PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 29 (43), 1981, pp. 215–219

GRAPHS HAVING PLANAR COMPLEMENTARY LINE (TOTAL) GRAPHS

Slobodan K. Simić[1]

The planarity of graphs (see [1] for basic definitions and notation) obtained by some graph operations (or graph valued functions) has bean considered in many papers by different authors. Here we shall mention only a few results. In [2] J. Sedlaček settled, historically the most famous result, which characterizes the graphs having planar line graphs. The same problem was treated by D. L. Greenwell and R. L. Hemminger [3] but with forbidden subgraphs involved. As far as the total graphs are concerned, the analogous problems were solved by M. Behzad [4] and J. Akiyama [5]. The purpose of this paper is to consider the planarity of the complements of line (total) graphs. The main feature of the latter problem is that planarity occurs now in the very restricted cases.

We shall first consider the graphs (isolated vertices will be ignored) having planar complementary line graphs. Denote the set of these graphs by $\mathcal{G}$, i.e. let

$$\mathcal{G} = \{G \mid L(G) \text{ is planar and } G \text{ has no isolated vertices}\}.$$

Thus we have to characterize the graphs from $\mathcal{G}$. For that purpose, the following definitions are helpful.

A sequence of mutually different edges $x_1, \dots, x_n$ $(n \geq 2)$ of a graph is an i-sequence if members x_i, x_{i+1} $(i = 1, \dots, n-1)$ are independent edges. The members x_1 and x_n are reffered as exterior ones, while others are interior. Two i-sequences are disjoint if they have no interior members in common.

Now, by restating Kuratowski's theorem, we get:

Proposition 1. Graph $G \in \mathcal{G}$, if and only if both (i) and (ii) hold:

(i) for any 5 edges $e_1, \dots, e_5$ of G, there does not exist a family of mutually disjoint i-sequences such that e_i, e_j $(1 \leq i < j \leq 5)$ are their exterior members;

[1]The results of this paper were communicated at International Colloqium Oberhof (DDR), 10–16 April 1977.

(ii) for any $3+3$ edges e_1, e_2, e_3, f_1, f_2, f_3 of G, there does not exist a family of mutually disjoint i-sequences such that e_i, f_i $(1 \leq i, j \leq 3)$ are their exterior members.

The immediate consequences of the above proposition are the following:

(a) if $G \in \mathcal{G}$, then $\chi(G) \leq 4$;

(b) if G_i is a component of some $G \in \mathcal{G}$, then $d(G_i) \leq 6$;

(c) if $G \in \mathcal{G}$ and $\Delta(G) \geq 9$, then $g(G) \leq \Delta(G) + 2$[2].

From (a) and (b) we conclude that the set of graphs $\mathcal{G}' = \{G \mid G \in \mathcal{G}$ and $\Delta(G) \leq 8\}$ is finite; on the other hand the graphs from $\mathcal{G}'' = \mathcal{G} \backslash \mathcal{G}'$ are of a very simple structure. Hence, we may expect to find much better insight into the graphs from $\mathcal{G}$, than Proposition 1 offers. For that purpose the following transformations on graphs are of interest:

t_1 – deletion of an edge,

t_2 – identification of a pair of nonadjacent vertices.

Now if $G \in \mathcal{G}$, denote by $\mathcal{J}(G)$ the set of all graphs obtained from G by combining t_1 and t_2 (isolated vertices are ignored, again). Then the next lemma is obvious.

Lemma 1. *If $G \in \mathcal{G}$, then $\mathcal{J}(G) \subseteq \mathcal{G}$.*

This lemma suggests the following definition. Graph G is called a t-maximal if for any other graph $G_0 \in \mathcal{G}$, $G \notin \mathcal{J}(G_0)$. Clearly, since $\mathcal{G}'$ is finite it can be described by the finite collection of t-maximal represetatives; on the other hand $\mathcal{G}''$ fails to satisfy that, but since the graphs of $\mathcal{G}''$ are very simple in structure (see (c)) it is possible (conditionally) to assume that the graph G_2 from Fig. 1 (n is being large enough) is a unique t-maximal graph from $\mathcal{G}''$. Thus we can describe all graphs from $\mathcal{G}$ by t-maximal representatives. Namely, we can prove:

