MEASURE AND INTEGRATION IN THE
ALTERNATIVE SET THEORY

Miodrag Rašković

With the analysis point of view, there is a strong similarity between the
saturated models of analysis (NA) and the Alternative Set Theory (AST). So,
notions as for example “infinite external set”, “internal set” and “hyper-finite set”
in NA have corresponding notions as “countable class”, “set” and “infinite” set in
AST (“infinite” in AST meaning).

The internal definition principle is in relation with the comprehension schema
in AST, and comprehension property in NA is in relation with the prolongation
axiom.

This similarity between NA and AST becomes complete understandable, if
one knows that the ultra-power of the set of hereditary-finite sets enriched with its
subsets is a model for AST (see [5]).

On the other hand, AST allows us to make a natural fundation of analysis.

Our intention will be translating the notions and theorems from [2] to AST.

For basic motivations, notions, axioms, definitions and theorems for AST, one
may consult [1]. We recall, that the class of natural numbers is $\mathbb{N}$,$^{1}$ finite natural
number is $FN$,$^{2}$ rational numbers is $RN$,$^{3}$ finite rational numbers is $FRN$,$^{4}$ and
bounded rational numbers is $BRN$. All of this classes are class-theoretically definable,
as theirs relations and operations $\leq, +$ and $\cdot$ are. The set of real numbers is
$\text{Real} = BRN/\equiv$, where by $\equiv$ “infinitely near” relations is denoted.

$\text{Real}$ has all topological and algebraical property as the classical reals does.
But for us, its selector $R \subseteq BRN$ is more useful than the such $\text{Real}$ itself. $R$ is

1We model natural numbers in the manner of von Neumann.
2$n \in FN$ iff $(\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n) (\forall X \subseteq n)$.
3$r \in RN$ iff $(\forall m, n \in Z) (r = (m, n))$ and we write $r = \frac{m}{n},$ where $Z = N \cup \{0, \alpha) \mid \alpha \in N.$
4$r \in FRN$ iff $(\exists m, n \in FN) (r = \frac{m}{n}),$ where $FN = FN \cup \{0, \alpha) \mid \alpha \in FN.$
5$r \in BRN$ iff $(\exists n \in FN) (|r| \leq n)$
6$x = y$ iff $(\forall n \in FN) (|x - y| \leq n)$
7A class $R$ is image of choice function on the class $\text{Real}$. 
topologically isomorphic with $\text{Real}$, so that we can translate all interesting properties from $\text{Real}$ to $R$.

For each $x \in R$, there is $\circ x \in \text{Real}$ so that $x \in \circ x$, and for each $y \in \circ x$ we have $st(y) = x$. Observe that $st(x)$ and $\circ x$ are not same.

For $A \subseteq R$, let $\circ A = \{x \in \text{Real} \mid (\exists y) (y \in A \land y \in x)\}$.

The topological notions as an interval, open and closed set, convergence and so on, we can define in the usual manner (if we exchange a set with a class), but the role of $N$ and $\text{RN}$ play $\mathcal{FN}$ and $\mathcal{FRN}$. The role of $^*R$ play $\mathcal{RN}$.

The set $R$ omits some algebraic characteristic, which $\text{Real}$ has. For example, in $R$, $(\exists x) (x = 2)$ is true, but not $(\exists x) (x = 2)$. However, this is not of an importance for us. $R$ is more natural then $\text{Real}$, because $R \subseteq \mathcal{BRN}$, while $\text{Real}$ is the class of classes.

We can replace the saturation with the following theorems (see [1]).

**Theorem 1.** Each countable class is proper semiset.

**Theorem 2.** Let $X$, $Y$ be countable classes such that $\cap X \subseteq \cup Y$. Then there is a set $u$ such that $\cup X \subseteq u \subseteq \cap Y$.

**Theorem 3.** Let $Z$ be set-theoretically definable class. Let $X$ be a countable subsemiset of $Z$. If $X$ is directed then there is a $u \in Z$ which is an upper bound of the elements of $X$ ordered by inclusion. If $X$ is dually directed then there is a $u \in Z$ which is a lower bound of the elements of $X$ ordered by $\subseteq$.

