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FOUR VARIATIONS ON A THEME OF S. B. PREŠIĆ CONCERNING SEMIGROUP FUNCTIONAL EQUATIONS

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1. Introduction. Let S_1 and S_2 be nonempty sets and suppose that:

(i) $g_1, \dots, g_n$ map S_1 into S_1;

(ii) G is the minimal semigroup generated by $g_1, \dots, g_n$;

(iii) J maps $S_1 \times S_2^n \{\top, \bot\}$;

(iv) $f \in S_2^{S_1}$.

In a number of papers ([1]–[7) S. B. Prešić considered various instances of the equation in f:

$$J(x, f(g_1 x), f(g_2 x), \dots, f(g_n x)) = \top \qquad (g_i x = g_i(x)) \tag{1.1}$$

where g_1 is the identity mapping, and under certain conditions determined its general solution. We briefly sketch Prešić results.

In [1] he constructed the general solution of the equation

$$f(x) = f(gx),$$

under the condition that g is a bijection, i.e. that G is a group. In papers [3], [4], [6] the general solution of the equation

$$a_1(x) f(x) + a_2(x) f(g_2 x) + \dots + a_n(x) f(g_n x) = F(x) \tag{1.2}$$

is determined under the condition that G is a group of order n. (Of course, in the case of equation (1.2) it is necessary to assume that S_2 is a field, and that $a_k : S_1 \to S_2$, $F : S_1 \to S_2$ are given).

In [5] Prešić determined the general solution of the equation

$$J(f(x_1, x_2, \dots, x_n), f(x_2, x_3, \dots, x_1), \dots, f(x_n, x_1, \dots, x_{n-1})) = \top$$

where $S_1 = A^n$, $x_i \in A$. Finally, in [7] Prešić constructed the general solution of the equation (1.1) under the condition that G is a group.

Hence, if G is a group, the problem of solving (1.1) is completely solved. However, Prešić also tried to solve (1.1) without additional conditions on the semigroup G. One such attempt is presented in [2]. Naturally, if the condtion "G is a group" is suppressed, one has to introduce some other conditions – in fact, a request on J, i.e. on the form of the equation (1.1). One such condition is given in [2], theorem 1. Also, paper [4], theorem 2, where the equation (1.2) with $F(x) = 0$ is considered, a condition for the coefficients a_1 and on the semigroup G is imposed which makes it possible to construct the general solution of that equation.

In this note we shall consider some special cases of the equation (1.1); in other words, we shall develop some variations on this theme of S. B. Prešić.

2. First variation. Consider the equation

$$J(f(g_1x), f(g_2x), \dots , f(g_nx)) = \top \tag{2.1}$$

and suppose that $G = \{g_1, g_2, \dots , g_n\}$ is a semigroup. Denote g_ig_j by g_{ij}, and let M be the matrix of the table of G, i.e. $M = \|g_{ij}\|_{n\times n}$. Denote the i-th column of M by $c_i(M)$.

If the following conditions are satisfied:

(i) there exists i $(1 \le i \le n)$ such that $c_i(M) = \|g_1g_2\cdots g_n\|^T$; in other words, the semigroup G has a right unit element;

(ii) for every i $(1 \le i \le n)$ we have

$$\{c_1(M), c_2(M), \dots , c_n(M)\} = \{c_1(Mg_i), c_2(Mg_i), \dots , c_n(Mg_i)\}$$

then every possible equation (2.1) has the general solution of the form

$$f(x) = F(\Pi(x), \Pi(g_1x), \Pi(g_2x), \dots , \Pi(g_nx))$$

where $\Pi\colon S_1 \to S_2$ is abitrary and $F\colon S_n^{n+1} \to S_2$ is constructed by the method described by Prešić [7].

Remark 2.1. If G is a group, conditions (i) and (ii) are satisfied.

Remark 2.2. For any semigroup G the inclusion

$$\{c_1(Mg_i), c_2(Mg_i), \dots , c_n(Mg_i)\} \subset \{c_1(M), c_2(M), \dots , c_n(M)\}$$

is valid for all i. Condition (ii) requests that $\subset$ is replaced by $=$.

Remark 2.3. If G is a monoid (i.e. a semigroup with identity) satisfying (ii), then G is a group.

Remark 2.4. There exists a semigroup satisfying (i) and (ii) which is not a group. An example is provided by $G = \{g_1, g_2, \dots , g_n\}$ defined by $g_kg_i = g_k$ for all i, $k = 1, 2, \dots, n$. It is a question whether this is the only semigroup satisfying (i) and (ii) which is not a group.

