PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 29 (43), 1981, pp. 29–37

MEASURABILITY OF STOCHASTIC PROCESS AND APPROXIMATE CONTINUITY OF ITS CORRELATION FUNCTION

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Abstract. The oscillation functions of second order process and of its correlation function are defined, and connections between properties of these functions and of the process are considered. Specially, relationships between mean square approximate continuity of the process (and separate approximate continuity of its correlation function) and measurability of that process are investigated.

1. Let $X = X(t)$, $t \in [0;1]$, be a real valued stochastic process of second order, defined on some fixed probability space $(\Omega, \mathcal{F}, P)$, and let $\Gamma = \Gamma(t,u) = (X(t), X(u)), t, u \in [0, : 1]$, be its correlation function. The functions $\omega = \omega(t)$ and $\omega' = \omega'(t), t \in [0;1]$, will be defined by

$$\omega(t) = \sup_{(t_n),(t_n{}')\in G_t} \overline{\lim_{n\to\infty}} \|X(t_n) - X(t'_n)\|, \tag{1}$$

$$\omega'(t) = \sup_{(t_n),(t_n{}'),(u_n),(u_n{}')\in G_t} \overline{\lim_{n\to\infty}} |\Gamma(t_n, u_n) - \Gamma(t_n{}', u_n{}')|^{1/2}, \tag{2}$$

where G_t denotes the set of all sequences converging to t; these functions ω and ω' are oscillation functions of X and Γ, respectively. It is clear that the mean square continuity of X at t is equivalent to equalities $\omega(t) = \omega'(t) = 0$.

Suppose that the function $g(t) = \Gamma(t,t)$, $t \in [0;1]$, is uniformly bounded by some contant $K > 0$, i.e., that

$$g(t) \le K \text{ for all } t \in [0;1]. \tag{3}$$

Then it is easy to see that the functions ω and ω' satisfy the following inequalities:

$$\frac{1}{2}\omega(t) \le \omega'(t) \le 2K^{1/2}\omega^{1/2}(t), \quad t \in [0;1]. \tag{4}$$

If we introduce notations

$$D_s = \{t\colon \omega(t) \geq s\}, \ \ {D_s}' = \{t\colon \omega'(t) \geq s\}, \ \ s \geq 0, \tag{5}$$

then (4) implies the following inclusions:

$$D_s \subset D'_{s/2} \subset D_{s^2/16K} \ \text{ for any } \ s > 0, \tag{6}$$

and, also, the equality

$$\{t\colon \omega(t) > 0\} = \{t\colon \omega'(t) > 0\}.$$

The last equiality means that the set of points t at which the function Γ is continuous on the diagonal $t = u$ is equal to the set of points at which the process X is mean square continuous.

If we denote by $G_t^-\,(G_t^+)$ the set of all sequences increasingly (decreasingly) converging to t, and if we change G_t in (1) and (2) by $G_t^-\,(G_t^+)$, we shall obtain the left (right) oscillation functions of X and Γ, which will be denoted by $\omega_- = \omega_-(t)$ and $\omega'_- = \omega'_-(t)$ ($\omega_+ = \omega_+(t)$ and $\omega'_+ = \omega'_+(t)$), respectively. The corresponding sets of the forms (5) we shall denote by D_s^-, ${D_s^-}'$, D_s^+, ${D_s^+}'$.

In the following we shall suppose, without loss of generality, that, if for some t there exists only one of limits $X(t-0) = \underset{s\to t-0}{\text{l.i.m.}} X(s)$, $X(t+0) = \underset{s\to t+0}{\text{l.i.m.}} X(s)$, then it is equal to $X(t)$, and if there exist both these limits, then the equality $X(t-0) = X(t)$ is satisfied; it is clear that this assumption is not a restriction, but rather a technical simplification.

The linear space of X will be denoted by $H(X)$. We say that the process X is measurable if it is measurable with respect to $\mathcal{F} \times \mathcal{B}_{[0;1]}$ where $\mathcal{B}_{[0;1]}$ is Borel σ-field over [0; 1]. If two stochastic processes, $X = X(t)$ and $Y = Y(t)$, $t \in [0;1]$, such that $P\{X(t) = Y(t)\} = 1$ for any t, we consider as equal, then it is know [3] that the process X is measurable if and only if its correlation function Γ is measurable and the linear space $H(X)$ is separable. The aim of that paper is to find some conditions, in terms of correlation function, under which the process is measurable, or under which it can be approximated by measurable processes. But, first of all we have to introduce the notion of approximate continuity of the process.

