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# MEASURE AND INTEGRATION IN THE ALTERNATIVE SET THEORY

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With the analysis point of view, there is a strong similarity between the saturated models of analysis (NA) and the Alternative Set Theory (AST). So, notions as for example "infinite external set", "internal set" and "hyper-finite set" in NA have corresponding notions as "countable class", "set" and "infinite" set in AST ("infinite" in AST meaning).

The internal definition principle is in relation with the comprehension schema in AST, and comprehension property in NA is in relation with the prolongation axiom.

This similarity between NA and AST becomes complete understandable, if one knows that the ultra-power of the set of hereditary-finite sets enriched with its subsets is a model for AST (see [5]).

On the other hand, AST allows us to make a natural fundation of analysis.

Our intention will be translating the notions and theorems from [2] to AST.

For basic motivations, notions, axioms, definations and theorems for AST, one may consult [1]. We recall, that the class of natural numbers is N,[1] finite natural number is $FN$,[2] rational numbers is $RN$[3], finite rational numbers is $FRN$[4] and bounded rational numbers is $BRN$.[5] All of this classes are class-teoreticly definable, as theirs relations and operations $\leq$, $+$ and $\cdot$ are. The set of real numbers is $Real = BRN/\doteq$, where by $\doteq$ "infinitely near"[6] relations is denoted.

*Real* has all topological and algebrical property as the classical reals does. But for us, its selector $R \subseteq BRN$[7] is more usuful than the such *Real* itself. $R$ is

[1] We model natural numbers in the manner of von Neumann.
[2] $n \in FN$ iff $(\forall X \subseteq n)$ ("$X$ is a set").
[3] $r \in RN$ iff $(\forall m, n \in Z)$ $(r = \langle m, n\rangle)$ and we write $r = \frac{m}{n}$, where $Z = N \cup \{\langle 0, \alpha\rangle \mid \alpha \in N\}$.
[4] $r \in FRN$ iff $(\exists m, n \in FZ)$ $\big(r = \frac{m}{n}\big)$, where $FZ = FN \cup \{\langle 0, \alpha\rangle \mid \alpha \in FN\}$.
[5] $r \in BRN$ iff $(\exists n \in FN)$ $(|r| \leq n)$
[6] $x \doteq y$ iff $(\forall n \in FN)$ $\big(|x-y| \leq \frac{1}{n}\big)$
[7] $A$ class $R$ is image of choice function on the class *Real*.

topologicly isomorphic with *Real*, so that we can translate all interesting properties from *Real* to $R$.

For each $x \in R$, there is $^\circ x \in Real$ so that $x \in^\circ x$, and for each $y \in^\circ x$ we have $st(y) = x$. Observe that $st(x)$ and $^\circ x$ are not same.

For $A \subseteq R$, let $^\circ A = \{x \in Real \mid (\exists y)\ (y \in A \wedge y \in x)\}$.

The topological notions as an interval, open and closed set, convergence and so on, we can define in the usual manner (if we exchange a set wigth a class), but the role of $N$ and $RN$ play $FN$ and $FRN$. The role of $^*R$ play $RN$.

The set $R$ omits some algebric characteristic, which *Real* has. For example, in $R$, $(\exists x)\ (x = 2)$ is true, but not $(\exists x)\ (x = 2)$. However, this is not of an importance for us. $R$ is more natural then *Real*, because $R \subseteq BRN$, while *Real* is the class of classes.

We can replace the saturation with the following theorems (see [1]).

THEOREM 1. *Each countable class is proper semiset.*

THEOREM 2. *Let $X$, $Y$ be countable classes such that $\cap X \subseteq \cup Y$. Then there is a set $u$ such that $\cup X \subseteq u \subseteq \cap Y$.*

THEOREM 3. *Let $Z$ be set-theoretically definable class. Let $X$ be a countable subsemiset of $Z$. If $X$ is directed then there is a $u \in Z$ which is an upper bound of the elements of $X$ ordered by inclusion. If $X$ is dually directed then there is a $u \in Z$ which is a lower bound of the elements of $X$ oredered by $\subseteq$.*

THEOREM 4. *Let $\{X_n, n \in FN\}$ be a sequence of revealed classes (for example, definable classes are revealed) such that for each $m \in FN$, $\cap\{X_n \mid n \leq m\}$ is non-empty. Then $\cap\{X_n \mid n \in FN\} \neq \emptyset$.*

## 1. Loeb aad Lebesque measure

Now, we will start to investigate Loeb measure in $AST$.

