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EQUATIONAL REFORMULATIONS OF INTUITIONISTIC PROPOSITIONAL CALCULUS AND CLASSICAL FIRST-ORDER PREDICATE CALCULUS

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1. The paper [1] of S. B. Prešić shows a possibility of assignment of an equational formal theory detoned by $\Theta(\sim)$, to any formal theory Θ. An essential relation between Θ and $\Theta(\sim)$ is given by assertions: (i) a binary predicate denoted by $\sim$ is formalization of the metatheoretic equiconsequence (or interdeducibility) relation of Θ (cf. [1] Theorem 1.) and (ii) Θ isomorphically embedded in $\Theta(\sim)$ by mapping $f\colon \mathit{For}(\Theta) \to \mathit{For}(\Theta(\sim))$ defined by $f(A) = A \sim \top$ (cf. [1] Lemma 3.) On the other hand, sufficient conditions (cf. [1] Condition 1. and Condition 2.) under which the converse of (ii) is valid, are given also there. Then the formal theory $\Theta(\sim)$, which we shall call an equational reformulation of Θ, is of particular importance for our further exposure. In other words, it is also established that every proof within the formal theory Θ can be translated into (completable) proof of $\Theta(\sim)$ and the converse too, provided that conditions 1. and 2. are fulfilled.

2. Let us assign the coresponding equational formal theory $I_0(\sim)$ to the intuitionistic propositional calculus I_0 formulated as in [2] p. 433. $I_0(\sim)$ will be equational reformulation of the formal theory I_0 too, because conditions 1. and 2. are satisfied (Cond. 2. is satisfied by *deduction theorem,* [2] p. 433). In this case, we should have in mind that

$$(0) \qquad \vdash_{I_0} A \Leftrightarrow B \ \textit{ iff } \ \vdash_{I_0(\sim)} A \sim B \ \text{ (by [1] Theorem 4. } (2^\circ))$$

for any propositional formulas A, B, where we write $A \Leftrightarrow B$ for $(A \Rightarrow B) \wedge (B \Rightarrow A)$.

3. The following are axioms of $I_0(\sim)$.

a) $A \sim A$; $A \,\&\, B \sim B \,\&\, A$; $A \,\&\, (B \,\&\, C) \sim (A \,\&\, B) \,\&\, C$; $A \,\&\, \top \sim A$;
$A \,\&\, (A \Rightarrow B) \sim A \,\&\, (A \Rightarrow B) \,\&\, B$;

b) $A \Rightarrow (B \Rightarrow A) \sim \top$; $(A \Rightarrow (B \Rightarrow C)) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C)) \sim \top$;
$A \Rightarrow (B \Rightarrow A \wedge B) \sim \top$; $A \wedge B \Rightarrow A \sim \top$; $A \wedge B \Rightarrow B \sim \top$;
$A \Rightarrow A \vee B \sim \top$; $B \Rightarrow A \vee B \sim \top$; $(A \Rightarrow C) \Rightarrow ((B \Rightarrow C) \Rightarrow$
$\Rightarrow (A \vee B \Rightarrow C)) \sim \top$; $(A \Rightarrow B) \Rightarrow ((A \Rightarrow \neg B) \Rightarrow \neg A) \sim \top$;
$\neg A \Rightarrow (A \Rightarrow B) \sim \top$.

Rules of inference of $I_0(\sim)$ are:

$$(CONGR) \qquad \frac{A \sim B}{B \sim A}, \frac{A \sim B \; B \sim C}{A \sim C}, \frac{A \sim B \; C \sim D}{A \& C \sim B \& D}.$$

Notice that these are axiom schemes and rule schemes each with infinitely many instances.

Using the relation (0) and known facts of the intuitionistic propositional calculus, it is not difficult to examine that the following formulas are theorems of $I_0(\sim)$:

$$A \vee B \sim B \vee A, A \wedge B \sim B \wedge A, A \vee (B \vee C) \sim (A \vee B) \vee C,$$
$$A \wedge (B \wedge C) \sim (A \wedge B) \wedge C, (A \wedge B) \vee B \sim B, A \wedge (A \vee B) \sim A,$$
$$A \wedge (A \Rightarrow B) \sim A \wedge B, (A \Rightarrow B) \wedge B \sim B, (A \Rightarrow B) \wedge (A \Rightarrow C) \sim A \Rightarrow B \wedge C,$$
$$(A \Rightarrow A) \wedge B \sim B, A \Rightarrow (\neg(A \Rightarrow A)) \sim \neg A.$$

This means that all axioms of the pseudo-Boolean algebras (cf. for example [3]) are satisfied in the formal theory $I_0(\sim)$.