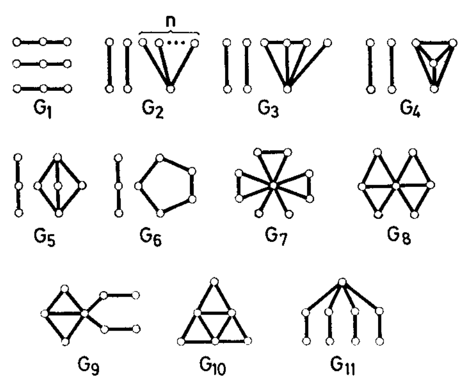

Fig. 1

[2]Recall that $p(G)$, $g(G)$, $\Delta(G)$, $d(G)$, $\chi(G)$, according to [1] denote the number of vertices, the number of edges, the maximal vertex degree, the diameter, the number of components of some graph G.

THEOREM 1. *Graph $G \in \mathcal{G}$, if and only if G is one of the graphs $G_1, \ldots, G_{11}$ of Fig. 1 or $G \in \mathcal{J}(G_i)$ for some i $(1 \leq i \leq 11)$.*

Outline of the proof. Insead of providing a complete proof (which is very elementary but long) we shall give only an idea and a step supposed to be the most illustrative.

Clearly, to prove the theorem it is sufficient to generate all t-maximal graphs of $\mathcal{G}$, or equivalently, to point out that the graphs of Fig. 1 are the entire collection. For that purpose we first fix some invariants of graphs from $\mathcal{G}$ and try to find all t-maximal graph under these restrictions. Namely, we can first fix the number of components (or just the number of nontrivial components) and later the maximal vertex degree. Proceeding in such a way and letting the above quantities to decrease, we shall obtain (up to order) the graphs of Fig. 1.

Here we shall deal only with the case when $\chi(G) = 1$ and $\Delta(G) = 5$. In fact we have to prove that G_9 is the only t-maximal graph under the above restrictions. So observe a vertex v having the degree five and its open neighbourhood $\underline{N}(v) = \{v_1, \ldots, v_5\}$ ($\overline{N}(v) = \underline{N}(v) \cup \{v\}$ is its closed neighbourhood). Now assume $u \notin \overline{N}(v)$. Then the following cases can occur:

Case 1: $\deg u \geq 4$. Assume first $\underline{N}(u) \subseteq \underline{N}(v)$. Then let $e_i = uv_i$ $(i = 1, 2, 3)$ and $f_{j-2} = vv_j$ $(j = 3, 4, 5)$. Now all pairs of edges e_i, f_j except e_3, f_1 are mutally disjoint i-sequences. Taking e_3, $e_3{}' = vv_1$, $e_3'' = uv_4$, f_1 we get the missing i-sequence which is disjoint from the previous ones. Hence, (ii) is contradicted and moreover if $\underline{N}(u) \not\subseteq \underline{N}(v)$ the same happens as can be easily seen[3].

Case 2: $\deg u = 3$. Now similarly to the foregoing, in order to avoid collisions with (ii), $\underline{N}(u) \subseteq \underline{N}(v)$. But then we get a graph that belongs to $\mathcal{J}(G_9)$. Hence to get a t-maximal graph we must add some more edges. But the latter contradicts (ii).

Case 3: $\deg u = 2$. We first note that at least one neighbour of u must be in $\underline{N}(v)$, since otherwise (ii) is being contradicted. If u is adjacent only to $v_1 \in \underline{N}(v)$, then, by (ii), any further edge (if it exists) must connect the vertices from $\underline{N}(v) \mid \{v_1\}$. According to (i) or (ii) at most one new edge may exist. Thus we get a graph which is not t-maximal since it belongs to $\mathcal{J}(G_7)$. So, assume that both neighbours of u are in $\underline{N}(v)$, and also let $\underline{N}(u) = \{v_1, v_2\}$. Suppose w is a vertex such that $w \neq u$ and $w \notin \overline{N}(v)$. Now, since w is not isolated, applying (ii) we get that w is adjacent to some vertex among v_3, v_4, v_5. If w is adjacent to only one vertex out of the mentioned ones then we get a graph belonging to $\mathcal{J}(G_7)$. Hence we must have more edges, but this contradicts either (i) or (ii). This implies that no vertex such as w may exist and that the only possibility that can give a t-maxiimal graph is to add more edges between vertices from $\underline{N}(v)$. Of course, because of (ii) v_1 and v_2 must be nonadjacent. Assume now the that two vertices among v_3, v_4, v_5 are adjacent. Then because of (ii) there cannot exist an edge that connects v_1 or v_2 to any vertex among v_3, v_4, v_5. Using (ii) again, it follows that

