PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 29 (43), 1981, pp. 39–44

CORRELATION FUNCTION AND SEPARABILITY OF LINEAR SPACE OF STOCHASTIC PROCESS

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Abstract. In this note necessary and sufficient conditions, in terms of correlation function Γ of X, for the separability of the linear space $H(X)$ of X are found.

1. Let $X = (X(t), t \in T)$ be a real-valued stochastic process of second order, such that for some $K > 0$

$$\|X(t)\| \leq K \text{ for all } t \in T, \tag{1}$$

where T is some interval from the real line R. Put

$$\Gamma(t,u) = (X(t), X(u)) = E(X(t)X(u)), \qquad t, u \in T,$$

and denote by $\mathcal{S}$ the set of all functions Γ_t, $\Gamma_t(\cdot) = \Gamma(t, \cdot)$, $t \in T$. If the distance r between functions Γ_t and Γ_s is defined by

$$r(\Gamma_t, \Gamma_s) = \sup_{u \in T} |\Gamma_t(u) - \Gamma_s(u)|, \tag{2}$$

then $(\mathcal{S}, r)$ becomes the metric space (the function r is well defined because all functions from $\mathcal{S}$ are bounded).

Denote by $H(X)$ the linear space of the process X, and by $\mathcal{R}(\Gamma)$ the reproducing kernet Hilbert space of the function Γ. It is known that $\mathcal{R}(\Gamma)$ consists of all functions f on T of the form $f(u) = E(xX(u))$, $u \in T$, for some $x \in H(X)$, and that the map $x \to f$ defines a scalar product preserving isomorphism between $H(X)$ and $\mathcal{R}(\Gamma)$, [1]; hence $\mathcal{S} \supset \mathcal{R}(\Gamma)$, where $\subset$ is the set theoretical inclusion. If the distance d in $H(X)$ is defined by $d(x,y) = \|x - y\|$, then from the previous it follows that the distance δ in $\mathcal{R}(\Gamma)$ is intoduced by $\delta(f,g) = d(x,y)$, where $f(u) = E(xX(u))$, $g(u) = E(yX(u))$, $u \in T$.

In this note shall find necessary and sufficient conditions which the correlation function Γ of X has to satisfy in order the linear space $H(X)$ to be separable.

2. Theorem 1. *The space $H(X)$, in which the distance is defined by d, is separable if and only if the metric space $(\mathcal{S}, r)$ is separable.*

Proof. By reason of the obvious inequality

$$r(\Gamma_t, \Gamma_s) \le K \cdot d(X(t), X(s)), \tag{3}$$

which holds for all $t, s \in T$, it follows that the separability of $H(X)$ implies the separability of $(\mathcal{S}, r)$.

Suppose, now, that the space $(\mathcal{S}, r)$ is separable, but that $H(X)$ is non-separable. Denote by $\mathcal{S}_0$ one at most countable from $\mathcal{S}$, which is everywhere dense in $\mathcal{S}$. From the assumption that $H(X)$ is non-separable it follows that for some $\delta > 0$ there is a set S from $[0;1]$ card $S = \chi_1$, such that

$$d(X(t), X(s)) > \delta, \quad t \neq s \quad t, s \in S.$$

Let ε be arbitrary positive constant less than $\delta^2/4$. If for different functions Γ_t, Γ_s, from $\mathcal{S}$ there exists one function Γ_u from $\mathcal{S}_0$ such that $r(\Gamma_t, \Gamma_u) < \varepsilon$ and $r(\Gamma_s, \Gamma_u) < \varepsilon$, then it will be $r(\Gamma_t, \Gamma_s) < 2\varepsilon$, which, specially, implies inequalities $|\Gamma(t,t) - \Gamma(s,t)| < 2\varepsilon$ and $|\Gamma(t,s) - \Gamma(s,s)| < 2\varepsilon$. Hence $d^2(X(t), X(s)) < \delta^2$, i.e., $d(X(t), X(s)) < \delta$. This means that to any t, such that $\Gamma_t \in \mathcal{S}_0$, there corresponds at most one u, $u \in S$, such that $r(\Gamma_t, \Gamma_u) < \varepsilon$. It follows that there exists a subset S_0 of S, such that card $S_0 = \chi_1$ and

$$r(\Gamma_t, \Gamma_u) \ge \varepsilon \text{ for any } \Gamma_t \in \mathcal{S}_0 \text{ and all } u \in S_0.$$

But, that contradicts the assumption that the set $\mathcal{S}_0$ is dense in $\mathcal{S}$. The proof is completed.