Let $x \approx n$ iff $(\exists f)(f\colon \xrightarrow[1-1]{on} x)$.

Let $\Omega$ be a set, and $^*\mathcal{P}(\Omega)$ the set of its subsets. It is easy to see that $^*\mathcal{P}(\Omega)$ is a field of sets.

Let $\bar{P}$ and $P$ be functions so that for $\Omega \approx n$, $A \in^* \mathcal{P}(\Omega)$ and $A \hat{\approx} m$, we have $\bar{P}(A) = \frac{|A|}{|\Omega|} = \frac{m}{n}$ and $P(A) = st\bar{P}(A)$.

If $A \subseteq \Omega$, we can define inner measure so that $P_{\text{inner}}(A) = \sup\{P(B) \mid B \subseteq A, B \text{ set}\}$, and outer measure $P_{\text{outer}}(A) = \inf\{P(B) \mid A \subseteq B, B \text{ set}\}$.

DEF. 1. A class $A \subseteq \Omega$ is Loeb measureble iff $P_{\text{inner}}(A) = P_{\text{outer}}(A)$ and let us put $P(A) = P_{\text{inner}}(A) = P_{\text{outer}}(A)$.

Let $L(\Omega)$ be a class of Loeb measurable classes (which are sets or semisets).

LEMMA 1. *For each a class $A$ we have:*

(1) $A \in L(\Omega)$ *iff for each* $\varepsilon > 0$ *there are classes* $B$ *and* $C$ *so that* $B \subseteq A \subseteq C$ *and* $P(C \backslash B) < \varepsilon$.

(2) $P_{\text{inner}}(A) = 1 - P_{\text{outer}}(\Omega \backslash A)$.

THEOREM 1. *(Loeb) The class* $L(\Omega)$ *is a* $\sigma$*-field and* $P$ *is a* $\sigma$*-additive function.*

*Proof.* A proof that $L(\Omega)$ is a field and $P$ is additive is as in (2).

To complete the proof, we must show that, if for each $i \in FN$, $A_i \supseteq A_{i+1}$, $A_i \in L(\Omega)$, $P(A_i) = r_i$, and $\lim_{\substack{i \to \infty \\ FN}} r_i = r$, then $\cap_{i \in FN} A_i \in L(\Omega)$ and $P(\cap_{i \in FN} A_i) = r$. We may apply the lemma 1 .... In this case is enough to choose sets $B$ and $C$, so that for $A = \cap_{i \in FN} A_i$ and $\varepsilon > 0$, we have $B \subseteq A \subseteq C$, $P(B) \geq r - \varepsilon$ and $P(C) \leq r + \varepsilon$.

The sequence $\{r_i\}_{i \in FN}$ is descending and $\lim_{\substack{i \to \infty \\ FN}} r_l = r$, so we can choose $n \in FN$[8] and a set $C$, so that $A_n \subseteq C$ and $P(C) < r_n + \frac{\varepsilon}{2}$. But then $A \subseteq A_n \subseteq C$ and $P(C) < r_n + \frac{\varepsilon}{2} < r + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = r + \varepsilon$, as required.

Let us find $B$. We can chose sets $B'_m \subseteq A_m$ with $P(B'_m) > r_m - 2_{\varepsilon}^{-m}$.

Let $B_m = B'_1 \cap \ldots \cap B'_m$. By induction, we can easily show that $P(B_m) > r_m - (1 - 2^{-m})\varepsilon$.

The class $X_m = \{B \in^* \mathcal{P}(\Omega) \mid \bar{P}(B) > r - \varepsilon \wedge B \subseteq B_m\}$ is revealed (and, more, a set). Also, we have $B_m \in X_m$, because $B_m \subseteq B_m$ and $\bar{P}(B_m) > r_m - (1 - 2^{-m}) > r - \varepsilon$. Therefore, $X_m \neq \emptyset$ and $X_1 \supseteq X_2 \supseteq \ldots$. So, by Theorem 4. we have $\cap_{m \in FN} X_m \neq \emptyset$.