[3]In further text we shall not display the i-sequences that are occurring in (i) or (ii).

at most two edges can exist connecting vertices v_3, v_4, v_5. But then we get a graph belonging to $\mathcal{J}(G_9)$. So any additional edge must connect vertices v_1 or v_2 to some vertex among v_3, v_4, v_5. The number of such edges, by (i) or (ii), cannot exceed one, while the remaming possibility gives a graph that belongs to $\mathcal{J}(G_7)$.

Case 4: $\deg u = 1$, i.e. all vertices not belonging to $\overline{N}(v)$ are of degree one. Assume first that there are at least three vertices of degree one not belonging to $\overline{N}(v)$. Clearly, their neighbours must be in $\underline{N}(v)$ and, by (ii), they are all different. If just three vertices of degree one exist, then by (i) or (ii) no more edges exist and the resulting graph belongs $\mathcal{J}(G_7)$. So assume that just two vertices of degree one exist outside $\overline{N}(v)$. If they have a common neighbour in $\underline{N}(v)$, say v_1, then any additional edge (if it exists) according to (ii) connects a pair of vertices from the set $\underline{N}(v)\backslash\{v_1\}$. More than one additional edge cannot exist according to (i) or (ii) while otherwise we get a graph belonging to $\mathcal{J}(G_7)$. Thus the vertices of degree one are adjacent to different vertices from $\underline{N}(v)$ and now following the analysis from case 3 we get G_9 as a t-maximal graph. Hence, it remains that at most one vertex of degree one outstide $\overline{N}(v)$ may exist. Now all additional edges (if they exist) connect the pairs of vertices from $\underline{N}(v)$ and by similar arguments we disprove the existence of any more t-maximal graph.

Corollary 1. If $G = \overline{L}(H)$ for some H is a planar graph, then G is an induced subgraph of some graph from Fig. 2.

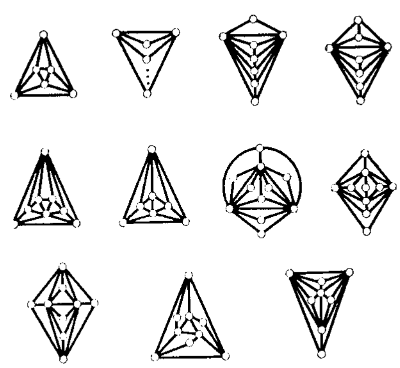

Fig. 2

With the complementary total graphs the situation is quite simple. Owing to the fact that $T(G)$ contains G and $L(G)$ as an induced subgraph we can easily check the validity of the following theorem.

Theorem 2. *If $\overline{T}(G)$ is a planar graph, then it holds:*

(a) *if $p(G) \leq 4$, then G is any graph except $2K_2$, C_4, K_4;*

(b) *if $p(G) > 4$, then G is one of the graphs $K_{1,2} \cup K_2$, $K_3 \cup 2K_1$, $K_{1,4}$.*

REFERENCES

[1] F. Harary, *Graph theory*, Reading 1969.

[2] J. Sedlaček, *Some properties of interchange graphs*, Theory of graphs and its applications, Academic Press, New York, 1962. pp. 145–150.

[3] D. L. Greenwell, R. L. Hemminger, *Forbidden subgraphs for graphs wtth planar line graphs*, Discrete Math 2. No. 1, (1972), 31–34.

[1] M. Behzad, *A criterion for the planarity of total graphs*, Proc. Cambridge Philos. Soc. 63 (1967), 679–681.

[5] J. Akiyma, *Forbidden subgraphs or planar total graphs, middle graphs and iterated line graphs*, J. Combinatorics Informations & System Sciences 3 (1978), 255–260.