**Theorem 4.** Let $\{X_n, n \in \mathcal{FN}\}$ be a sequence of revealed classes (for example, definable classes are revealed) such that for each $m \in \mathcal{FN}$, $\cap\{X_n \mid n \leq m\}$ is non-empty. Then $\cap\{X_n \mid n \in \mathcal{FN}\} \neq \emptyset$.

1. **Loeb and Lebesque measure**

Now, we will start to investigate Loeb measure in $\mathcal{AST}$.

Let $x \approx n$ iff $(\exists f) (f: \frac{n}{1+x} \rightarrow x)$.

Let $\Omega$ be a set, and $^*\mathcal{P}(\Omega)$ the set of its subsets. It is easy to see that $^*\mathcal{P}(\Omega)$ is a field of sets.

Let $\bar{P}$ and $P$ be functions so that for $\Omega \approx n$, $A \in^* \mathcal{P}(\Omega)$ and $A \approx m$, we have $\bar{P}(A) = |A|_n = \frac{1}{n}$ and $P(A) = st\bar{P}(A)$.

If $A \subseteq \Omega$, we can define inner measure so that $P_{\text{inner}}(A) = \sup\{P(B) \mid B \subseteq A, B \text{ set}\}$, and outer measure $P_{\text{outer}}(A) = \inf\{P(B) \mid A \subseteq B, B \text{ set}\}$.

**Def. 1.** A class $A \subseteq \Omega$ is Loeb measurable iff $P_{\text{inner}}(A) = P_{\text{outer}}(A)$ and let us put $P(A) = P_{\text{inner}}(A) = P_{\text{outer}}(A)$.

Let $L(\Omega)$ be a class of Loeb measurable classes (which are sets or semisets).

**Lemma 1.** For each a class $A$ we have:
(1) $A \in L(\Omega)$ iff for each $\varepsilon > 0$ there are classes $B$ and $C$ so that $B \subseteq A \subseteq C$ and $P(C \setminus B) < \varepsilon$.
(2) $P_{\text{inner}}(A) = 1 - P_{\text{outer}}(\Omega \setminus A)$.

**Theorem 1.** (Loeb) The class $L(\Omega)$ is a $\sigma$-field and $P$ is a $\sigma$-additive function.

**Proof.** A proof that $L(\Omega)$ is a field and $P$ is additive is as in (2).

To complete the proof, we must show that, if for each $i \in FN$, $A_i \supseteq A_{i+1}$, $A_i \in L(\Omega)$, $P(A_i) = r_i$, and $\lim_{i \to \infty} r_i = r$, then $\cap_{i \in FN} A_i \in L(\Omega)$ and $P(\cap_{i \in FN} A_i) = r$. We may apply the lemma 1 .... In this case is enough to choose sets $B$ and $C$, so that for $A = \cap_{i \in FN} A_i$ and $\varepsilon > 0$, we have $B \subseteq A \subseteq C$, $P(B) \geq r - \varepsilon$ and $P(C) \leq r + \varepsilon$.

The sequence $\{r_i\}_{i \in FN}$ is descending and $\lim_{i \to \infty} r_i = r$, so we can choose $n \in FN^8$ and a set $C$, so that $A_n \subseteq C$ and $P(C) < r_n + \frac{\varepsilon}{2}$. But then $A \subseteq A_n \subseteq C$ and $P(C) < r_n + \frac{\varepsilon}{2}$. Let $B_m = B_1 \cap \ldots \cap B_m$. By induction, we can easily show that $P(B_m) > r_m - (1 - 2^{-m})\varepsilon$.

The class $X_m = \{B \in ^* \mathcal{P}(\Omega) \mid \tilde{P}(B) > r - \varepsilon \wedge B \subseteq B_m\}$ is revealed (and, more, a set). Also, we have $B_m \subseteq B_m$ and $\tilde{P}(B_m) > r_m - (1 - 2^{-m}) > r - \varepsilon$. Therefore, $X_m \neq \emptyset$ and $X_1 \supseteq X_2 \supseteq \ldots$. So, by Theorem 4, we have $\cap_{m \in FN} X_m \neq \emptyset$.