The process X is said to be aproximately mean square continuous at $t \in [0;1]$ if there is a set $E_t \in \mathcal{B}_{[0;1]}$, such that X is mean square continuous at t in E_t, and E_t has the unit metric density at t. Similarly, we shall say that the correlation function Γ is approximately continuous at $(t,u) \in [0;1] \times [0;1]$ if there is a set $E_t^u \in \mathcal{B}_{[0;1]} \times \mathcal{B}_{[0;1]}$, such that Γ is continuous at (t,u) in E_t^u, and E_t^u has the unit metric density at (t,u), [7, 8]. It is easy to see that, if the process X is approximately mean square continuous at t, then its correlation function Γ is approximately continuous at (t,t).

2. It is known [2] that the functions ω and ω' (and, also, the other oscillation functions, which are defined at the end of the preceeding section) are upper semicontinuous. In [2] it is proved that the mean square continuity from the left of the process X at any point outside of one at most countable set implies that the set $D_s^+ = \{t\colon \omega_+(t) \geq s\}$ is nowhere dense for any $s > 0$ (which means that, under the same conditions, the set D_s is also nowhere dense for any $s > 0$). In the next theorem we shall give different conditions under which any set D_s, $s > 0$ (and, by (6), the set $D_s{}'$, $s > 0$, also), will be nowhere dense.

THEOREM 1. *The set D_s (and $D_s{}'$, also) is nowhere dense for any $s > 0$ if any of the following conditions is satisfied:*

(I) *The process X is approximately mean square continuous;*

(II) *For any t from one set E, which is everywhere dense in* $[0;1]$, *there exists at least one of limits $X(t-0)$ and $X(t+0)$.*

PROOF. Suppose that (I) does not imply the statement of the theorem, i.e. that, for some $s > 0$, there exists an interval $[a;b] \subset [0;1]$ in which the set D_s is dense. From the fact that the set D_s is closed, [2], it follows $[a;b] \subset D_s$. Let δ, $0 < \delta < 1/2$, be arbitrary but fixed number, and let t_1 be arbitrary point from $(a;b)$; denote by E_1 some set with the follows two properties: (a) X is continuous at t_1 in that set E_1; (b) E_1 has the unit metric density at t_1. There exists a closed interval $I_1 \subset [a;b]$ containing t_1, such that

$$\frac{m(I_1 \cap E_1)}{m(I_1)} > 1 - \delta,$$
$$\|X(t) - X(t_1)\| < s/6 \text{ for any } t \in I_1 \cap E_1.$$

There is $t_2 \in I_1$ such that

$$\|X(t_2) - X(t_1)\| > 5s/6$$

(really, if it is not the case, then the inequality $\omega(t) \leq 5s/6$ holds for all $t \in I_1$, contrary to the hypothesis $\omega(t) > s$ in $[a;b]$). Denote by E_2 the set having the above properties (a) and (b) with respect to that point t_2. There exists a closed interval $I_2 \subset I_1$ containing t_2, such that

$$\frac{m(I_2 \cap E_2)}{m(I_2)} > 1 - \delta,$$
$$\|X(t) - X(t_2)\| < s/6 \text{ for any } t \in I_2 \cap E_2.$$

If we continue the described procedure, we shall obtain one sequence (t_n) of points and corresponding sequence (I_n) of closed intervals, any of which is contained in the proceeding one. Note that the inequality

$$\|X(t_n) - X(t_{n+1})\| > 5s/6 \tag{7}$$

holds for all $n = 1, 2, \ldots$. Let $\tilde{t}$ be the point which is contained in any interval of the sequence (I_n). Denote by $\tilde{E}$ some set which, with respect to $\tilde{t}$, has the above properties (a) and (b). There exists a natural number n_0 such that

$$\frac{m(\tilde{E}\cap I_n)}{m(I_n)} > 1-\delta \ \text{ for any } \ n \geq n_0, \tag{8}$$

$$\|X(t)-X(\tilde{t})\| < s/6 \ \text{ for any } \ t \in I_{n0}\cap \tilde{E}. \tag{9}$$

