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ON SOME SOLUTION OF EQUATION $\varphi(x) + \varphi(f(x)) = F(x)$ UNDER THE CONDITION THAT F SATISFIES $F(f^p(x)) = F(x)$[1]

Mirjana Malenica

I

The results of this paper are connected with solutions $\varphi(x)$ of the functional equation

$$\varphi(x) + \varphi(f(x)) = F(x) \tag{1}$$

where $f(x)$ and $f(x)$ are given functions. Many mathematicians have been concerned with equation (1). Some special cases of this equation have been considered by H. Steinhaus [8]. G, H. Hardy [4], R. Raclis [7] and others. Equuations of more general type have been examined by M. Ghermanencu [3] and Kitamura [5].

Some papers of M. Kuczma and M. Barjaktarević are of special interest to this paper. I the paper [6] M. Kuczma proves that under some natural assumptions equation (1) possesses infinitely many solutions which continuous for every x this is not a root of the equation

$$f(x) = x. \tag{$*$}$$

In the same paper, under assumption that the solution is continuous for $x = x_0$. satisfying equation $(*)$, the author proves the existence of a most such colution. Further, he proves that with addition assumptions such a solution exists and it is given by an explicit formula.

M. Barjaktarević in his paper [1] proves four theorems. Two of them, using different regular methods of summability for the series

$$\frac{1}{2}F(b) + \sum_{v=0}^{\infty}(-1)^v\{F(f^v(x)) - F(b)\}, \tag{2}$$

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give sufficient conditions for the existence of a solution of equation (1). The other two theorems give sufficient conditions for existence of a unique continuous solution of (1) in $(a, b]$.

The author of this paper, using the same method, examined some solutions of the equation (1) in the papers [9], [10] and [11]. In [9] and [10], using different regular methods of the summability of the series (2), sufficient i. e. necessary and sufficient conditions for the existence of a solution of the equation (1) are given. Also, sufficient and necessary conditions are given, under which the cosidered regular methods are translative from the right on a set

$$\mathcal{U} = \left\{ \{s_n(x)\}_{n=0}^{+\infty} \right\}_{x \in [a,b]}, \tag{3}$$

in the sence of the definition of translation given in [2], page 21, where $s_n(x)$ is a partial sum of range n of the series (2).

In the [11] equation (1) is considered under the condition that the function F has a special form for which the series (2), in general, diverges. Also conditions, are given the series (2) to be T-sumable ($T = (a_{kn})$ a regular matrix transformation) to φ so that the following relations are valid

$$\begin{gathered} T s_n(x) \to \varphi(x) \\ \varphi(f(x)) + \varphi(x) = F(x). \end{gathered}$$

Connected with this, in [11] theorem I is proved.

In this paper we consider a similar cases depending on F as in [11], but with a more general function F.

Theorem I of [11] is related to $p = 1$ (see paragraph II) and the results of this paper are related to the case $p > 1$. There are some specific differences between these two cases.

In theorem I of [11] and troughout this paper it is assumed that

1. $f(x)$ is a continuous strongly monotonic function on $[a, b]$ and that

$$f(a) = a, \;\; f(b) = b; \;\; f(x) > x, \;\; x \in (a, b);$$

2. $f^0(x) = x$, $f^{y+1}(x) = f(f^y(x))$, $v \in \{0, \pm 1. \pm 2, \ldots\}$.

II

Let $p \in \{2, 3, \ldots\}, p$ fixed and let the function $F(x)$ have the properties

$$\begin{cases} F(f^p(x)) = F(x) & x \in [a, b] \\ F(f^i(x)) \neq F(x) & x \in [a, b),\, i = 1, 2, \ldots, p-1, p > 1 \\ F(x) = g(x) & x \in [x_0, f(x_0)),\, x_0 \in (a, b) \end{cases} \tag{4}$$

where $g(x)$ is an arbitrary chosen functions with domain $[x_0, f(x_0))$.