Of course, rules $\frac{A \sim B}{\neg A \sim \neg B}$ and $\frac{A \sim B C \sim D}{A \circ C \sim B \circ D}$ are valid in $I_0(\sim)$, where $\circ$ can be each of the following symbols $\vee$, $\wedge$ and $\Rightarrow$.

In accordance with [3] (cf. p. 58, 124),we can introduce a partial ordering relation in the pseudo-Boolean algebra $\langle A, \cap, \cup, \supset, - \rangle$: $a \leq b$ *iff (def)* $a \cup b = b$. Also $c \leq a \supset b$ *iff* $a \cap c \leq b(00)$. Let $1 = (def)\ a \supset a$ and $0 = (def) - 1$.

Lemma 1. *Let $\langle A, \cup, \cap, \supset, - \rangle$ be the psudo-Boolean algebra. Then for every $a, b, c \in A$:*

(1) $a \cap 1 = a$, $a \cup 0 = a$; (2) $0 \leq a \leq 1$; (3) $a \cap b \leq b$, $a \leq a \cup b$; (4) *if* $b \leq c$, *then* $a \cap b \leq a \cap c$; (5) $a \cap (a \supset b) = a \cap (a \supset b) \cap b$; (6) $a \supset (b \supset a) = 1$;
(7) $(a \supset (b \supset c)) \supset ((a \supset b) \supset (a \supset c)) = 1$; (8) $(a \cap b) \supset a = 1$;
(9) $(a \cap b) \supset b = 1$: (10) $a \supset (b \supset (a \cap b)) = 1$; (11) $a \supset (a \cup b) = 1$;
(12) $b \supset (a \cup b) = 1$; (13) $(a \supset c) \supset ((b \supset c) \supset ((a \cup b) \supset c)) = 1$;
(14) $(a \supset b) \supset ((a \supset -b) \supset -a) = 1$; (15) $-a \supset (a \supset b) = 1$.

Proof. We will prove, for example, (2), (3), (6) and (13).

(2) $(a \cap 1) \cup 1 = 1$ (by axiom of PBA (pseudo-Boolean algebra))
iff $a \cap 1 \leq 1$ (by definition of $\leq$)
$a \leq 1$ (by (1))

Also $-1 = 0 \leq -a$ (by [3] Ch. I, 12.3.). So, for every $a \in A$ $0 \leq a \leq 1$.

(3) $(a \cap b) \cup b = b$ (by axiom of PBA)

iff (3.1) $a \cap b \leq b$ (by definition of $\leq$)

$a = a \cap (a \cup b)$ (by axiom of PBA)

$\leq a \cup b$ (by (3.1))

(6) $b \cap a \leq a$ (by (3))

iff $a \leq b \supset a$ (by (00))

iff $a \cap 1 \leq b \supset a$ (by (1))

iff $1 \leq a \supset (b \supset a)$ (by (00))

But $a \supset (b \supset a) \leq 1$ (by (2)). So, $1 = a \supset (b \supset a)$ (by antisymmetry of $\leq$).

(13) $(a \supset c) \cap (b \supset c) = (a \cup b) \supset c$ (by [3] Ch. I, 12.2. (17))

then $(b \supset c) \cap (a \supset c) \leq (a \cup b) \supset c$

iff $(a \supset c) \leq (b \supset c) \supset ((a \cup b) \supset c)$ (by (00))

iff $(a \supset c) \cap 1 \leq (b \supset c) \supset ((a \cup b) \supset c)$ (by (1))

iff $1 \leq (a \supset c) \supset ((b \supset c) \supset ((a \cup b) \supset c))$ (by (00))

So, $1 = (a \supset c) \supset ((b \supset c) \supset ((a \cup b) \supset c))$ (by (2) and antisymmetry of $\leq$).