It is easy to see that it must be

$$(\tilde{E}\cap I_{n_0})\cap E_{n_0} \neq \emptyset; \tag{10}$$

namely, from $(\tilde{E}\cap I_{n_0})\cap E_{n_0} = \emptyset$ it follows $\tilde{E}\cap I_{n_0} \subset E^c_{n_0}\cap I_{n_0}$ (A^c denotes the set theoretical complement of A), which implies $m(\tilde{E}\cap I_{n_0}) \leq m(E^c_{n_0}\cap I_{n_0}) < \delta\cdot m(I_{n_0})$, contrary to (8). Suppose that the inequality

$$\tilde{E}\cap(E_{n_0+1}\cap I_{n_0+1}) \neq \emptyset$$

also holds; from that inequality and from (9) and (10) it follows that there exist points $t' \in \tilde{E}\cap I_{n_0}\cap E_{n_0}$ and $t'' \in \tilde{E}\cap I_{n_0+1}\cap I_{n_0+1}$ for which inequalities $\|X(t')-X(\tilde{t})\| < s/6$, $\|X(t'')-X(\tilde{t})\| < s/6$ are satisfied, which implies

$$\|X(t')-X(t'')\| < s/3. \tag{11}$$

But, from (7) it is easy to obtain

$$5s/6 < \|X(t_{n_0})-X(t_{n_0+1})\| < s/3 + \|X(t')+X(t'')\|,$$

that is

$$\|X(t')-X(t'')\| > s/2,$$

contrary to (11). Thus we showed the equality

$$\tilde{E}\cap(E_{n_0+1}\cap I_{n_0+1}) = \emptyset,$$

which also implies

$$\tilde{E}\cap I_{n_0+1} \subset E^c_{n_0+1}\cap I_{n_0+1}. \tag{12}$$

From the inequality

$$m(E_n\cap I_n) > (1-\delta)m(I_n),$$

which because of the described construction, holds for any $n = 1, 2, \ldots$, and from (12), it is easy to obtain

$$m(\tilde{E}\cap I_{n_0+1}) \leq \delta\cdot m(I_{n_0+1}),$$

which is in contradiction to (8). The proof of the first part is completed.

Now we shall prove that (II) implies the statement of the theorem. For arbitrary $t \in (0;1)$ and $h > 0$, any of the intervals $(t-h;t)$, $(t;t+h)$, $(t-h;t+h)$

will be denoted by $i_{t,h}$. Let a and b be arbitrary points from (0; 1), such that $a < b$. For arbitrary $s > 0$ there is $t \in (a; b)$ such that the inequality

$$\sup_{u,v \in i_{t,h}} \|X(u) - X(v)\| < s \tag{13}$$

holds for some $h > 0$ and a corresponding interval $i_{t,h}$ (really, from the assumption that, for some $s > 0$, such t and h do not exist, it follows that there is no t in $(a; b)$ for which at least one of limits $X(t-0)$, $X(t+0)$ exist, which contradicts the assumption (ii). The inequality (13) implies

$$\omega(u) < s \text{ for any } u \in i_{t,h},$$

where $i_{t,h}$ is the same interval as in (13). From the fact that the points a and b are arbitrarily chosen it follows that the set D_s is nowhere dense in [0; 1] for any $s > 0$, as we wanted to prove.

Corollary 1.1. *If for the process X any of the conditions* (I) *and* (II) *is satisfied, then the set of discontinuities of X is of the first category.*

Corollary 1.2. *Approximately mean square continuous process is continuous at one everywhere dense in* [0; 1] *set of the power of the continuum.*

Theorem 2. *If a process X satisfies any of the conditions* (I) *and* (II), *then the set of discontinuities of X is equal to the set of discontinuities of its oscillation function ω.*

Proof. The set of discontinuities of X is equal to the set $Q = \{t : \omega(t) > 0\}$, [2]. From the fact that the function ω is continuous at any point t at which the equality $\omega(t) = 0$ holds, it follows that the set P of discontinuities of ω is contained in Q. Suppose that there is a $t_0 \in Q$, such that $t_0 \notin P$. From that assumption, for the function ω is continuous at t_0, it follows that there is $h > 0$ such that

$$\omega(t) > 0 \text{ for } t \in (t_0 - h); t_0 + h);$$

but, this inequality implies $(t_0 - h; t_0 + h) \subset Q$, which contradicts the fact that the set Q is of the first category, [6]. That proves the theorem.