From (1) it follows that any function f from $\mathcal{R}(\Gamma)$ is bounded, which means that the distance r can be, in natural way, extended from $\mathcal{S}$ to the whole space $\mathcal{R}(\Gamma)$. Thus, we can consider two metrics, r and δ, on the linear space $\mathcal{R}(\Gamma)$, and from Theorem 1 we obtain the following result.

Corollary 1.1. *The distances r and δ are equivalent.*

Remark. This Corollary can be proved without refering to Theorem 1. Namely, from (3) it follows that the distance δ is stronger than the distance r, and from the obvious inequality $\delta(f,g) \le \sqrt{2} \cdot \sqrt{r(f,g)}$, which holds for all $f, g \in \mathcal{R}(\Gamma)$, it follows that the distance $\sqrt{r}$ is stronger than the distance δ; but since r and $\sqrt{r}$ are equivalent distances, [2], Corollary 1.1 is proved.

Theorem 1 permits us to determine the dimension of the linear space of some process in the case that only its correlation function is known, and not that process itself.

3. Denote by $\mathcal{B}$ the family of all Borel sets from R, and by $\mathcal{B}(T)$ the corresponding family of sets induced by $\mathcal{B}$ on T. It is easy to see that the measurability of Γ with respect to $\mathcal{B}(T) \times \mathcal{B}(T)$ is neither necessary nor sufficient condition for the separability of $H(X)$, as it is shown in next examples.

EXAMPLE 1. Put $T = [0;1]$ and let A be some set from T. The indicator function of a set B will be denoted by I_B. It is easy to see that the function $\Gamma{:}\, T \times T \to R$, defined by

$$\Gamma(t,u) = I_{A\times A}(t,u) \cdot \min(t,u) + I_{\bar{A}\times\bar{A}}(t,u),$$

is non-negative definite, which means that it is the correlation function of some process X, [4]. The space $(\mathcal{S}, r)$ is separable, disregarding the set A is measurable or not. But, the function Γ will be $\mathcal{B}$-measurable if and only if the set A is from $\mathcal{B}(T)$.

EXAMPLE 2. It is easy to transform the previous example so that the new function Γ corresponds to some process X with non-separable space $H(X)$. Really, if we define the functions Γ by

$$\Gamma(t,u) = I_B(t,u), \quad (t,u) \in T \times T,$$

where $B = \{(s,v){:}\, s = v \text{ or } (s,v) \in A \times A\}$ and A is the set from Example 1, then the space $(\mathcal{S}, r)$ is not-separable if and only if card $A = \chi_1$ or card $\bar{A} = \chi_1$ disregarding this set A is measurable or not.

4. Let us define the map $\gamma{:}\, T \to \mathcal{S}$ by

$$\gamma(t) = \Gamma_t, \quad t \in T.$$

Denote by $\mathcal{B}(\mathcal{S})$ the σ-algebra of all Borel sets in $\mathcal{S}$, i.e., in the metric space $(\mathcal{S}, r)$. The connection between $\mathcal{B}(T)$-measurability of the map γ and separability of $H(X)$ will be investigated in the next theorem.