Let is $B \in \cap_{m \in FN} X_m$. Then, for each $m \in FN$, $B \subseteq B_m$ and $\bar{P}(B) \doteq P(B) > r - \varepsilon$. For each $m \in FN$, we have $B_m \subseteq A_m \cdot SoB \subseteq \cap_{m \in FN} A_m$, as required.

THEOREM 2. *For each* $A \in L(\Omega)$, *there is* $B \in^* \mathcal{P}(\Omega)$ *so that* $P(A \triangle B) = 0$.

The proof of the Theorem follows by the Theorem II.

THEOREM 3. *If* $P(A) = 0$ *and a semiset* $A$ *is a countable union of sets, then there is a set* $B \supseteq A$, *so that* $P(B) = 0$.

The proof follows by the Theorem 3.

Our intention will be to define Lebesques measure on $R$ and *Real*, and to show its conection with Loeb's measure.

Let $H \in N \backslash FN$ and $\triangle t = \frac{1}{H}$. Then, the class $T = \{0, \triangle t, 2\triangle t, \ldots, 1\} = \{\langle k, H \rangle \mid k \in H\}$ is a set.

Let $[s,t] = \{x \in R \mid s \leq x \leq t\}$ and $st_T : T \to [0,1]$, where $st_T = st \restriction T$. Further on, we will write $st$ instead of $st_T$, if it will not bring us to an ambiguity.

For $A \subseteq [0,1]$ we have $st^{-1}(A) \subseteq T$ and $st(st^{-1}(A)) = A$, while for $B \subseteq T$ we have $st(B) \subseteq [0,1]$ and $st^{-1}(st(B)) \supseteq B$.

Let ${}^*[s,t] = \{x \in RN \mid s \leq x \leq t\}$ and $(s,t) = [s,t] \backslash \{s,t\}$.

[8] So that $r < r_n + \frac{\varepsilon}{2}$

Now, we give the definition of Lebsque measure $\mu$, on [0, 1].

For $A = [s,t]$ we have $\mu([s,t]) = \mu((s,t)) = t - s$.

If $A \subseteq [0,1]$, and $A$ is open, then $A$ can be written (uniquely) as a countable union of disjoint open intervals, $A = \cup_{n \in FN} A_n$, and then $\mu(A) = \sum_{n \in FN} \mu(A)$.

If $A \subseteq [0,1]$ is closed, then $[0,1] \backslash A$ is open, so $\mu(A) = 1 - \mu([0,1] \backslash A)$.

For all other $A \subseteq [0,1]$, we define $\mu_{\text{inner}}(A) = \sup\{\mu(B) \mid B \subseteq A,\, B \text{ is closed}\}$ and $\mu_{\text{outer}}(A) = \inf\{\mu(B) \mid A \subseteq B,\, B \text{ is open}\}$.

The class of Lebesque measurable subclasses is $\text{Leb}\,[0,1] = \{A \subseteq [1,0] \mid \mu_{\text{inner}}(A) = \mu_{\text{outer}}(A)\}$, and for $A \in \text{Leb}\,[0,1]$ we have $\mu(A) = \mu_{\text{outer}}(A) = \mu_{\text{inner}}(A)$.

Now, we give Fisher's theorem, which is connection between Loeb's and Lebesque's measure. We need several lemmas for the proof of the theorem. However we omit theirs proofs, which are very similar to proofs in (2).

Only, we must use the Theorem 4. instaed of $\omega$-saturation.

LEMMA 2. *For each $r \in [0,1]$, $st^{-1}(\{r\}) = \{t \in T \mid t \doteq r\} \in L(T)$ and has Loeb measure but any set containing it has a positive Loeb measure.*

LEMMA 3. *For all $s,t \in [0,1]$, so that $s \leq t$ we have $P(T \cap^* [s,t]) = t - s$ and $P(st^{-1}[s,t] = \mu([s,t]) = t - s$.*

LEMMA 4. *If $A \subseteq [0,1]$ and $A$ is closed, then $st^{-1}(A) \in L(T)$ and $\mu(A) = P(st^{-1}(A))$.*

LEMMA 5. *Let $B \subseteq T$ be set. Then $st(B)$ is closed.*

THEOREM 5. *(Fisher) for each $A \subseteq [0,1]$, $A \in \text{Leb}\,[0,1]$ iff $st^{-1}(A) \in L[T)$, and in this case, we have $\mu(A) = P(st^{-1}(A))$.*

The proof follows by the lemmas above.