3. One of problems, connected with relationship between approximate continuity of X, is the problem of determining conditions which the functions $\Gamma_t(\cdot) = \Gamma(t, \cdot)$, $t \in [0; 1]$, have to satisfy in order the process X to be approximately mean square continuous. The following two theorems deal in that problem.

Denote by $\mathcal{S}$ the set of all functions $\Gamma_t(\cdot) = \Gamma(t, \cdot)$, $t \in [0; 1]$, and by $E_{u,t}$ the set in which the function Γ_u is continuous at the point t, and which has the unit metric density at t.

Theorem 3. *If all functions from $\mathcal{S}$ are approximately continuous and the set $\cap_{u \in [0;1]} E_{u,t}$ has the unit metric density at t for any t, then the function Γ (and the proces X, also) is approximately continuous.*

Proof. Let t be arbitrary point from $[0;\ 1]$. Put $E_t = \cap_{u\in[0;1]} E_{u,t}$. For arbitrary $u \in [0;1]$ we have

$$\|X(t) - X(u)\|^2 \le |\Gamma_t(t) - \Gamma_t(u)| + |\Gamma_u(t) - \Gamma_u(u)|.$$

From $u \in E_t$ it follows $u \in E_{t,t}$, which implies

$$|\Gamma_t(t) - \Gamma_u(u)| \to 0 \ \text{ when } \ u \to t \ \text{ and } \ u \in E_t.$$

Also, from $u \in E_t$ it follows $u \in E_{s,t}$ for any s, and specially for $s = u$, which implies

$$|\Gamma_u(t) - \Gamma_u(u)| \to 0 \ \text{ when } \ u \to t \ \text{ and } \ u \in E_t.$$

Thus we proved that the process X is mean square continuous at t along the set E_t, which, by reason of [7] and (4), means that the function Γ is a approximately continuous.

Lemma 1. *Let t be arbitrary point from R, and E_1, E_2 some sets whose metric densities at t are equal to unity. Then the metric density of the $E = E_1 \cap E_2$ at t is also equal to unity.*

Proof. For any $\varepsilon > 0$ there are $h_1 > 0$, $h_2 > 0$, such that

$$m(E_i \cap [t-h; t+h]) \ge 2h(1 - \varepsilon/2) \ \text{ for any } \ 0 < h \le h_i, \ \ i = 1, 2.$$

Hence

$$m(E \cap [t-h; t+h]) \ge 2h(1 - \varepsilon) \ \text{ for any } \ h \le \min\{h_1, h_2\},$$

which completes the proof.

Theorem 4. *If all functions from $\mathcal{S}$ and the function $g(t) = \Gamma(t,t)$, $t \in [0;1]$, are approximately continuous, then the function Γ (and the process X, also) is approximately continuous.*

Proof. For arbitrary $u \in [0;1]$ it will be

$$\|X(t) - X(u)\|^2 \le 2|\Gamma_t(t) - \Gamma_t(u)| + |g(t) - g(u)|.$$

The first of the terms on the right side of that inequality will converge to zero when $u \to t$ and $u \in E_{t,t}$. Denote by $\tilde{E}_t$ the set in which the function g is continuous at t, and which has the unit metric density at t. Put $\hat{E}_t = \hat{E}_{t,t} \cap \tilde{E}_t$. It is clear that the process X is mean square continuous at t in $\hat{E}_t$, and in Lemma 1 it is proved that the set $\hat{E}_t$ has the unit metric density at t. The theorem is proved.

Theorem 5. *If all functions from $\mathcal{S}$ are approximately continuous, then the process X is approximately mean square continuous almost everywhere.*

Proof. From the approximate continuity of all functions from $\mathcal{S}$ it follows [4] that the function Γ is measurable, and, also [5], that the function g is measurable.