Let $B \in \cap_{m \in FN} X_m$. Then, for each $m \in FN$, $B \subseteq B_m$ and $\tilde{P}(B) = P(B) > r - \varepsilon$. For each $m \in FN$, we have $B_m \subseteq A_m$. So $B \subseteq \cap_{m \in FN} A_m$, as required.

**Theorem 2.** For each $A \in L(\Omega)$, there is $B \in ^* \mathcal{P}(\Omega)$ so that $P(A \Delta B) = 0$.

The proof of the Theorem follows by the Theorem II.

**Theorem 3.** If $P(A) = 0$ and a semiset $A$ is a countable union of sets, then there is a set $B \supseteq A$, so that $P(B) = 0$.

The proof follows by the Theorem 3.

Our intention will be to define Lebesque’s measure on $R$ and $\text{Real}$, and to show its conection with Loeb’s measure.

Let $H \in FN \setminus FN$ and $\Delta t = \frac{1}{2}$. Then, the class $T = \{0, \Delta t, 2\Delta t, \ldots, 1\} = \{\langle k, H \rangle \mid k \in H \}$ is a set.

Let $[s, t] = \{x \in R \mid s \leq x \leq t\}$ and $st_T : T \to [0, 1]$, where $st_T = st \upharpoonright T$. Further on, we will write $st$ instead of $st_T$, if it will not bring us to an ambiguity.

For $A \subseteq [0, 1]$ we have $st^{-1}(A) \subseteq T$ and $st(st^{-1}(A)) = A$, while for $B \subseteq T$ we have $st(B) \subseteq [0, 1]$ and $st(st^{-1}(B)) \supseteq B$.

Let $^{*}[s, t] = \{x \in RN \mid s \leq x \leq t\}$ and $(s, t) = [s, t] \setminus \{s, t\}$.\footnote{So that $r < r_m + \frac{\varepsilon}{2}$}
Now, we give the definition of Lebesgue measure \( \mu \) on \([0, 1]\).

For \( A = [s, t] \) we have \( \mu([s, t]) = \mu((s, t)) = t - s \).

If \( A \subseteq [0, 1] \), and \( A \) is open, then \( A \) can be written (uniquely) as a countable union of disjoint open intervals, \( A = \bigcup_{n \in F} A_n \), and then \( \mu(A) = \sum_{n \in F} \mu(A_n) \).

If \( A \subseteq [0, 1] \) is closed, then \( [0, 1] \setminus A \) is open, so \( \mu(A) = 1 - \mu([0, 1] \setminus A) \).

For all other \( A \subseteq [0, 1] \), we define \( \mu_{\text{inner}}(A) = \sup \{ \mu(B) \mid B \subseteq A, B \text{ is closed} \} \) and \( \mu_{\text{outer}}(A) = \inf \{ \mu(B) \mid A \subseteq B, B \text{ is open} \} \).

The class of Lebesgue measurable subclasses is \( \text{Leb}[0, 1] = \{ A \subseteq [1, 0] \mid \mu_{\text{inner}}(A) = \mu_{\text{outer}}(A) \} \), and for \( A \in \text{Leb}[0, 1] \) we have \( \mu(A) = \mu_{\text{outer}}(A) = \mu_{\text{inner}}(A) \).

Now, we give Fisher’s theorem, which is connection between Loeb’s and Lebesque’s measure. We need several lemmas for the proof of the theorem. However we omit theirs proofs, which are very similar to proofs in (2).

Only, we must use the Theorem 4, instead of \( \omega \)-saturation.

**Lemma 2.** For each \( r \in [0, 1] \), \( st^{-1}(\{r\}) = \{ t \in T \mid t \sim r \} \in L(T) \) and has Loeb measure but any set containing it has a positive Loeb measure.

**Lemma 3.** For all \( s, t \in [0, 1] \), so that \( s \leq t \) we have \( P(T \cap [s, t]) = t - s \) and \( P(st^{-1}[s, t]) = \mu([s, t]) = t - s \).

**Lemma 4.** If \( A \subseteq [0, 1] \) and \( A \) is closed, then \( st^{-1}(A) \in L(T) \) and \( \mu(A) = P(st^{-1}(A)) \).