We say ${}^\circ A \subseteq^\circ [0,1]$ is Lebesque measurable iff $A \in \text{Leb}\,[0,1]$. Let us define ${}^\circ\mu$ with ${}^\circ\mu({}^\circ A) = \mu(A)$.

By Fishers theorem, we have ${}^\circ\mu(A) = P(st^{-1}(A))$.

## 2. Loeb measurable functions

First, will introduce two notions, and will give the theorem, which connects them.

DEF. 2. A class function $F$ is Loeb measurable iff for each $r \in R$, we have $\{w \mid F(w) \leq r\} \in L(\Omega)$.

A set function $f$ is lifting of a function $F \colon \Omega \to R$ iff $f \colon \Omega \to RN$ and $st\, f(w) = F(w)$ almost sure (on a set of measure 1).

THEOREM 5. *A class function $F$ is Loem measurable iff it has a lifting $f$. Moreover, if for each $w \in \Omega$, $|F(w)| \leq n$, then we can find a lifing $f$ such that $|f(w)| \leq n$.*

PROOF: Let $f$ lifts $F$ on $x$, where $x = \{w \in \Omega \mid stf(w) = F(w)\}$ and $P(x) = 1$. For $r \in R$, a class $\{w \in x \mid F(w) \leq r\} = \cap \{w \in x \mid f(w) \leq r + \frac{1}{n}\}$ is measurable, as countable intersection of the measurable sets.

Let $Q = \{g_n \mid n \in FN\}$ be a sequence of finite rationals and $F$ a Loeb measurable function.

The classes $A_n = \{w \mid F(W) \leq g_n\}$ are Loeb measurable, for each $n \in FN$. By the Theorem 2. there is a set $B_n$ so that $P(B_n \triangle A_n) = 0$ and for $g_m \leq g_n$ we have $B_m \subseteq B_n$.

For each $n \in FN$, there is an function $f_n$ such that:

$$(*) \qquad (\forall m \leq n)(x \in B_m \leftrightarrow f_n(x) \leq g_m).$$

It is enough to take $f_n$, so that $f_n(B_0) = \{g_0\}$ and $f(B_{m+1} \backslash B_m) = \{g_m\}$, for each $1 \leq m \leq n$.

Let $X_n = \{f \mid$ "$f$ is function" and $(*)\}$. Then, we have $X_n \neq \emptyset$ and $X_1 \supseteq X_2 \ldots$. According the Theorem 4., there is $f \in \cap_{n \in FN} X_n$. So we have $stf(w) = F(w)$ for we $\Omega \backslash \cup_{n \in FN} (A_n \triangle B_n)$ and

$$\begin{aligned} P\left(\Omega \backslash \bigcup_{n \in FN} (A_n \triangle B_n)\right) &= P(\Omega) - P\left(\bigcup_{n \in FN} (A_n \triangle B_n)\right) \\ &= 1 - \sum_{n \in FN} P(A_n \triangle B_n) = 1. \end{aligned}$$

So we have st $f(w) = F(w)$ almost surely.

The rest we can prove trivialy, if we bound $f$ in $X_n$ by $n$.

DEF. 4. A set function $f$ is lifting of $F_0 \colon [0,1] \to R$ iff $f \colon T \to RN$ and $st\, f(t) = F_0(st(t))$ almost surely on $T$.

LEMMA 6. *A class function $F_0 \colon [0,1] \to R$ is Lebesque measurable iff $F_0$ has a lifting $f \colon T \to RN$.*

PROOF:

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[dr, "F"] & RN \arrow[d, "st"] \\ {[0,1]} \arrow[u, "st^{-1}"] \arrow[r, "F_0"] & R \end{tikzcd}$$

Let us define $F \colon T \to R$ by $F(t) = F_0(st^{-1}(t))$. By Fisher's theorem (Theorem 4) $F_0$ is Lebesque measurable iff $F$ is Loeb measurable iff (by Theorem 5.) $F$ has a lifting $f$.