That means [8] that the functions Γ and g are almost everywhere approximately continuous (first of them with respect to the Lebesgue measure on $\mathcal{B}_{[0;1]}\times\mathcal{B}_{[0;1]}$, and the last one with respect to the Lebesgue measure on $\mathcal{B}_{[0;1]}$). Hence there is a set S from [0; 1] of measure zero, such that the function Γ is approximately continuous at any point (t,t), $t\in S^c$ (for, if t is some point at which g is approximately continuous, then the approximate continuity of Γ at (t,t) follows from Theorem 4), i.e., the process X is approximately mean square continuous almost everywhere, as we wanted to show.

4. Let us denote by $\omega^-_{\Gamma_t}=\omega^-_{\Gamma_t}(u)$, $\omega^+_{\Gamma_t}=\omega^+_{\Gamma_t}(u)(\omega^-_g=\omega^-_g(u),\omega^+_g=\omega^+_g(u))$ the left and right oscillation function of the function $\Gamma_t(g)$, and introduce, also, the following notations:

$$\begin{aligned}\Delta^-_g &= \{u\colon \omega^-_g(u)>0\},\\ \Delta^+_g &= \{u\colon \omega^+_g(u)>0\},\\ \Delta^-_\Gamma &= \{u\colon \omega^-_{\Gamma u}(u)>0\},\\ \Delta^+_\Gamma &= \{u\colon \omega^+_{\Gamma u}(u)>0\}.\end{aligned}$$

Theorem 6. *Suppose that all functions from $\mathcal{S}$ are approximately continuous and that the set S from Theorem 5 is closed. If the equalities*

$$\Delta^-_\Gamma\cap\Delta^+_\Gamma=\emptyset, \tag{14}$$
$$\Delta^-_g\cap\Delta^+_g=\emptyset, \tag{15}$$
$$\Delta^-_g\cap\Delta^+_\Gamma=\emptyset, \tag{16}$$
$$\Delta^+_g\cap\Delta^-_\Gamma=\emptyset \tag{17}$$

are satisfied, then the set of discontinuites of the process X has zero Lebesgue measure.

Proof. Denote $D^-=\{t\colon\omega^-(t)>0\}$, $D^+=\{t\colon\omega_+(t)>0\}$ and show that

$$(D^-\cap D^+)\cap S^c=\emptyset. \tag{18}$$

Let t be arbitrary point from (0; 1), such that $t\in S^c$. Suppose that $t\in D^-$ and prove that then $t\notin D^+$.

From the assumption that, also, $t\in D^+$, it follows that there is some $\varepsilon>0$ such that in any neighbourhood $(t;t+h)$ of t there is at leat one u such that the inequality

$$\|X(t)-X(u)\|>\varepsilon$$

is satisfied. From that it can be obtained, as in Theorem 4,

$$\varepsilon<\|X(t)-X(u)\|\le\sqrt{2}|\Gamma_t(t)-\Gamma_t(u)|^{1/2}+|g(t)-g(u)|^{1/2},$$

which implies that at least one of the inequalities

$$\omega^+_{\Gamma_t}(t)>\varepsilon/2, \tag{19}$$
$$\omega^+_g(t)>\varepsilon/2 \tag{20}$$

must be satisfied. If (19) is satisfied, that, by reason of (14) and (16), means that $t \notin \Delta_\Gamma^-$ and $t \notin \Delta_g^-$, which is equivalent to $\omega_-(t) = 0$, contrary to our assumption $t \in D^-$. If (20) is satisfied, then, by reason of (15) and (17), it will be $t \notin \Delta_g^-$ and $t \notin \Delta_\Gamma^-$, which, as before, contradicts our assumption $t \in D^-$. Thus, we proved that arbitrary point $t \in S^c$ does not belong to both D^- and D^+, which means that (18) is true.

As the set S is closed, the set S^c can be written as the union of at most countably many disjoint open intervals $I_2, I_2, \dots$:

$$S^c = \bigcup_{i=1}^{\infty} I_i. \tag{21}$$

Put ${}^iD^- = D^- \cap I_i$ and ${}^iD^+ = D^+ \cap I_i$, $i = 1, 2, \dots$. The equality

$${}^iD^- \cap {}^iD^+ = \emptyset, \quad i = 1, 2, \dots, \tag{22}$$

which follows from (18), means that at any point $t \in I_i$ there exists at least one of limits $X(t-0)$, $X(t+0)$. We are going to show that, for any $s > 0$, the set $D_s^i = I_i \cap D_s$, $i = 1, 2, \dots$, contains at most countably many elements; it will imply, by reason of (21), that the set $S^c \cap D_s$ has at most countably many elements, i.e., that the process X has at most countably many discontinuities, which proves the theorem.