THEOREM 2. *The following statements are equivalent:*

(I) *All functions from $\mathcal{S}$ are Borel measurable and the space $H(X)$ is separable.*

(II) *The map γ is Borel measurable.*

PROOF. In order to prove that (I) implies (II) it is enough to show that, for arbitrary $t \in T$ and any $\varepsilon > 0$, the set

$$U_{t,\varepsilon} = \{u{:}\, \Gamma_u \in B(\Gamma_t; \varepsilon)\}$$

belongs to the σ-algebra $\mathcal{B}(T)$, where $B(\Gamma_t; \varepsilon)$ denotes the closed ball in $\mathcal{S}$ whose centre is in Γ_t and radius is equal to ε:

$$B(\Gamma_t; \varepsilon) = \{\Gamma_u \in \mathcal{S}{:}\, r(\Gamma_t, \Gamma_u) \le \varepsilon\}\,.$$

Let us put

$$U^0_{t,\varepsilon;s} = \{u \colon |\Gamma_t(s) - \Gamma_u(s)| \le \varepsilon\}\,,$$
$$U^0_{t,\varepsilon} = \bigcap_{s\in S} U^0_{t,\varepsilon;s},$$

where S denotes one at most countable set from T, such that the set $\{\Gamma_s, s \in S\}$ is dense in $\mathcal{S}$ (the countability of such a set S is implied by the assumption on separability of $H(X)$ and by Theorem 1).

From the assumption on measurability of all functions from $\mathcal{S}$ it follows that the set $U^0_{t,\varepsilon;s}$ is measurable for all t, $s \in T$ and any $\varepsilon > 0$. That implies measurability of the set $U^0_{t,\varepsilon.}$ In order to show that the set $U_{t,\varepsilon}$ is measurable, it is enough to show that the equality $U_{t,\varepsilon} = U^0_{t,\varepsilon}$ holds for all $t \in T$ and $\varepsilon > 0$. That, practically, means that we have only to show that the inclusion

$$U^0_{t,\varepsilon} \subset U_{t,\varepsilon} \tag{4}$$

holds (the oposite inclusion is obvious).

For any $s \in T$ and any $\delta > 0$ let us denote by $\bar{s} = \bar{s}(s;\delta)$ some element (not necessarily unique) from S, such that the inequality

$$\sup_{t\in T} |\Gamma_t(s) - \Gamma_t(\bar{s})| < \delta$$

is satisfied (the existence of such an element follows from the assumption that the set $\{\Gamma_s, s \in S\}$ is dense in $\mathcal{S}$ and from the equality $\Gamma_t(s) = \Gamma_s(t)$ which holds for all t, $s \in T$). Denote by u an arbitrary element from $U^0_{t,\varepsilon}$; let us show that the inequality $r(\Gamma_t, \Gamma_u) \le \varepsilon$ holds, that is that $u \in U_{t,\varepsilon}$.

For arbitrary $s \in T$ we have

$$|\Gamma_t(s) - \Gamma_u(s)| \le$$
$$\le |\Gamma_t(s) - \Gamma_t(\bar{s})| + |\Gamma_t(\bar{s}) - \Gamma_u(\bar{s})| + |\Gamma_u(\bar{s}) - \Gamma_u(s)| < \varepsilon + 2\delta.$$

Thus, for any $\delta > 0$ the function Γ_u belongs to the open ball $B(\Gamma_t; (\varepsilon + 2\delta)-) \equiv \{\Gamma_v \colon r(\Gamma_t, \Gamma_v) < \varepsilon = 2\delta\}$, i.e., $u \in U_{t,(\varepsilon+2\delta)-} \equiv \{v \colon \Gamma_v \in B(\Gamma_t; (\varepsilon + 2\delta)-)\}$. This implies that

$$u \in \bigcap_{\delta>0} U_{t,(\varepsilon+2\delta)-},$$

which is equivalent to $u \in U_{t,\varepsilon.}$ That proves (4), and the proof of the first part is completed.

Now we shall prove that (I) is the consequence of (II). From (II), [3] and Theorem 1 it follows that the space $H(X)$ is separable, which means that it is enough to show that for arbitrary $t \in T$ and arbitrary open set $C \subset T$, the set

$$A_t = \{u \colon \Gamma_t(u) \in C\}$$

belongs to $\mathcal{B}(T)$. Put

$$\tilde{A}_t = \{\Gamma_u : u \in A_t\}$$

and prove that the set $\tilde{A}_t$ belongs to $\mathcal{B}(\mathcal{S})$; from that and from the obvious equality $\gamma^{-1}(\tilde{A}_t) = A_t$, it will follow, by reason of (II), that the set A_t belongs to $\mathcal{B}(T)$.