EXAMPLE 2.1. Let G be the semigroup defined in Remark 2.4. and consider the equation

$$a_1 f(g_1 x) + a_2 f(g_2 x) + \cdots + a_n f(g_n x) = 0. \tag{2.2}$$

(Here we suppose that S_2 is a field and that $a_k \in S_2$). The general solution of the equation (2.2) is given by

$$f(x) = F(\Pi(x),\ (\Pi(g_1 x),\ \Pi(g_2 x), \dots, \Pi(g_n x)),$$

where $\Pi\colon S_1 \to S_2$ is arbitrary, and F is defined by

$$\begin{aligned} F(u_1, u_2, \dots, u_{n+1}) &= u_1 \ \text{ if } \ a_1u_2 + a_2u_3 + \cdots + a_nu_{n+1} = 0 \\ &= 0 \ \text{ if } \ a_1u_2 + a_2u_3 + \cdots + a_nu_{n+1} \neq 0 \end{aligned}$$

In particular, if $a_1 + a_2 + \cdots + a_n \neq 0$, then the general solution of (2.2) can be written as

$$(f(x) = \Pi(x) - \frac{1}{a_1 + \cdots + a_n}(a_1\Pi(g_1x) + a_2\Pi(g_2x) + \cdots + a_n\Pi(g_nx)).$$

3. Second variation. If instead of the equation (2.1), we consider the more general equation (1.1), it can be shown that the condition "G is a group" cannot be replaced by the weaker condition "G is a semigroup satisfying (i) and (ii)" if we want to preserve the form of the general solution as given by Prešić [7].

4. Third variation. Since the condition "G is a group" cannot be satisfactorily weakened (the weaker condition given in Section 2 is not very useful) it is natural to look for some suitable condition which can be placed on the function J which will ensure solvability of the considered equuation. Before we give some examples, we introduce some notations.

If $J\colon S_2^n \to \{\top, \bot\}, g_i\colon S_1 \to S_2$ and if the considered equation is

$$J(f(g_1x), f(g_2x), \dots, f(g_nx)) = \top \qquad (g_1 \text{ is right unit})$$

then replacing x by $g_1x, g_2x, \dots, g_nx$, we obtain the system

$$J(f(g_{1k}x), f(g_{2k}x), \dots, f(g_{nk}x)) = \top \qquad (k = 1, 2, \dots n) \tag{4.1}$$

where $g_{ij}x = g_i(g_j(x))$. Instead of the system (4.1) it is more convenient to operate with the "algebraic" system

$$J(u_{1k}, u_{2k}, \dots, u_{nk}) = \top \qquad (k = 1, 2, \dots, n) \tag{4.2}$$

where u_{ij} is corresponded to $g_{ij}(x)$.

We give two illustrations of the possibilities which arise by specifying J and/or G.

ILLUSTRATION 1. Suppose that S_2 is a field, $a_i \in S_2$, and consider the equation

$$a_1 f(g_1 x) + a_2 f(g_2 x) + \cdots + a_n f(g_n x) = 0 \tag{4.3}$$

Suppose, further, that the corresponding system (4.2), which in this case reads

$$a_1 u_{1k} + a_2 u_{2k} + \cdots + a_n u_{nk} = 0 \qquad (k = 1, 2, \ldots, n)$$

reduces to one equation only. In other words, we suppose that there exist $\alpha_k (k = 2, 3, \ldots, n)$ such that

$$a_1 u_{1k} + a_2 u_{2k} + \cdots + a_n u_{nk} = \alpha_k (a_1 u_{11} + a_2 u_{21} + \cdots + a_n u_{n1}) \qquad (k = 2, 3, \ldots, n)$$

where some (or all) α_k may be 0.

If $a_1 + a_2\alpha_2 + \cdots + a_n\alpha_n \neq 0$, the general solution of (4.3) is given by

$$f(x) = \Pi(x) - \frac{1}{a_1 + a_2\alpha_2 + \cdots + a_n\alpha_n}(a_1\Pi(g_1 x) + a_2\Pi(g_2 x) + \cdots + a_n\Pi(g_n x))$$

where $\Pi\colon S_1 \to S_2$ is arbitrary.

REMARK 4.1. The semigroup G, defined in Remark 2.4. is such that its corresponding "algebraic" system always reduces to one equation only.