**Lemma 5.** Let \( B \subseteq T \) be set. Then \( st(B) \) is closed.

**Theorem 5.** (Fisher) for each \( A \subseteq [0, 1] \), \( A \in \text{Leb}[0, 1] \) iff \( st^{-1}(A) \in L(T) \), and in this case, we have \( \mu(A) = P(st^{-1}(A)) \).

The proof follows by the lemmas above.

We say \( \circ A \subseteq [0, 1] \) is Lebesque measurable iff \( A \in \text{Leb}[0, 1] \). Let us define \( \circ \mu \) with \( \circ \mu(A) = \mu(A) \).

By Fishers theorem, we have \( \circ \mu(A) = P(st^{-1}(A)) \).

### 2. Loeb measurable functions

First, will introduce two notions, and will give the theorem, which connects them.

**Def. 2.** A class function \( F \) is Loeb measurable iff for each \( r \in R \), we have \( \{ w \mid F(w) \leq r \} \in L(\Omega) \).

A set function \( f \) is lifting of a function \( F: \Omega \to R \) iff \( f: \Omega \to RN \) and \( st f(w) = F(w) \) almost sure (on a set of measure 1).

**Theorem 5.** A class function \( F \) is Loen measurable iff it has a lifting \( f \). Moreover, if for each \( w \in \Omega \), \( |F(w)| \leq n \), then we can find a lifting \( f \) such that \( |f(w)| \leq n \).
**Proof:** Let \( f \) lifts \( F \) on \( x \), where \( x = \{ w \in \Omega \mid stf(w) = F(w) \} \) and \( P(x) = 1 \). For \( r \in \mathbb{R} \), a class \( \{ w \in x \mid F(w) \leq r \} = \bigcap \{ w \in x \mid f(w) \leq r + \frac{1}{n} \} \) is measurable, as countable intersection of the measurable sets.

Let \( Q = \{ g_n \mid n \in FN \} \) be a sequence of finite rationals and \( F \) a Loeb measurable function.

The classes \( A_n = \{ w \mid F(W) \leq g_n \} \) are Loeb measurable, for each \( n \in FN \). By the Theorem 2, there is a set \( B_n \) so that \( P(B_n \triangle A_n) = 0 \) and for \( g_m \leq g_n \) we have \( B_m \subseteq B_n \).

For each \( n \in FN \), there is an function \( f_n \) such that:

\[(\forall m \leq n)(x \in B_m \leftrightarrow f_n(x) \leq g_m).\]

It is enough to take \( f_n \), so that \( f_n(B_0) = \{ g_0 \} \) and \( f(B_{m+1}\setminus B_m) = \{ g_m \} \), for each \( 1 \leq m \leq n \).

Let \( X_n = \{ f \mid "f \ is \ function" \ and \ (\ast) \} \). Then, we have \( X_n \neq \emptyset \) and \( X_1 \supseteq X_2 \ldots \). According the Theorem 4., there is \( f \in \cap_{n \in FN} X_n \). So we have \( stf(w) = F(w) \) for \( w \in \Omega \setminus \bigcup_{n \in FN} (A_n \triangle B_n) \) and

\[
P \left( \Omega \setminus \bigcup_{n \in FN} (A_n \triangle B_n) \right) = P(\Omega) - P \left( \bigcup_{n \in FN} (A_n \triangle B_n) \right) = 1 - \sum_{n \in FN} P(A_n \triangle B_n) = 1.
\]

So we have \( stf(w) = F(w) \) almost surely.

The rest we can prove trivially, if we bound \( f \) in \( X_n \) by \( n \).

**Def. 4.** A set function \( f \) is lifting of \( F_0: [0,1] \rightarrow RN \) iff \( F:T \rightarrow RN \) and \( stf(t) = F_0(st (t)) \) almost surely on \( T \).

**Lemma 6.** A class function \( F_0: [0,1] \rightarrow R \) is Lebesque measurable iff \( F_0 \) has a lifting \( f:T \rightarrow RN \).