Suppose that $\tilde{A}_t$ has the power of the continuum (for, if $\tilde{A}_t$ is at most countable set, then it is Borel mesurable, and the statement is proved). From the separability of $H(X)$ (and from Theorem 1) it follows that $\mathcal{S}$ contains at most countably many isolated elements, which implies that we have to show that arbitrary non-isolated element $\Gamma_u \in \mathcal{S}$ from $\tilde{A}_t$ must be an interior element for $\tilde{A}_t$.

From the assumption that the set C is open, and from $\Gamma_t(u) \in C$, it follows that there is $\delta > 0$ such that

$$(\Gamma_t(u) - \delta; \Gamma_t(u) + \delta) \subset C. \tag{5}$$

Let us show that for arbitrary $0 < \varepsilon < \delta$, the ball

$$B(\Gamma_\varepsilon; u) \equiv \{\Gamma_v \in \mathcal{S} : r(\Gamma_u, \Gamma_v) < \varepsilon\}$$

belongs to $\tilde{A}_t$; that will mean that Γ_u is an interior element for $\tilde{A}_t$. From $\Gamma_v \in B(\Gamma_u; \varepsilon)$, (5) and

$$|\Gamma_t(u) - \Gamma_t(v)| \le r(\Gamma_u, \Gamma_v) < \varepsilon < \delta,$$

it follows

$$\Gamma_t(v) \in C,$$

which is equivalent to $\Gamma_v \in \tilde{A}_t$, as we wanted to prove. Thus we proved that any $\Gamma_u \in \tilde{A}_t$ is interior element for $\tilde{A}_t$ or isolated element of $\mathcal{S}$, which means that $\tilde{A}_t \in \mathcal{B}(\mathcal{S})$. The proof is completed.

It is known [5] that the function $\tilde{d}$, defined by

$$\tilde{d}(t,s) = d(X(t), X(s)), \qquad (t,s) \in T \times T$$

is a pseudo-metric on T. Denote by $\tilde{\gamma}$ the map from T into $H(X)$, defined by $\tilde{\gamma}(t) = X(t)$, $t \in T$.

The following theorem represents the consequence of already proved theorems and represents one characterization of the separability of $H(X)$ by means of properties of the correlation function of X.

THEOREM 3. *(a) If T is arbitrary set from R, then the following statements are equivalent:*

(i) $(T, \tilde{d})$ *is separable;*

(ii) $(H(X), d)$ *(or, equivalently,* $(\mathcal{R}(\Gamma), \delta)$*) is separable;*

(iii) $(\mathcal{S}, r)$ *is separable.*

(b) If T is an interval from R and all functions Γ_t, $t \in T$, are Borel measurable, then any of the statements (i)–(iii) is equivalent to each of the following statements:

(iv) The map γ is Borel measurable:

(v) The map $\tilde{\gamma}$ is Borel measurable.

PROOF. The only non-trivial part is the equivalence of (iv) and (v). Define the new map $\hat{\gamma}$, from $\mathcal{S}$ into $H(X)$, by

$$\hat{\gamma}(\Gamma_t) = X(t) \qquad \Gamma_t \in \mathcal{S},$$

and show that this map is $\mathcal{B}(\mathcal{S})$-measurable. It is enough to show that, if $B(X(t);\varepsilon)$ is the set defined by

$$B(X(t);\varepsilon) \equiv \{X(s)\colon d(X(t),X(s)) \leq \varepsilon, s \in T\},$$

then the set $\hat{\gamma}^{-1}(B(X(t);\varepsilon))$ is from $\mathcal{B}(\mathcal{S})$ for any $\varepsilon > 0$. That immediately follows from the facts that between $\mathcal{S}$ and $\{X(t), t \in T\}$ there exists a scalar product preserving isomorphism, and that the distances r and δ are equivalent (also on $\mathcal{S}$,of course), which means that they generate the same Borel δ-algebra on $\mathcal{S}$. The proof is completed.

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