So, all axioms of $I_0(\sim)$ are satisfied in the psuedo-Boolean algebra $\langle For(I_0), \vee, \wedge, \Rightarrow, \neg\rangle$. Of course, rules of inference (CONGR) are valid too.

The consequence of the above assertions is the following statement.

Theorem 1. $\vdash_{\overline{I_o(\sim)}} A \sim B$ *iff* $A = B$ *in the pseudo-Boolean algebra* $\langle For(I_0), \vee, \wedge, \Rightarrow, \neg\rangle$.

4. Now, similarly as in the preceding case, we will assing the corresponding formal theory $K(\sim)$ to the classical first-order predicate calculus K (formulated as in [2] p. 108, with the axioms for equality

(1) $x = x$

(2) $x = y \Rightarrow (A \Rightarrow A(x/y))$,

and all generalizations of (1) and (2)). The formal theory $K(\sim)$ will be an equational reformulation of K, because conditions 1. and 2. are satisfied (Cond. 2. is satisfied by *deduction theorem* [2] p. 109).

Using [1] Theorem 4. (2°) again we have (000)$\vdash_{\overline{K}} A \Leftrightarrow B$ *iff* $I_{\overline{K(\sim)}} A \sim B$ for any first-order formulas A, B.

5. The axioms of $K(\sim)$ are as follows:

a) the same as **3**. a);

b) $A \sim \top$ (where A is any axiom of the classical propositional calculus); $\forall x(A \Rightarrow B) \Rightarrow (\forall x A \Rightarrow \forall x B) \sim \top$; $\forall x A \Rightarrow A(x/t) \sim \top$ (where t is any term free for x in A); $A \Rightarrow \forall x A \sim \top$ (where the variable x is not free in A); $x = x \sim \top$; $x = y \Rightarrow (A \Rightarrow A(x/y)) \sim \top$; $\forall x \top \sim \top$.

Rules of inference of $K(\sim)$ are (CONGR) also.

Using the relation (000) and known facts of the classical first-order predicate calculus, we can establish that following formulas are theorems of $K(\sim)$:

$$A \vee B \sim B \vee A;\ \ A \wedge B \sim B \wedge A;\ \ A \vee (B \vee C) \sim (A \vee B) \vee C;$$
$$A \wedge (B \wedge C) \sim (A \wedge B) \wedge C;\ \ (A \wedge B) \vee B \sim B;\ \ A \wedge (A \vee B) \sim A;$$
$$(A \vee B) \wedge C \sim (A \wedge C) \vee (B \wedge C);\ \ (A \wedge B) \vee C \sim (A \vee C) \wedge (B \vee C);$$
$$A \vee \neg A \sim \top;\ \ A \wedge \neg A \sim F\ \text{ (where we write } F \text{ for } \neg\top);\ \ \exists x F \sim F;$$
$$A \vee \exists x A \sim \exists x A;\ \ \exists x (A \wedge B) \sim \exists x A \wedge B\ \text{ (the variable } x \text{ is not free in } B);$$

$\exists x \exists y A \sim \exists y \exists x A$; $A(x/y) \wedge (\neg A)(x/y) \sim F$ (y is any variable free for x in A); $x = x \sim \top$; $x = y \sim \exists z\ (x = z \wedge z = y)$. It means, all axioms of the cylindric algebras (cf. [4]) are satisfied in the formal theory $K(\sim)$. Naturally, rules $\frac{A \sim B}{\neg A \sim \neg B}$, $\frac{A \sim B}{\exists x A \sim \exists x B}$ and $\frac{A \sim B\, C \sim D}{A \circ C \sim B \circ D}$ are valid in $K(\sim)$ too ($\circ$ is $\vee$ or $\wedge$).

