \) is a subsemigroup of \( S \), so \( S \setminus S_a \) is a prime ideal of \( S \), hence it is two-sided, (Lemma 2.2.). Let \( a, b \in S \). As \( S \) does not have proper prime ideals it follows that \( S \setminus S_a = \emptyset \), i.e. \( S = S_a \) and there exist \( u \in S^1 \) and \( h \in N \) such that \( a^h = ub \). Analogously \( b^k = va \), \( k \in N, v \in S^1 \).

Hence, \( S \) is a left archimedean semigroup. If can be proved, in a similar way, that \( S \) is a right archimedean semigroup.

Conversely, let \( S \) be a left and right archimedean semigroup. Then \( S \) is weakly commutative. Let \( S \) has a proper prime ideal \( I \) and let \( a \in I, b \in S \setminus I \). Then there exist \( x \in S \) and \( n \in N \) such that \( b^n = ax \in I \), so \( b \in I \), which is impossible.

**Lemma 2.3.** Let \( S \in \pi \) be a archimedean semigroup with the idempotent \( e \), then \( eS \) is a group and \( eS = Se = SeS \) hold.

**Proof.** Let \( a \in eS \). Then \( a = ex \) for some \( x \in S \). From this we have \( ea = e^2x = ex = a \), so \( e \) is a left identity for \( eS \). \( S \) is an archimedean semigroup, then it exists \( y \in S \) such that \( e = ya \), (Lemma 2.1.) i.e. \( e = (ey)a \). Hence, \( a \) has in \( eS \) an inverse element relatively to \( e \). It follows that \( eS \) is a group with identity \( e \). For arbitrary \( a \in eS, a = ex \) holds, \( x \in S \), so \( a = eex \in SeS \). Similarly for
arbitrary $b \in S e S$ is $b = u e v$, ($u, v \in S$), i.e. $b = u(ev)e = (uev)e \in S e$, because $e$ is an identity in $e S$. Hence,

\[(2.6)\quad e S \subset S e S \subset S e.\]

We prove that

\[(2.7)\quad S e \subset S e S \subset e S\]

analogously. From (2.6) and (2.7) we have that $e S = S e = S e S$.

By using the Theorem of Clifford (Theorem 4; 19, [2]) it can be easily verified.

**Lemma 2.4.** If $S$ is an ideal extension of a weakly commutative archimedean semigroup with identity by a nil-semigroup, then $S$ is weakly commutative semigroup.

The following theorem is an extension of the result of T. Tamura and N. Kimura [10].

**Theorem 2.3.** Let $S$ be a semigroup. Then $S$ is a weakly commutative archimedean with an idempotent if and only if $S$ is a group or $S$ is an ideal extension of a group by a nil-semigroup.

**Proof.** Let $S$ be a weakly commutative archimedean semigroup with the idempotent $e$. If $S$ is simple, then $S$ is a group (Corollary 1.1.). If $S$ is not simple, take the ideal $I = S e S$ and the factor-semigroup of Rees $S/I$. From Lemma 2.3, $I$ is a group. Since $S$ is an archimedean semigroup, so for every $a \in S$, $b \in I$ there exist a natural number $n$ and $x \in S$ such that $a^n = b x$ holds, (Lemma 2.1.). From this we have that $a^n \in I$. Hence, $S/I$ is a nil-semigroup. If $e$ is a zero in $S$ then $S/I \cong S$, so $S$ is a nil-semigroup itself, because $I$ contains only $e$.

Conversely, let $S$ be an extension of the group $I$ by a nil-semigroup $Q$. From Lemma 2.4, $S$ is a weakly commutative semigroup. Obviously, $S$ contains only one idempotent (an identity from $I$). Let us prove that $S$ is an archimedean semigroup. The semigroup $S/I$ is a nil-semigroup, and, as $S/I \cong Q$, it follows that for arbitrary $a, b \in S$ there exist $h$ and $k$ such that $a^h, b^k \in I$. But, $I$ is a group, so there exist $x, y \in I$ such that $a^h = b^k x, b^k = a^h y$. Hence, $S$ is a right archimedean semigroup, so from Lemma 2.1. it is archimedean. The assertion follows immediacy if $S$ is a group.

**Lemma 2.5.** Let $S$ be an archimedean weakly commutative semigroup without idempotents. Then $a \neq ab$, for every $a, b \in S$.

**Proof.** Let $S$ be an archimedean weakly commutative semigroup without idempotents. Assume opposite, i.e. $a = ab$. Then for $a$ and $b$ there exist $x \in S$ and $n \in N$ such that $b^n = ax$ holds, (Lemma 2.1.), and so $a = ab = ab^2 = \cdots = ab^n$, so $a = a^2 x$. Hence, the element $a$ is a right regular, so it is regular, (Theorem 1.1.). It follows that $S$ has an idempotent, which is impossible.
3. $Q_r$-Semigroups

**Definition 3.1.** [6]. A semigroup $S$ is a power joined if for every $a, b \in S$ there exist $m, n \in N$ such that $a^m = b^n$.

Obviously, a power joined semigroup is weakly commutative.

Immediately follows

**Lemma 3.1.** Let $S$ be a semigroup. Then the following conditions are equivalent:

(i) $S$ is power joined,

(ii) Every ideal from $S$ is a power joined semigroup,

(iii) Every right (left) ideal of $S$ is a power joined semigroup.

T. E. NORDAHL, [5] considered commutative $Q$-semigroups. We give here the definition of $Q_r$-semigroup, which is another generalization of a power joined semigroup.

**Definition 3.2.** A semigroup $S$ is $Q_r$-semigroup ($Q_r$-semigroup) if every proper right (left) ideal of $S$ is a power joined semigroup.