EXAMPLE 4.1. Consider the real equation

$$af(x,y) + bf(y,x) + cf(x,x) + df(y,y) = 0 \tag{4.4}$$

with $a + b + c + d = 0$[1], i.e. the equation

$$af(x,y) + bf(y,x) + cf(x,x) - (a+b+c)f(y,y) = 0. \tag{4.5}$$

The equation (4.5) leads to the system

$$\begin{aligned} au + bv + cw - (a+b+c)z &= 0 \\ bu + av - (a+b+c)w + cz &= 0 \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

[1] If $a + b + c + d \neq 0$, the equation (4.4) is easily reduced to the (cyclic) group equation $af(x,y) + bf(y,x) = 0$.

and it will reduce one equation only, if

$$\frac{a}{b}=\frac{b}{a}=-\frac{c}{a+b+c}=-\frac{a+b+c}{c}. \tag{4.6}$$

The system (4.6) yields two possibilities:

(i) $a=b$, $c=-a$;

(ii) $a+b=0$, c abitrary.

In case (i) the equation (4.5) becomes

$$f(x,y)+f(y,x)-f(x,x)-f(y,y)=0$$

and has general solution

$$f(x,y)=\Pi(x,y)-\frac{1}{2}(\Pi(x,y)+\Pi(y,x)-\Pi(x,x)-\Pi(y,y)),$$

while in case (ii) we obtain the equation

$$af(x,y)-af(y,x)+cf(x,x)-cf(y,y)=0$$

with the general solution

$$f(x,y)=\Pi(x,y)-\frac{1}{2a}(a\Pi(x,y)-a\Pi(y,x)+c\Pi(x,x)-c\Pi(y,y)),$$

where in both cases Π is an arbitrary function

Illustration 2. Suppose that $G=\{g_1,g_2,\dots,g_m,g_{m+1},\dots,g_n\}$ is a monoid satisfying

$$g_ig_j=g_j \text{ for all } i(1\le i\le n) \text{ and } j=m+1,\dots,n,$$

and consider the equation

$$J_1(f(x),f(g_2x),\dots,f(g_nx))=J_2(f(x),f(g_2x),\dots,f(g_nx)) \tag{4.7}$$

where $J_1(a,a,\dots,a)=J_2(a,a,\dots,a)$ for all a.

If the system

$$J_1(u_{1k},u_{2k},\dots,u_{nk})=J_2(u_{1k},u_{2k},\dots,u_{nk}) \quad (k=1,2,\dots,n)$$

implies

$$u_1=F(u_{m+1},\dots,u_n) \quad (u_i\text{corresponds to}f(g_ix))$$

where

$$F(F(a_1,\dots,a_1),\ F(a_2,\dots,a_2),\dots,F(a_{n-m},\dots,a_{n-m}))=F(a_1,\dots,a_{n-m}),$$

then the general solution of (4.7) is given by

$$f(x) = F(\Pi(g_{m+1}x), \dots, \Pi(g_n x)),$$

where Π is an arbitrary function.

EXAMPLE 4.2. Consider the real functional equation

$$\begin{aligned} f(x,y,z)^2 + f(y,y,z)^2 + f(z,z,z)^2 = \\ = f(x,y,z)f(y,y,z) + f(y,y,z)f(z,z,z) + f(z,z,z)f(x,y,z). \end{aligned} \tag{4.8}$$

If $g_1(x,y,z) = (x,y,z)$, $g_2(x,y,z) = (y,y,z)$, $g_3(x,y,z) = (z,z,z)$, then clearly $g_i g_3 = g_3$ $(i = 1,2,3)$. The corresponding "algebraic" system

$$\begin{aligned} u^2 + v^2 + w^2 - uv - vw - wu = 0 \\ v^2 + w^2 - 2vw = 0 \\ 0 = 0 \end{aligned}$$

yields $u = v = w$, and hence the general solution of (4.8) is

$$f(x,y,z) = \Pi(z,z,z)$$

where Π is an arbitrary function, or equaivalently,

$$f(x,y,z) = \Phi(z)$$

where Φ is an arbitrary function.