Further, sometimes use the following condition (5) on $F(x)$:

$$(5)\qquad \begin{cases} \text{there are points } x_i \;\; l \in \{0,1,\dots,p\}, \text{ in } (a,b) \text{ so that} \\ \Delta = \begin{vmatrix} 1F(x_0) & F(f(x_0)) & F(f^2(x_0)) & \dots & F(f^{p-1}(x_0)) \\ 1F(x_1) & F(f(x_1)) & F(f^2(x_1)) & \dots & F(f^{p-1}(x_1)) \\ 1F(x_2) & F(f(x_2)) & F(f^2(x_2)) & \dots & F(f^{p-1}(x_2)) \\ \dots & \dots & \dots & \dots & \dots \\ 1F(x_p) & F(f(x_p)) & F(f^2(x_p)) & \dots & F(f^{p-1}(x_p)) \end{vmatrix} \neq 0. \end{cases}$$

We will be concerned with finding solution $\varphi(x)$ on $[a,b]$ of equation (1) that have form

$$(6)\qquad \varphi(x) = a + b_0 F(x) + b_1 F(f(x)) + \cdots + b_{p-1}(x))F(f^{p-1}(x)).$$

In connction with this the following theorem is valid.

THEOREM 1. *For the equation (1), in which function $F(x)$ has the properties (4), to have a soloution $\varphi(x)$ on $[a,b]$ of the form (6), it is sufficient and, if F also satisfies condition (5), it is neccesary that*

$$(7)\qquad \begin{cases} p \;\; \textit{is odd number} \\ a = 0, \;\; b_i = \dfrac{1}{2}(-1)^i, \quad i \in \{0,1,\dots,p-1\} \end{cases}$$

i.e. that the function $\varphi(x)$ has the form

$$(8)\qquad \varphi)x) = \sum_{i=0}^{p-1} (-1)^i F(f^i(x)).$$

PROOF. To prove that the conditions (7) are sufficient it is enough to substitute (8) into (1) in place of $\varphi(x)$. Let us prove that the conditions (7) are neccesary under the assumption that (4) and (5) are satisfied.

Now, according to (6), we get

$$(9)\quad \varphi(x) + \varphi(f(x)) - F(x) = 2a + (b_{p-1} + b_0 - 1)F(x) + \sum_{i=1}^{p-1} (b_i + b_{i-1})F(f^i(x)).$$

For $\varphi(x)$ defined with (6), to be a solution of our equation it is neccesary (and sufficient) that the right of (9) is identically equal to zero on $[a,b]$. Seting the right side of (9) equal to $x_i (i = 0,1,\dots,p)$, we get $p+1$ linear equation

$$(10)\quad \beta_0 F(x_i) + \beta_1 F(f(x_i)) + \cdots + \beta_{p-1} F(f^{p-1}(x_i)) + 2a = 0 \quad (i = 0,1,\dots,p),$$

where for the sake of simplicity, we put

$$(11) \qquad \begin{cases} \beta_0 = b_{p-1} + b_0 - 1 \\ \beta_i = b_i + b_{i-1} \qquad (i = 1, \dots, p-1). \end{cases}$$

System (10) is the system of $p+1$ linear equation with $p+1$ unknown elements

$$2a, \beta_0.\beta_1, \dots, \beta_{p-1}.$$

Since the determinante Δ of this system by assumption is different than zero, system (10) has only the trivial solution

$$(12) \qquad a = 0, \beta_i = 0 \qquad (i = 0, 1, \dots, p = 1).$$

System (12) reduce to a system of linear equation which for p odd number gives (7). For p an even number she system is contradictory.

REMARK 1. We note that for the case $p \geq 2$ and p an even number $\varphi(x)$ of the form (8) is not a solution of (1). Namely, we have

$$\varphi(x) + \varphi(f(x) = 0$$

which contradics (5) in which $\Delta \neq 0$.

This proves the theorem.

We note the followwing: If $F(x)$ has properties (4), then the series (2) obvioulsy diverges since

$$a_n(x) = (-1)^n \{F f^n(x)) - F(b)\}$$

doesn't converge to zero as $n \to \infty$.

Because of this, it is interest to ask if there exists a regular method of summability $T = (a_{kk})$ with which the series (2) wil be summable to T-sum $\varphi(x)$, that is the solution of the considered equation, taking into account that $F(x)$ has properties in (4). In the connection with that we have

THEOREM 2. *Let the following condition be satisfied:*

a) function F has properties (4) *and $p > 2$ is odd number,*

b) the series (2) is T-sumable to sum $\varphi(x)$, where $T = (a_{k,n})$ is a regular matrix transformation

c) the limit

$$\lim_{k\to\infty} B_r^k \text{ exists for each } r \in \{0, 1, \dots, 2p-2\},$$

where

$$B_r^k = \sum_{n=1}^{\infty} a_{k.n.2p+r}, \qquad r \in \{0, 1, \dots, 2p-2\}.$$

For the functions $\varphi(x)$ to be a solution of the equation (1) it is sufficient and, if F in addition satisfies the conditions in (5) it is neccessary that

$$a = 0,\ \ b_i = \frac{1}{2}(-1)^i,\ \ i \in \{0, 1, \dots, p-1\} \tag{13}$$

and solution $\varphi(x)$ is given by (6).