**Proof:**

Let us define \( F:T \rightarrow R \) by \( F(t) = F_0(st^{-1}(t)) \). By Fisher's theorem (Theorem 4) \( F_0 \) is Lebesque measurable iff \( F \) is Loeb measurable iff (by Theorem 5.) \( F \) has a lifting \( f \).
REMARK: If $f : T \to RN$ is almost surely finite set, then $f$ is a lifting of some $F$, where is $F(w) = stf(w)$. Such $F$ is Loeb measurable.

DEF. 5. A set function $f$ is a uniform lifting of $F$ iff for all $w \in T$, $st f(w) = F(w)$.

THEOREM 6. A class function $F$ has a uniform lifting $f$ iff $\{ w \mid f(w) \leq r \}, \{ w \mid f(w) > r \} \in \{ x \mid \exists A : F \in \mathcal{P}(T)^{1}X = \cap_{n \in FNA} \} \text{ for all } r \in R$.

The proof follows by Theorem 4.

Let $F_0 : T \to \text{Real}$, so that $\circ F(w) = F_0(w)$ The function $F_0$ is Loeb measurable iff for all $r \in R$, we have $\{ w \in T \mid F(w) \leq r \} \in L(T)$.

3. Integration

Let $\Omega$ be infinite set, then there is $n \in N$ and a set function $f$ so that $f: n \to \Omega$. Then, we have $\sum_{w \in \Omega} F(w) = \sum\sum_{m=1}^{n} F(f(m))$.

A bounded and Loeb measurable function $F: \Omega \to R$ is simple iff range $(F)$ is finite.

DEF. 6. Let $F$ be bounded and measurable function. If $F$ is simple, then $\int_{\Omega} F(w) dw = \sum_{r \in \text{range}(F)} r \cdot F(F^{-1}(\{r\}))$. In general

\[ \int_{\Omega} F(w) dw = \sup \left\{ \int_{\Omega} G(w) dw \mid G \text{ is simple and } G \leq F \right\} \]

THEOREM 7. (Loeb) Let $F: \Omega \to R$ be bounded and Loeb measurable function and let $f : \Omega \to RN$ be a bounded lifting of $F$, then

\[ \int_{\Omega} F(w) dw = st \sum_{w \in \Omega} f(w) \Delta w. \]

The proof is similar to the proof in (2).

THEOREM 8. Let $F : [0, 1] \to R$ be bounded Lebesgue measurable and let $f$ be a lifting function of $f$. Then, we have $\int_{\Omega} F(t) dt = \sum_{w \in \Omega} f(w) \Delta w$.

The proof follows easily by Theorem 4 and Theorem 7.

Let $F : \Omega \to RN$ be Loeb measurable and non-negative. Let us denote $\min(F(w), n)$ by $(F \wedge n)(w)$.

Then, we define $\int_{\Omega} F(w) dw = \lim_{n \to \infty} \int_{\Omega}(F \wedge n)(w) dw$.

A function $F$ is Loeb integrable iff $\int_{\Omega} F(w) dw$ is finite.

We can denote $\max(F(w), 0)$ by $F^+(w)$ and $\min(F(w), 0)$ by $F^-(w)$. Then we have $F = F^+ + F^-$. 
In general case, for unbounded $F$, we define

$$\int_{\Omega} F(w)dw = \int_{\Omega} F^+(w)dw - \int_{\Omega} F^-(w)dw.$$  

So, we see that a function $F$ is Loeb integrable iff both $F^+$ and $F^-$ are.

**Def. 7.** Let $f$ be non-negative set function so that $f: \Omega \to RN$. The function $f$ is $s$-integrable iff sum $\sum_{w \in \Omega} f(w) \Delta w$ is finite, and

$$\lim_{n \to \infty} st \sum_{w \in \Omega} (f \wedge n)(w) \Delta w = st \sum_{w \in \Omega} f(w) \Delta w.$$  

In general, a function $f$ is $s$-integrable iff both $f^+$ and $f^-$ are.

**Theorem 9.** A function $F$ is Loeb integrable iff $F$ has an $s$-integrable lifting $f$.

The proof is similar to the proof in (2).

For $F_0: \{0,1\} \to \text{Real}$ and $F_0(w) =^o F(w)$ we can define

$$\int_0^1 F(t)dt =^o \left( \int_0^1 F(t)dt \right).$$

REFERENCES