EXAMPLE 4.3. Let $g: S_1 \to S_2$ where S_2 is a field and let $\{i, g, \dots, g^{n-1}\}$ be a cyclic group of order n. Furthermore, let $h: S_1 \to S_2$ be such that $gh = h$, $h^2 = h$. Then, clearly $g^k h = k$ for all k, and the mappings g, h generate the semigroup $G = \{i, g, \dots, g^{n-1}, h, hg, \dots, hg^{n-1}\}$. Consider the equation

$$\begin{aligned} a_1 f(x) + a_2 f(gx) + \cdots + a_n f(g^{n-1}x) \\ + b_1 f(hx) + b_2 f(hgx) + \cdots + b_n f(hg^{n-1}x) = 0 \end{aligned} \tag{4.9}$$

where $\sum_{v=1}^n a_v + \sum_{v=1}^n b_v = 0$.

If

$$D = \begin{vmatrix} a_1 & a_2 & \cdots & a_n \\ a_n & a_1 & & a_{n-1} \\ \vdots & & & \\ a_2 & a_3 & & a_1 \end{vmatrix} \neq 0,$$

then the general solution of the equation (4.9) is

$$f(x) = \frac{1}{D}(D_1\Pi(hx) + D_2\Pi(hgx) + \cdots + D_n\Pi(hg^{n-1}x))$$

where D_1, D_2, $\dots D_n$ are the determinants obtained from D by replacing the first column of D by $(-b_1, -b_n, \dots, b_2), (-b_2, -b_1, \dots, -b_3), \dots, (-b_n, -b_{n-1}, \dots, -b_1)$ reespectively.

REMARK 4.2. A useful case of the equation (4.9) is provided by

$$S_1 = R^n,\ \ S_2 = R,\ \ g(x_1, x_2, \dots, x_n) = (x_2, x_3, \dots, x_1),$$
$$h(x_1, x_2, \dots, x_n) = (x_1, x_1, \dots, x_1).$$

5. A special functional equation. We shall now apply some of the previous results to the real equation

$$af(x,y) + bf(y,x) + cf(x,x) + df(y,y) = 0. \tag{5.1}$$

We distiguish between several cases (excluding the trivial case $a = b = c = d = 0$).
$(I) a^2 = b^2 > 0$.

(I.i) $a + b + c + d \neq 0$. Then (5.1) implies $f(x,x) = 0$, and it reduces to the cyclic equation

$$af(x,y) + bf(y,x) = 0$$

which has the following general solution

(I.i.a)	$f(x,y) = 0$	$(a \neq b^2)$
(I.i.b)	$f(x,y) = \Pi(x,y) - \Pi(y,x)$	$(a = b)$
(I.i.c)	$f(x,y) = \Pi(x,y) + \Pi(y,x)$	$(a + b = 0)$.
(I.ii)	$d + b + c + d = 0$.	

(I.ii.a) If $a^2 \neq b^2$, we arrive at a special case of Example 4.2. and the general solution of (5.1) is

$$f(x,y) = \frac{bd - ac}{a^2 - b^2}\Pi(x,x) + \frac{bc - ad}{a^2 - b^2}\Pi(y,y).$$

(I.ii.b) If $a + b = 0$, this is a special case of Example 4.1, and the general solution of (5.1) is

$$f(x,y) = \Pi(x,y) - \frac{1}{2a}(a\Pi(x,y) + b\Pi(y,x) + c\Pi(x,x) + d\Pi(y,y)).$$

(I.ii.c) If $a = b$, we distinguish between:

(I.ii.c.1) $a + c = 0$. This is again a special case of Example 4.1; the equation becomes

$$f(x,y) + f(y,x) - f(x,x) - f(y,y) = 0$$

and its general solution is

$$f(x,y)=\Pi(x,y)-\frac{1}{2}(\Pi(x,y)+\Pi(y,x)-\Pi(x,x)-\Pi(y,y)).$$

(I.ii.c.2) $a+c\neq 0$. Then (5.1) becomes

$$f(x,y)+f(y,x)+\alpha f(x,x)-(2+\alpha)f(y,y)=0\left(\alpha=\frac{c}{a}\right) \tag{5.2}$$

which together with

$$f(y,x)+f(x,y)+\alpha f(y,y)-(2+\alpha)f(x,x)=0$$

implies $f(x,x)=f(y,y)$, and (5.2) reduces to

$$f(x,y)+f(y,x)-2f(x,x)=0. \tag{5.3}$$

The general solution of (5.3) is given by [4]:

$$f(x,y)=\Pi(x,y)-\Pi(y,x)+K$$

where K is an arbitrary constant, or, equivalently

$$f(x,y)=\Pi(x,y)-\Pi(y,x)+2\Pi(k,k).$$

where $k\in R$ is fixed.

(II) $a^2=b^2=0$.