Proof. Because of theorem 1, it is sufficient to prove that, under the conditions of the present theorem, functions $\varphi(x)$ has the from (6).
Numely, under conditions of the present theorem we have

$$s_n(x) = -\frac{1}{2}(-1)^n F(b) + \sum_{v=0}^{n}(-1)^v F_v \ \text{ with } \ F_v = F(f^v(x)),$$

$$s'_k(x) = \frac{1}{2}F(b)\left\{\sum_{n=0}^{\infty} a_{k,2n} - \sum_{n=0}^{\infty} a_{k,2k+1}\right\} + \sum_{n=0}^{\infty} a_{kn}\left(\sum_{v=0}^{n}(-1)^v F_v\right).$$

If we put

$$n = 2p \cdot m_n + r_n \quad (r_n = 0, 1, \dots, 2p-1;\ \ m_n \in \{0, 1, 2, \dots\}),$$

it is easy to prove

$$\sum_{v=0}^{n}(-1)^v F_v = \begin{cases} \displaystyle\sum_{v=0}^{r_n}(-1)^v F_v, & p \text{ is odd number } r_n \in \{0, 1, \dots, 2p-1\}. \\ \displaystyle 2m_n \sum_{v=1}^{p-1}(-1)^v F_v + \sum_{v=0}^{r_n}(-1)^v F_v, & p \text{ is even number} \end{cases}$$

Now for p odd number

$$\begin{aligned} s'_k(x) = &-\frac{1}{2}F(b)\left\{\sum_{n=0}^{\infty} a_{k,2n} - \sum_{n=0}^{\infty} a_{k,2n+1}\right\} \\ &+ \sum_{n=0}^{\infty} a_{k,2pm_n+r_n}\left(\sum_{v=0}^{n}(-1)^v F_v\right), \quad r_n \in \{0, 1, \dots, 2p-1\}, \end{aligned}$$

$$\begin{aligned} \varphi(x) = \lim_{k\to\infty} s'_k(x) = -\frac{1}{2}F(b)\{B_0 + B_2 + \dots + B_{2p-2} + B_1 - B_3 - \dots B_{2p-1}\} + \\ + \sum_{=0}^{2p-1} B_r\left(\sum_{v=0}^{r}(-1)^v F_v\right), \end{aligned}$$

where

$$B_r^k = \sum_{n=0}^{\infty} a_{k,n2p+r}, \quad r \in \{0, 1, \dots, 2p-2\},$$

so that

$$\sum_{n=0}^{\infty} a_{k,2n} = B_0^k + B_2^k + \cdots + B_{2p-2}^k, \quad \sum_{n=0}^{\infty} a_{k,2n+1} = B_1^k + B_3^k + \cdots + B_{2p-1}^k,$$

and

$$B_r = \lim_{k\to\infty} \bar{B}_r^k, \quad r \in \{0, 1, \dots, 2p-2\}$$

and

$$\lim_{k\to\infty} \sum_{n=0}^{\infty} a_{k,2n} = B_0 + B_2 + \cdots + B_{2p-2}, \quad \lim_{k\to\infty} \sum_{n=0}^{\infty} a_{k,2n+1} = B_1 + B_3 + \cdots + B_{2p-1}.$$

Because of this, taking into account (4), after a long but stratight forward calculation we get

$$\varphi(x) = a + b_0 F_0 + b_1 F_1 + b_2 F_2 + \cdots + b_{p-1} F_{p-1},$$

where the coefficients a, b_i $(i = 0, 1, \dots, p-1)$ are given by

$$a = \frac{1}{2} F(b)(B_1 + B_3 + \cdots + B_{2-1} - B_0 - B_2 - \cdots - B_{2p-2}$$

$$b_j = (-1)^j \sum_{i=0}^{p-1} B_{i+j} \qquad (j = 0, 1, 2, \dots, p-1).$$

Therefore $\varphi(x)$ is of the form (6).

REMARK 2. We note that the case $p = 1$, in a more general form, is included in theorem I of [11].

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Milena Malenica
71000 Sarajevo
Ilija Engela 19
Jugoslavija