$Q_r$-semigroup is $Q$-semigroup. The converse is not true. For example, the semigroup $S$ given by

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is a $Q$-semigroup. But, the right ideal $\{a, d\}$ is not a power joined semigroup, so $S$ is not a $Q_r$-semigroup.

The following theorem describes weakly commutative $Q_r$-semigroups.

**Theorem 3.1.** Let $S$ be a semigroup. Then $S$ is a weakly commutative $Q_r$-semigroup if and only if one of the tree possibilities hold:

1° $S$ is a power joined semigroup,

2° $S$ is a group,

3° $S = M \cup G$ and the identity $e$ of the group $G$ is a left identity of $S$ and $M$ is the unique maximal prime ideal of $S$ and $M$ is a power joined semigroup.

**Proof.** Let $S$ be an archimedean weakly commutative $Q_r$-semigroup. Then $S$ has one idempotent at most, (Corollary 2.1). If $S$ does not have an idempotent, then from Lemma 2.5. for every $a \in S$ is $a \not\in aS$. From this, we have that $aS$ is a proper right ideal of $S$. Hence, $aS$ is a power joined semigroup. For $b \in S$ there exists $n \in N$ such that $b^n \in aS$, (Lemma 2.1) and Obviously there exists
$m \in N$ such that $a^m \in aS$. $aS$ is a power joined semigroup, then there exist $r$, $s \in N$ such that $a^{mr} = b^{ns}$. Hence, in this case $S$ is a power joined semigroup. If $S$ has an idempotent $e$, then from Lemma 2.3. $eS$ is a group-ideal of $S$. If $eS \neq S$, then $eS$ is a proper ideal of $S$, $eS$ is power joined, so $eS$ is a periodic group. So, $S$ is a nil-extension of a periodic group, (Theorem 2.3.). From this $eS$ is a power joined semigroup with one idempotent. If $es = S$, then $S$ is a group. If $S$ is not an archimedean semigroup then from Theorem 2.1. $S$ has a proper prime ideal. Denote with $M$ the union of all proper prime ideals of $S$. Then $M$ is a maximal prime ideal of $S$ and $M$ is a power joined semigroup. If $M = S$, then $S$ is a power joined semigroup. If $M \not= S$, then for $x \in S \setminus M$ is $x^2 \in S \setminus M$ and as $M$ is a maximal ideal of $S$, $M \cup J(x) = M \cup J(x^2) = S$, so $x = x^2$ or $x = x^2t$ or $x = t_1x^2$ or $x = t_2x^2t_3$, for some $t_1, t_2, t_3 \in S \setminus M$. From the Theorem 1.1 we have that in each of these cases $x$ is a regular element, i.e. $S \setminus M$ is a regular semigroup, so it contains idempotents. It can be easily verified that $S \setminus M$ has only one idempotent. Hence, $S \setminus M$ is a group. So if $M \not\subseteq S$, then

(*)

$$S = M \cup G$$

where $M$ is a unique maximal prime ideal which is a power joined semigroup and $G$ is a group. We distinguish now two cases:

(i) $eS = S$. Then for each $x \in S$ is $x = es$, for some $z \in S$ and $ex = e(es) = x$. Hence, $e$ is a left identity of $S$, and in this case $S$ is of the type $3^0$.

(ii) $eS \not\subseteq S$. Then $eS$ is a power joined semigroup. From (*) we have that

$$eS = eM \cup eG = eM \cup G.$$ 

It follows that $G \subseteq eS$, so $S = M \cup eS$ which means that in this case $S$ is a power joined semigroup.

Conversely, let 3$^0$ holds. If $a, b \in M$ then there exist $x, y \in M$ and $n \in N$ such that $(ab)^n = xa = by$, because $M$ is power joined. If $a, b \in G$ then $(ab)^n = xa = by$ for some $x, y \in G$ and $n \in N$. If $a \in M, b \in G$, then $bab \in M$, so

$$(ab)^{2k} = [a(bab)]^k = xa = baby, \text{ for some } k \in N \text{ and some } x, y \in M.$$ 

Hence, $S$ is a weakly commutative semigroup. Take an arbitrary proper right ideal $R$ from $S$. If $R \subset M$, then $R$ is a power joined semigroup, so $S$ is a $Q_r$-semigroup. If $R \cap G \neq \emptyset$, then $G \subset R$ and we have that $e \in R$. Hence, $R = S$, which is impossible. In the other cases the assertion immediately follows.

Note that Lemma 3.1, holds if we change the term “ideal” with the term “quasi-ideal” (“bi-ideal”), (for definitions of a quasi-ideal and bi-ideal see [2] or [9]). Hence, the notion of a power joined semigroup could be generalised in the following way:

**Definition 3.3.** A semigroup $S$ is a $Q_r$-semigroup ($Q_r$-semigroup) if every proper quasi-ideal (bi-ideal) of $S$ is a power joined semigroup.
Denote with \( P, Q_b, Q_r, Q_l, Q_q, Q \) the classes of all power joined, \( Q_{br}, Q_{r-}, Q_{l-}, Q_{q-}, Q \)-semigroups. The we have

**Lemma 3.2.** \( P \subset Q_b \subset Q_q \subset Q_r \cup Q_l \subset Q \).

From the Theorem 3.1, its dual theorem and lemma 3.2, immediately follows

**Theorem 3.2.** Let \( S \) be a weakly commutative archimedean semigroup with no idempotents, then the following conditions are equivalent:

(i) \( S \) is power joined,
(ii) \( S \) is \( Q_b \)-semigroup,
(iii) \( S \) is \( Q_r \)-semigroup,
(iv) \( S \) is \( Q_l \)-semigroup,
(v) \( S \) is \( Q_q \)-semigroup.

**References**