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ANTI-INVERSE RINGS

Blagoje Cerović

In [1], anti-inverse semigroups are studied. In this paper we consider rings for which the multiplicative semigroup is anti-inverse. For the class of anti-inverse rings, we determine a basis class in the sense of E. S. Ljapin, (Proposition 2.4.).

1. Let $(R, +, \cdot)$ be a ring in which the following condition holds:

$$(\forall x \in R)(x^n = x) \tag{1}$$

It is well known that: If for every element x of a ring R there exists $n(x) > 1$, so that $x^{n(x)} = x$, then R is a commutative ring. (N. Jacobson, [2]).

LEMMA 1.1. *If R is a ring with* (1) *and R has identity, then;*

a) Every non-invertible element of R is a zero divisor.

b) Every prime ideal of R is maximal.

c) R is semiprimitive, i.e. the Jacobson radical of R is (0).

(See [3], Propositions 3. and 4., pp. 57.).

LEMMA 1.2. *Let R be a ring with* (1). *Then*

$$(\forall x \in R)((2^n - 2)x = 0) \tag{2}$$

For any x, $y \in R$ we have

$$\begin{aligned} x + y = \sum_{k=2}^{n-1} \binom{n}{k} x^{n-k} y^k = x^n + y^n + \sum_{k=2}^{n-1} \binom{n}{k} x^{n-k} y^k \\ = (x + y)^n = x + y. \end{aligned}$$

From this

$$\sum_{k=1}^{n-1} \binom{n}{k} x^{n-k} y^k = 0.$$

For $x = y$ we obtain

$$\sum_{k=1}^{n-1} \binom{n}{k} x^n = 0$$

Therefore,

$$(2^n - 2)x = 0.$$

Lemma 1.3 *Let R be a ring with* (1) *and I an ideal of R. Then the following conditions are equivalent:*

(a) I is a maximal ideal of R

(b) I is a prime ideal of R

(c) $(\forall x \in R)(x^{n-1} \in I \wedge 1 - x^{n-1} \notin I) \wedge (x^{n-1} \notin I \wedge x^{n-1} \in I)$

Proof. $(a) \Leftrightarrow (b)$. Tis follows by Lemma 1.1.

$(b) \Rightarrow (c)$. For any $x \in R$ we have $x^{n-1} + (1 - x^{n-1}) = 1 \notin I$ i.e. the elements x^{n-1} and $1-x^{n-1}$ are not both in ideal I. Futher on we have $x^{n-1}(1-x^{n-1}) = 0 \in I$ and since I is a prime ideal we have that (c) holds.

$(c) \Rightarrow (a)$. For any $x \in R\backslash I$ from $xx^{n-1} = x$ we have $x^{n-1} \notin I$. Then $1 - x^{n-1} \in I$. Further on, $1 = (1 - x^{n-1}) + x^{n-1} \in I + x^{n-1}R \subset I + xR$. Let P be an ideal of R and $I \subset P$, $I \neq P$. For $x \in P\backslash I$ holds $I + xR \subset P$. But, since $1 \in I + xR$, we have $P = R$, i.e. I is maximal ideal of R.

Lemma 1.4. *Let R be a ring with* (1). *If R is an integral domain, then R is a field.*

Proof. Let $x \neq 0$ be any element of integral domain R. Then from $x^n = x$ we have $x(x^{n-1} - 1) = 0$, and from this $x^{n-1} = 1$, i.e. every element $x \neq 0$ from integral domain R is invertible and its inverse element is x^{n-2}.

2. In this section we consider one class of rings for which (1) holds. In [1] the class $\mathcal{A}$ of anti-inverse semigroups are studied, i.e. the class of all semigroups for which

$$(\forall x)(\exists y)(xyx = y \wedge yxy = x)$$

holds.

One characterization of the class $\mathcal{A}$ ([1], Theorem 2.1) is: Let S be a semigroup. Then

$$S \in \mathcal{A} \Leftrightarrow (\forall x \in S)(\exists y \in S)(x^2 = y^2 \wedge yx = x^3y \wedge x^5 = x).$$

Definition 2.1 The ring $(R, +, \cdot)$ is called anti-inverse if for every $x \in R$ there exists its anti-invesre element $y \in R$, i.e.

$$(\forall x \in R)(\exists y \in R)(xyx = y \wedge yxy = x).$$

EXAMPLE 2.1. The ring $(R, +, \cdot)$ given by following tables

$+$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

is an anti-inverse ring. The element a is its own anti-inverse. For b the anti-inverse element is c.

EXAMPLE 2.2 Every Bolean ring $(B, +, \cdot)$ is an anti-inverse ring, Indeed, the element $x \in B$ is its own anti-inverse, since $xxx = x$.

EXAMPLE 2.3. The ring Z_6 is an anti-inverse ring.

By $\mathcal{AR}$ we denote the class of anti-inverse rings.

Immediately, from the Theorem of Jacobson and the Theorem 2.1. [1] follows the

PROPOSITION 2.1 *Every anti-inverse ring is a commutative ring.*

PROPOSITION 2.2 *Let R be a ring. Then*

$$R \in \mathcal{AR} \Leftrightarrow (\forall x \in R)(x^3 = x).$$

PROOF. If for every $x \in R$, $x^3 = x$ holds, then every element of R is its own anti-inverse i.e. $R \in \mathcal{AR}$.

Conversely, let x be an arbitrary element of the ring R and $y \in R$ its anti-inverse element. By Theorem 2.1. [1] we have $x^2 = y^2$, and from commutativity and $yxy = x$ holds $y^2x = x$. From this we have $x^3 = x$.

PROPOSITION 2.3. *Let $R \in \mathcal{AR}$. Then the following conditions are equivalent:*

(a) R is a field.

(b) R has two elements 0 and 1 or three elements 0, 1 and -1.

PROOF. $(a) \Rightarrow (b)$. If R is a field, then (0) is a unique maximal ideal of R. By Lema 1.3. for arbitrary element $x \in R$ holds

$$x^2 = 0 \text{ and } 1 - x^2 \neq 0 \tag{3}$$

or

$$1 - x^2 = 0 \text{ and } x^2 \neq 0. \tag{4}$$

Let $x \neq 0$ be an element of R. Only 0 satisfies the condition (3), so for this x holds (4). From $1 - x^2 = 0$ we have $(1 - x)(1 + x) = 0$. From this $x = 1$ or $x = -1$. If

$1 \neq -1$, then the field R has three elements 0, 1 and -1 and the operations $+$ and $\cdot$ are defined by the following way

$+$	0	1	-1
0	0	1	-1
1	1	-1	0
-1	-1	0	1

$\cdot$	0	1	-1
0	0	0	0
1	0	1	-1
-1	0	-1	$1.$

If $1 = -1$, then the field R has two elements 0 and 1 and the operations $+$ ane $\cdot$ are defined by the following way

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

$b \Rightarrow (a)$. It is clear that every anti-inverse ring with two or three elements is a field.

COROLLARY. *If an anti-inverse ring is a field, then it is isomorphic to Z_2 or Z_3.*

REFERENCES

[1] S. Bogdanović, S. Milić, V. Pavlović, *Anti-inverse semigroups,* Publ. Inst. Math. 24 (38) (1978), 19–28.

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[3] И. Ламбек, *Кольца и модули*, "Мир", Москва, 1971.