Remark: If $f: T \to RN$ is almost surely finite set, then $f$ is a fifting of some $F$, where is $F(w) = st f(w)$. Such $F$ is Loeb measurable.

Def. 5. A set function $f$ is a uniform lifting of $F$ iff for all $w \in T$, $st\, f(w) = F(w)$.

Theorem 6. *A class function $F$ has a uniform lifting $f$ iff $\{w \mid f(w) \le r)$, $\{w \mid f(w) \le r\} \in \{x \mid \exists A(A: FN \to^* \mathcal{P}(T)^1 X = \cap_{n \in FN} A\}$ for all $r \in R$.*

The proof follows by Theorem 4.

Let $F_0: T \to Real$, so that ${}^\circ F(w) = F_0(w)$. The function $F_0$ is Loeb measurable iff for all $r \in R$, we have $\{w \in T \mid F(w) \le r\} \in L(T)$.

## 3. Integration

Let $\Omega$ be infinite set, then there is $n \in N$ and a set function $f$ so that $f: n \xrightarrow[1-1]{on} \Omega$. Then, we have $\sum_{w \in \Omega} F(w) = \sum_{m=1}^{n} F(f(m))$.

A bounded and Loeb measurable function $F: \Omega \to R$ is simple iff range $(F)$ is finite.

Def. 6. Let $F$ be bounded and measurable function. If $F$ is simple, then $\int_\Omega F(w)dw = \sum_{r \in \text{range}(F)} r \cdot P(F^{-1}(\{r\}))$. In general

$$\int_\Omega F(w)dw = \sup \left\{ \int_\Omega G(w)dw \mid G \text{ is simple and } G \le F \right.$$

Theorem 7. *(Loeb) Let $F: \Omega \to R$ be bounded and Loeb measurable function and let $f: \Omega \to RN$ be a bounded lifting of $F$, then*

$$\int_\Omega F(w)dw = st \sum_{w \in \Omega} f(w) \triangle w.$$

The proof is similar to the proof in (2).

Theorem 8. *Let $F: [0,1] \to R$ be bounded Lebesgue measurable and let $f$ be a lifting function of $f$. Then, we have $\int_\Omega F(t)dt = st \sum_{w \in \Omega} f(w) \triangle w$.*

The proof follows easily by Theorem 4. and Theorem 7.

Let $F: \Omega \to RN$ be Loeb measurable and non-negative. Let us denote $\min(F(w), n)$ by $(F \wedge n)(w)$.

Then, we define $\int_\Omega F(w)dw = \lim_{n \to \infty} \int_\Omega (F \wedge n)(w)dw$.

A function $F$ is Loeb integrable iff $\int_\Omega F(w)dw$ is finite.

We can denote $\max(F(w), 0)$ by $F^+(w)$ and $\min(F(w), 0)$ by $F^-(w)$. Then we have $F = F^+ + F^-$.

In general case, for unbounded $F$, we define

$$\int\limits_{\Omega} F(w)dw = \int\limits_{\Omega} F^{+}(w)dw - \int\limits_{\Omega} -F^{-}(w)dw.$$

So, we see that a function $F$ is Loeb integrable iff both $F^{+}$ and $F^{-}$ are.

DEF. 7. Let $f$ be non-negative set function so that $f\colon \Omega \to RN$. The function $f$ is $s$-integrable iff sum $\sum_{w\in\Omega} f(w)\triangle w$ is finite, and

$$\lim_{\substack{n\to\infty \\ FN}} st \sum_{w\in\Omega} (f \wedge n)(w)\triangle w = st \sum_{w\in\Omega} f(w)\triangle w.$$

In general, a function $f$ is $s$-integrable iff both $f^{+}$ and $f^{-}$ are.

THEOREM 9. *A function $F$ is Loeb integrable iff $F$ has an $s$-integrable lifting $f$.*

The proof is similar to the proof in (2).

For $F_0\colon [0,1] \to \mathit{Real}$ and $F_0(w) =^{\circ} F(w)$ we can define

$$\int\limits_0^1 F(t)dt =^{\circ} \left( \int\limits_0^1 F(t)dt \right).$$

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