Suppose that the set D_s^i, for some i and some $s > 0$, has countinuously many elements. Denote by $\hat{D}_s^i$ the perfect subset of D_s^i; in $\hat{D}_s^i$ there exists at least one point t, such that some sequences (t'_n), (t''_n), $t'_n < t$, $t''_n > t$, $n = 1, 2, \dots$, of points from $\hat{D}_s^i$ converge to t, [7]. These sequences, clearly, are such that the inequalities $\omega(t'_n) \geq s$ and $\omega(t''_n) \geq s$ are satisfied for all $n = 1, 2, \dots$, which is equivalent to $t \in {}^iD^- \cap {}^iD^+$. From that and from (22) it follows that the set $\hat{D}_s^i$ is empty, i.e., that the set D_s^i has at most countably many elements. The proof is completed.

Corollary 6.1. *If the process X is such that $D^- \cap D^+ = \emptyset$, then it has at most countably many discontinuities.*

Corollary 6.2. *The process X, for which the equality $D^- \cap D^+ = \emptyset$ is satisfied, has separable linear space $H(X)$ (see [1]).*

Lemma 2. *Let η be arbitrary subspace of $H(X)$, and let a new process X_1 be defined as the projection of X on η:*

$$X_1(t) = P_\eta X(t) \quad t \in [0; 1]. \tag{23}$$

If all functions from $\mathcal{S}$ are approximately continuous, then the correlation function Γ_1 of X_1 is measurable.

Proof. From the approximate continuity of functions from $\mathcal{S}$ follows, by reason of Theorem 5, that X is almost everywhere approximately continous. If X

is approximately continuous at t, and if E_t is a set in which X is continuous at t, and whose metric density at t is equal to unity, then it will be

$$\|X_1(t) - X_1(u)\| = \|P_\eta(X(t) - X(u))\| \leq \|X(t) - X(u)\| \to 0, \;\; u \to t, \;\; u \in E_i,$$

which is equivalent to the approximate continuity of X_1 at t. Thus the process X_1, defined by (23), is almost everywhere approximately continuous, which implies the approximate continuity of Γ_1 at almost all points of the set $[0;1] \times [0;1]$, which, by reason of [8], means that Γ_1 is measurable function. The proof is completed.

THEOREM 7. *If all conditions from Theorem* 6 *are satisfied, then for every* $\varepsilon > 0$*, there exists a measurable stochastic process* $X_\varepsilon = X_\varepsilon(t)$*,* $t \in [0;1]$*, such that*

$$m\{t\colon X(t) \neq X_\varepsilon(t)\} < \varepsilon. \tag{24}$$

PROOF. In Theorem 6 it is proved that the set T of discontinuities of X has zero Lebesgue measure. For every $\varepsilon > 0$ there exist an open set T_ε such that $T \subset T_\varepsilon$ and $m(T_\varepsilon) < \varepsilon$. We can suppose, without loss of generality, that the set T_ε^c is perfect. The new space H_ε we shall define by

$$H_\varepsilon = \bar{\mathcal{L}}\{X(t), \quad t \in T_\varepsilon^c\},$$

where $\bar{\mathcal{L}}\{\cdot\}$ denotes the closure of the linear manifold spanned by elements in the parentheses. From the continuity of X on T_ε^c it follows the separability of H_ε. It is easy that the process X_ε defined by

$$X_\varepsilon(t) = P_{H_\varepsilon} X(t), \quad t \in [0;1],$$

will satisfy (24), and that, obviously, will be $H(X_\varepsilon) = H_\varepsilon$. That, means of [3] and Lemma 2, implies the measurability of the process X.

COROLLARY 7.1. *If the correlation function* Γ *of* X *is measurable, and the equality* $D^- \cap D^+ = \emptyset$ *is satisfied, then the process* X *is measurable.*

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