The equation (5.1) becomes

$$cf(x,x)+df(y,y)=0, \tag{5.4}$$

and we distinguish between two cases:

(II.i) $c+d\neq 0$. Then (5.4) becomes

$$f(x,x)=0, \tag{5.5}$$

which is a special case of Example 4.1. Hence, the general solution of (5.5) is

$$f(x,y)=\Pi(x,y)-\Pi(x,x).$$

(II.ii) $c+d=0$. The equation (5.4) becomes

$$f(x,x)=f(y,y),\ \text{ i.e. }\ f(x,x)=f(k,k) \qquad (k\text{ fixed}).$$

Its general solution is esily established to be

$$f(x,y)=\Pi(x,y)-\Pi(x,x)+\Pi(k,k),$$

where $k \in R$ is fixed.

Hence, in all cases the general solution of (5.1) can be expressed as a linear combination of $\Pi(x, y)$, $\Pi(y, x)$, $\Pi(x, x)$, $\Pi(y, y)$, $\Pi(k, k)$, where Π is an abitrary function, and $k \in R$ is fixed.

6. Fourth variation. As we mentioned in the previous section, in [4] Prešić considered, as an example, the real equation

$$f(x, y) + f(y, x) - 2f(x, x) = 0 \tag{6.1}$$

and remarked that its general solution

$$f(x, y) = \Pi(x, y) - \Pi(y, x) + K$$

(Π arbitrary function, K arbitrary constant) cannot be expressed by means of the semigroup $G = \{g_1, g_2, g_3, g_4\}$, where $g_1(x, y) = (x, y)$, $g_2(x, y) = (y, x)$, $g_3(x, y) = (x, x)$, $g_4(x, y) = (y, y)$.

Suppose that k is a fixed real number. It is easily shown that the general solution of (6.1) can be written in the form

$$f(x, y) = \Pi(x, y) - \Pi(y, x) + 2\Pi(k, k),$$

i.e. the equation (6.1) is solvable within the semigroup $G' = \{g_1, g_2, g_3, g_4, g_5\}$ where $g_5(x, y) = (k, k)$.

This example shows that it may be possible to extend the initial semigroup G into a semigroup $G_e \supset G$ which is such that it allows the equation to be solved within it. Indeed, B. Alimpić [8] showed that any equation of the form

$$J(f(x, y), f(y, x), f(x, x)f(y, y)) = \top$$

can be solved within semigroup $G'' = \{g_1, \dots, g_9\}$, where $g_1, \dots, g_5$ are defined as above, and $g_6(x, y) = (x, k)$, $g_7(x, y) = (k, x)$, $g_8(x, y) = (y, k)$, $g_9(x, y) = (k, y)$.

In the previous section we showed that any linear equation (5.1) can be solved within the semigroup G' (the wider semigroup G'' is not needed).

This suggests two questions:

(i) For a given equation (1.1) unsolvable within G, does there exist an extended semigroup $G_e \supset G$ such that the equation is solvable within G_e.

(ii) If the answer to (i) is affirmative, is it possible to find the minimal extended semigroup G_m such that the equation is solvable within G_m.

REFERENCES

[1] S. B. Prešić: *Sur l'équation fonctionelle* $f(x) = f(g(x))$. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 61–No. 64 (1961), 29–31.

[2] S. B. Prešić, *Sur l'équation fonctionelle* $f(x) = H(x, f(x), f(\theta_1 x), \dots, f(\theta_n x))$. Ibid. No. 115–No. 121 (1963), 17–20.

[3] S. B. Prešoć, *Méthode de résolution d'une classe d'équations fonctionnelles linéaires.* C. R. Acad. Sci. Paris 257 (1963), 2224-2226.

[4] S. B. Prešić, *Méthode de résolution d'une classe d'équations fonctionelles lineaires.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 115–No. 121 (1963), 21–28.

[5] S. B. Prešić, *A method for solving a class of cyclic functional équations.* Mat. Vesnik 5 (20) (1968), 375–377.

[6] S. B. Prešić, *Méthod de résolution d'une classe d'équations fonctionnelles linéaires et non homogénes.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 247–No. 273 (1969), 67–72.

[7] S. B. Prevšić, *Opšta grupna funkcionalna jednačina.* Mat. Vesnik 7 (22) (1970), 317–320.

[8] B. P. Alimpić, *On models of certain formulas of the predicate calculus of first order.* Ibid. 5 (20) (1968), 347–351.

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