LEMMA 2. *If we let $a \supset b$ and $o_k a$ denote $-a \cup b$ and $-c_k - a$, respectively, and if $\langle A, \cup, \cap, -, 0, 1, c_k, d_{km} \rangle_{k,m<\alpha}$ is the cylindric algebra of dimension α, then for every $a, b, c \in A$ and $k, m < \alpha$:*

(1) $a \cap 1 = a$; (2) $a \cap (a \supset b) = a \cap (a \supset b) \cap b$; (3) $a \supset (b \supset a) = 1$;
(4) $(a \supset (b \supset c)) \supset ((a \supset b) \supset (a \supset c)) = 1$; (5) $(a \cap b) \supset a = 1$;
(6) $(a \cap b) \supset b = 1$; (7) $a \supset (b \supset (a \cap b)) = 1$;
(8) $a \supset (a \cup b) = 1$; (9) $b \supset (a \cup b) = 1$;
(10) $(a \supset c) \supset ((b \supset c) \supset ((a \cup b) \supset c)) = 1$;
(11) $(-a \supset b) \supset ((-a \supset -b) \supset a) = 1$;
(12) $o_k(a \supset b) \supset (o_k a \supset o_k b) = 1$;
(13) $a \supset o_k a = 1$ (where form of a is $c_k b$ or $o_k b$)
(14) $o_k a \supset s^k_m a = 1$; (15) $d_{kk} = 1$; (16) $d_{km} \supset (a \supset s^k_m a) = 1$;
(17) $o_k 1 = 1$ (*s^k_m is m-for-k substitution*).

PROOF. (1) – (11) is provable in BA (Boolean algebra).

(12)
$$\begin{aligned} 1 &= (c_k - a \cup c_k - b) \cup -(c_k - b) \ \text{ (in } BA) \\ &= c_k(-a \cup -b) \cup -(c_k - b) \ \text{ (by [4] Theorem 1.2.6.)} \\ &= c_k((a \cap -b) \cup -a) \cup -(c_k - b) \ \text{ (in } BA) \\ &= (c_k(a \cap -b) \cup c_k - a) \cup -(c_k - b) \ \text{ (by [4] Th. 1.2.6.)} \\ &= o_k(a \supset b) \supset (o_k a \supset o_k b) \ \text{ (in } BA) \end{aligned}$$

(13)
$$\begin{aligned} 0 &= -c_k b \cap c_k b \ \text{ (in } BA) \\ &= c_k(-c_k b \cap c_k b) \ \text{ (by axiom of } CA \text{ (cylindric algebra))} \\ &= c_k - c_k b \cap c_k b \ \text{ (by axiom of } CA) \end{aligned}$$

Let a be denote for $c_k b$. Then $0 = c_k - a \cap a$, i.e. $1 = -(c_k - a \cap a) = a \supset o_k a$. Similarly, when form of a is $o_k b$.

(14) Let $k = m$. $o_k a \supset s^k_m a = c_k - a \cup s^k_m a$

$$
\begin{aligned}
&= c_k - a \cup a \ \text{ (by def. of substitution [4] 1.5.1.)}\\
&= (-a \cup c_k - a) \cup a \ \text{ (by axiom of } CA)\\
&= 1 \ \text{ (in } BA)
\end{aligned}
$$

Let $k \neq m$. $c_k - a \cup s^k_m a = c_k - a \cup c_k(d_{km} \cap a)$ (by def. of substitution [4] 1.5.1)

$$
\begin{aligned}
&= c_k(-a \cup (d_{km} \cap a)) \ \text{ (by [4] Th. 1.2.6.)}\\
&= c_k(-a \cup d_{km}) \ \text{ (in } BA)\\
&= c_k - a \cup c_k d_{km} \ \text{ (by [4] Th. 1.2.6.)}\\
&= c_k - a \cup 1 \ \text{ (by [4] Th. 1.3.2.)}\\
&= 1 \ \text{ (in } BA)
\end{aligned}
$$

(15) is an axiom of CA.

(16) and (17) can be proved similarly.

Consequently, all axioms of $K(\sim)$ are satisfied in the free cylindric algebra of (first-order) formulas (with equality) $\langle For(K), \vee, \wedge, \neg, F, \top, \exists x_k, x_k = x_m\rangle_{k,m<\omega}$. Rules of inference (CONGR) are valid too.

The immediate consequence of the above assertions is the following statement.

THEOREM 2. *$I\frac{}{K(\sim)}A \sim B$ iff $A = B$ in the free cylindric algebra of formulas $\langle For(K), \vee, \wedge, \neg, F, \top, \exists x_k, x_k = x_m\rangle_{k,m<\omega}$.*

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